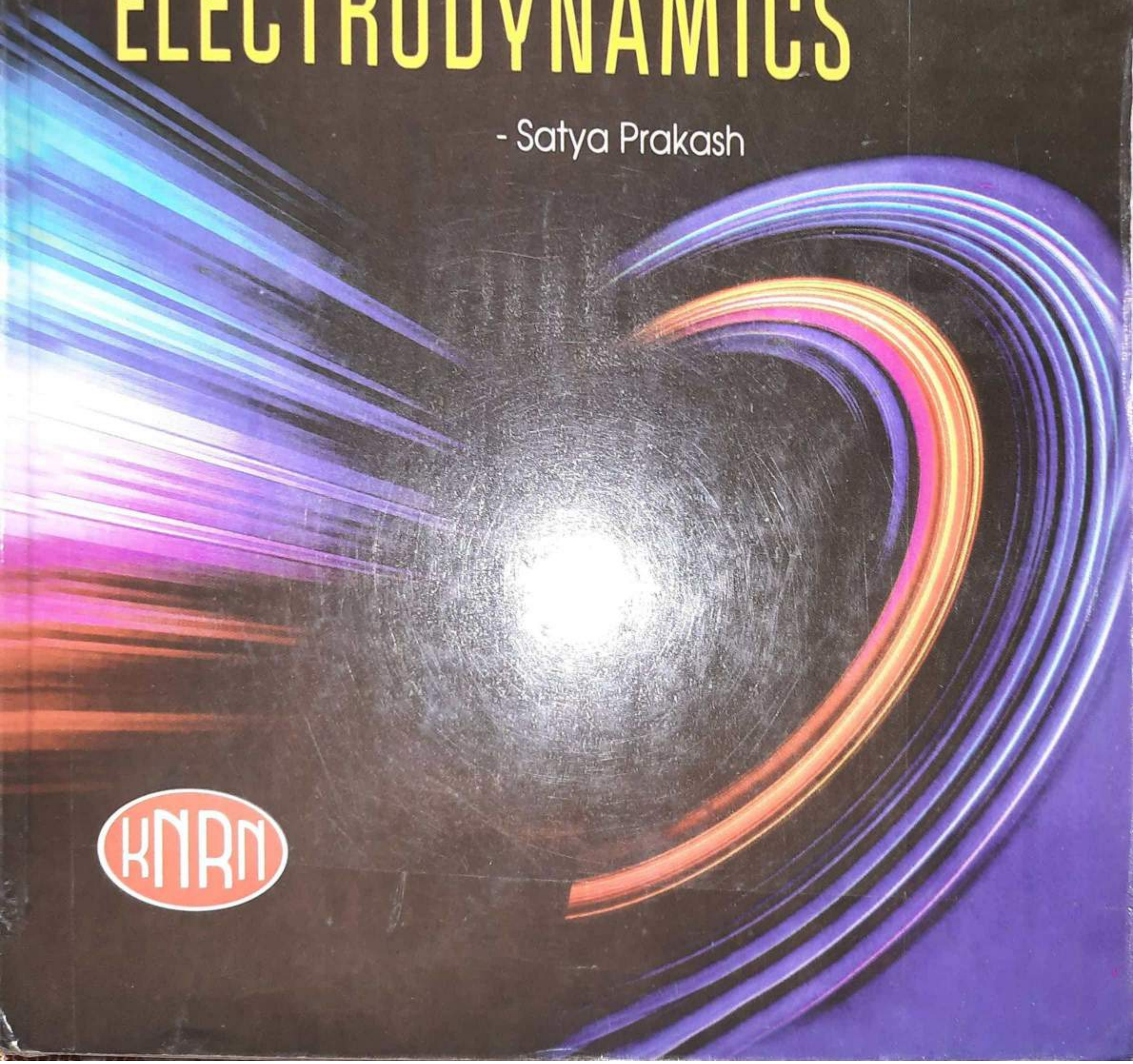
ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS

- Satya Prakash



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ELECTROMAGNETIC THEORY
AND
ELECTRODYNAMICS

(INCLUDING ELECTROSTATICS AND MAGNETOSTATICS)

[For T.D.C., Honours & Post-Graduate Students of All Indian Universities]

MUKUL
SINGH
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By

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PREFACE TO THE ELEVENTH EDITION

I fell great pleasure in bringing out the eleventh edition of the book "Electromagnetic Theory and Electrodynamics". In this edition the book has been throughly revised and following changes have been made.

- The book is brought in large size as desired by students and modern trend.
- Two new chapters titled "Steady Currents" and "Propagation and Radio Waves" have been added to cover the latest syllabi.
- A number of new topics e.g. radiation due to quadrupole, antiferromagnetism and Radiation damping have been added at appropriate places.
- A large number of solved and unsolved problems taken from latest university papers have been added at appropriate places.
- The misprints and omissions have been removed as far as possible.

I am thankful to students and professors for giving warm reception to the book and sending valuable criticism and suggestions for improvement of the book. I am grateful to publishers Kedar Nath Ram Nath Meerut and Vibgyor computer for bringing the book in its present form.

It is hoped that the book is its present from will be more beneficial to the readers. Suggestions for further improvement of the book are cordially invited.

Gyan Vihar, Bijnor —*SATYA PRAKASH*

PREFACE TO THE FIRST EDITION

The present book **"ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS"** is meant for honours and post-graduate students of all Indian Universities. It has been my experience that the students feel very difficult to prepare the revised courses in electromagnetic theory and electrodynamics because of nonavailiability of a book on the subject in simple language explaining the basic ideas of the subfect relating to their previous knowledge and the necessary mathematical steps to arrive at the final results. This book has been written according to revised syllabi of various Indian universities to fullfil requests of my students and continuous inspiration of my colleagues teaching in various post-graduate colleges and universities.

The subject matter has been built up from the basic experim\ental laws and well known accepted concepts. The study of electrodynamics needs mathematical concepts, identities and theorems of vector and tensor analysis. That is why the first chapter is devoted to review these concepts, identities and theorems. In order to keep the book self-sufficient and to develop the subject of electrodynamics from experimental laws five chapters from second to seventh have been devoted to electrostatics and magnetostatics explaining fundamental concepts, basic equations, theorems and solutions of various problems. That is Maxwell's equations of electrodynamics are derived from the experimental laws of Coulomb, Ampere and Faraday, instead of being postulated initially. Chapter 8 explains the origin and derivation of Maxwell's equations and illustrates the propagation of electromagnetic waves in free space and in various media. The next two chapters (9, 10) illustrate applications of electromagnetic waves to reflection, refraction (including wave guides and resonant cavities), dispersion and scattering. The introduction of electromagnetic potentials and the Hertz solution of the wave equation in spherical coordinates are treated in the conventional way in chapter 11 titled "***Electromagnetic Potentials and Spherical Waves.***"

The important subject of "emission of radiation by accelerated point charge and other radiating systems such a osscillating dipole, current element, half-wave antenna and antenna arrays" is discussed in detail chapter 12 titled, ***"Electrodynamics of a moving charge and radiating systems."*** Finally chapter 15 is devoted to relativistic electrodynamics which mainly describes Lorentz covariant formulation of Maxwell's equations and Lorentz force equation, electromagnetic fields of a point charge and invariants of electromagnetic fields.

Rationalised M.K.S. system of units has been used throughout, simply because the majority of modern reference books and papers are now written in this system. The subject matter has been discussed physically and mathematically in a systematic way keeping in mind the difficulties of students. The efforts have been made to make treatments simple, lucid and rather comprehensive. The obscure and difficult points have been explained in simple language and with necessary diagrams in a manner such that the students may feel no difficulty in following the subject. A large number of problems selected from various examination papers and other standard souces have been solved at proper places. In the end of each chapter the selected set of "***Questions and Problems***" has been given. The students are advised to test themselves by a diligent attempt at their solutions.

Though there are a large number of books on this topic, but every book has its own importance and this is the only book written according to latest syllabi of various universities and keeping in mind the actual difficulties of students. Though there is nothing to claimed as original but the subject matter has been arranged in such a way that a student can easily follow the subject.

I am thankful to authors and publishers of various books which I have freely consulted in preparation of manuscripts. I am greatful to Prof. M. P. Tyagi, Dr. V. P. Arora, Dr. J. P. Agarwal, Dr. A. K. Jain of Vardhman College Bijnor. Prof. S. S. Sangal of Hindu College, Moradabad, Prof. V. K. Sharma of K. G. K. College Moradabad for their invaluable suggestions and fruitful blessings. I fail in my duties if I do not thank the publishers for taking keen interest in getting out the book published well in time and the printers for nice and early printing.

In spite of great care some errors and omissions might have crept in. I shall be greatful to the teachers and students who will point them out.

Suggestions and opinions for improvement of the book are cordially invited.

—SATYA PRAKASH

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ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS

1

Preliminary Mathematical Concepts

(Vectors and Tensors)

1·0. Introduction.

"Electrodynamics" is that branch of Physics in which we study the microscopic behaviour of charges in motion and their interaction with matter. The study and derivation of laws and basic equations of elctrodynamics require a few mathematical concepts of vector and tensor analysis. The aim of the present chapter is to review these mathematical concepts in brief.

1·1. Review of Vector Algebra.

A vector is a *'directed line segment'* having direction and magnitude both. Accordingly a vector **A** can be represented by an arrow $\overrightarrow{OP}$ of finite length directed from *initial point O to terminal point P*, the length of arrow represents the magnitude of vector **A** and the arrow-head denotes the direction of vector **A**. In printed matter a vector quantity is represented by the clarendon letter **A**. The magnitude of modulus of vector **A** is represented by $|\mathbf{A}|$ or simply A. The unit vector along vector **A** is represented by $\hat{A}$.

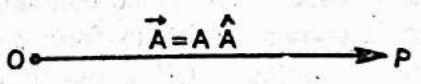


Fig. 1.1

The formulation of a law of physics in terms of vector is independent of the choice of axes of reference.

1·1. (a) Addition of vectors.

The manner in which the vector quantities are added is expressed by the *triangular law or parallelogram law of vector addition.*

Let there be two given vectors **a** and **b**. To add vector **b** to vector **a**, the initial point of vector **b** is placed at the terminal point of vector **a**. then an arrow drawn from the intial point of **a** to the terminal point of **b** represents the vector sum (**c**) of two given vectors **a** and **b** *i.e.*, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ or $\mathbf{c} = \mathbf{a} + \mathbf{b}$ (fig. 1.2).

The sum is independent of choice of position of *O*. This law of vector addition is called the *triangular law of vector addition.*

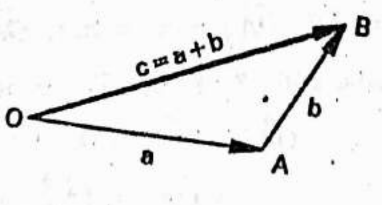


Fig. 1.2

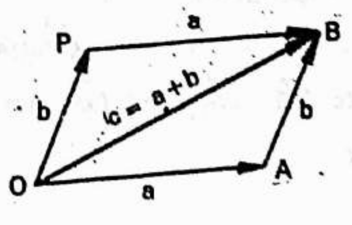


Fig. 1.3

Now the parallelogram is completed as shown in fig. 1·3. Then the parallelogram law of vector addition may be stated as :

If the two adjacent sides of a parallelogram represent the two given vectors **a** and **b** in magnitude and direction, then the diagonal of the parallelogram will represent the vector sum (**c**) of the two given vectors **a** and **b.**

The vector addition obeys the following properties :

(i) The vector addition is *commutative i.e.* $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

(ii) The vector addition is associative *i.e.*

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{c}) + \mathbf{b}.$$

1·1. (b). Multiplication of a Vector by a Scalar.

The product of a scalar s and vector **a** is a vector whose modulus is $|s|$ a and whose direction is same as that of **a.** If **a** and **b** are vectors and s and t are scalars, then

(i) $s\mathbf{a} = \mathbf{a}s$.

(ii) $s(t\mathbf{a}) = (st)\,\mathbf{a}$.

(iii) $(s+t)\,\mathbf{a} = s\mathbf{a} + t\mathbf{a}$.

(iv) $s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}$.

(v) If **a** and **b** are two non-zero collinear vectors, then there exists *a* non-zero scalar s such that

$$\mathbf{a} = s\mathbf{b}$$

conversely relation of this type implies that **a** and **b** are collinear.

1·1. (c). Orthogonal Resolution of Vectors ; Direction Cosines.

Let. OX, OY, OZ be three non-coplanar lines, each being perpendicular to the palne contianing the other two. The system of axes is chosen, so as to form a righthanded cartesian coordinate system such that if OX is turned towards OY about OZ, through a small angle, the right handed screw would advance along positive direction of OZ.

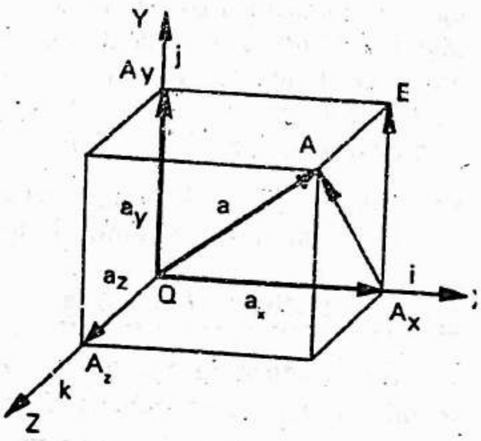


Fig. 1.4

Let **OA** represent the given vector **a** such that the lengths of projections of **a** along X, Y and Z axes be a_x, a_y and a_z, respectively respresented in fig. 1.4 by $\overrightarrow{OA}_x, \overrightarrow{OA}_y$ and $\overrightarrow{OA}_z$. To resolute vector **a** along X, Y, Z axes the initial point may be taken at any point in space. Here we have taken the initial point of **a** at origin of coordinate system merely for convenience.

If is customary to take $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as unit vectors along X, Y and Z axes respectively, so that

$$\overrightarrow{OA}_x = \mathbf{a}_x = a_x\mathbf{i},\ \overrightarrow{OA}_y = \mathbf{a}_y = a_y\mathbf{j},\ \overrightarrow{OA}_z = \mathbf{a}_z = a_z\mathbf{k}$$

where $\overrightarrow{OA}_x, \overrightarrow{OA}_y$ and $\overrightarrow{OA}_z$ are collinear with X, Y and Z axes respectively.

Now
$$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{OA}_x + \overrightarrow{A_xA}\\ &= \overrightarrow{OA}_x + (\overrightarrow{A_xE} + \overrightarrow{EA})\\ &= \overrightarrow{OA}_x + \overrightarrow{OA}_y + \overrightarrow{OA}_z \quad (\text{since } \overrightarrow{A_xE} = \overrightarrow{OA}_y, \overrightarrow{EA} = \overrightarrow{OA}_z)\end{aligned}$$

i.e.
$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \qquad ...(1)$$

This equation represents vector **a** in terms of rerctangular components.

By geometry the magnitude of **a** may be expressed as

$$a = |a| = |\overrightarrow{OA}| = \sqrt{[(\overrightarrow{OA}_x)^2 + (\overrightarrow{OA}_y)^2 + (\overrightarrow{OA}_z)^2]}$$

i.e.

$$a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}. \quad ...(2)$$

Now unit vector along **a** may be expressed as

$$\hat{a} = \frac{\mathbf{a}}{a} = \frac{a_x}{a}\,\mathbf{i} + \frac{a_y}{a}\,\mathbf{j} + \frac{a_z}{a}\,\mathbf{k} \quad ...(3)$$

obviously $\frac{a_x}{a}, \frac{a_y}{a}, \frac{a_z}{a}$ are cosines of angles which vector **a** makes with X, Y and Z axes respectively. That is why $\frac{a_x}{a} = \cos\alpha$, $\frac{a_y}{a} = \cos\beta$ and $\frac{a_z}{a} = \cos\gamma$ are called *direction cosines* of vector **a** relative to X, Y, Z axes respectively. Substituting these values in (3), we get

$$\hat{a} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \quad ...(4)$$

Taking modulus of (4) on either side, we get

$$|\hat{a}| = \sqrt{(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)}$$

i.e.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad ...(5)$$

i.e. sum of squares of direction cosines of any vector is unity.

1·1. (d) Product to two Vectors.

As a vector quantity involves magnitude and direction both ; therefore it is difficult to assign a definite meaning to the product of two vectors, whether the product is a scalar or a vector quantity. Accordingly there are two types of product of two vectors :

1. The scalar product of two vectors a and b is denoted by. $\mathbf{a} \cdot \mathbf{b}$ and is defined as a scalar quantity which is equal to the product of the magnitudes of two given vector and the cosine of angle between their directions *i.e.*

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta. \quad (1)$$

The scalar product is also called *dot product.*

Characteristics of scalar product of two vectors.

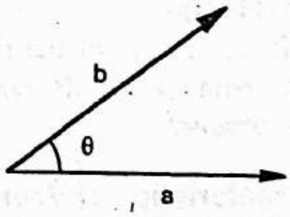


Fig. 1.5

(i)

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$$

$$\mathbf{b} \cdot \mathbf{a} = ba\cos(-\theta) = ab\cos\theta$$

Thus

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

i.e. the scalar product of two vector is commutative.

(ii) If $\mathbf{a} \cdot \mathbf{b} = 0$, then either of two vectors is a null vector or the vectors are mutually perpendicular *i.e.* either. $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\theta = \frac{\pi}{2}$. For *two mutually perpendicular vectors* $\mathbf{a} \cdot \mathbf{b} = ab\cos\frac{\pi}{2} = 0$; therefore if **i, j, k** are unit veectors along three mutually perpendicular axes, then $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.

Obviously two given vectors are orthogonal if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

(iii) The scalar product of two *equal vectors* is given by

$$\mathbf{a} \cdot \mathbf{a} = aa\cos\theta = a^2$$

i.e. the square of any vector is equal to the square of its modulus; thus if **i, j, k** are unit vectors

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1.$$

(iv) *Scalar product of two vectors obeys associative law i.e.*

$$(m\mathbf{a})\cdot(n\mathbf{b}) = mn\ \mathbf{a}\cdot\mathbf{b} = n\mathbf{a}\cdot m\mathbf{b}$$

(v) *Scalar product obeys distributive law of multiplication i.e.*

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$$

(vi) ***Scalar product in terms of components.*** If (a_x, a_y, a_z) and (b_x, b_y, b_z) are components of **a** and **b** along (X, Y, Z) axes respectively, then

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \text{ and } \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

so that

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z.$$

2. **Vector Product of two Vectors a and b** is represented by $\mathbf{a}\times\mathbf{b}$ and is defined as a vector whose moduluus is equal to the product of magnitudes of given vectors and the sine of angle between their directions and whose direction is perpendicular to the plane containing **a** and **b** *i.e.*

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\hat{n}$$

where θ is the angle between **a** and **b** and $\hat{n}$ is a unit vector along $\mathbf{a}\times\mathbf{b}$. The correct direction of $\hat{n}$ is determined by right-hand screw rule ; shown in fig. 1·6 (b) and 1·6 (c).

According to **right hand screw rule** if right-handed screw is rotated from the direction of **a** to that of **b** through a small angle, then the direction, in which the screw advances, is the correct direction of $\hat{n}$ (or $\mathbf{a}\times\mathbf{b}$) [fig. 1·6 (b)].

This law may be stated in a different way as :

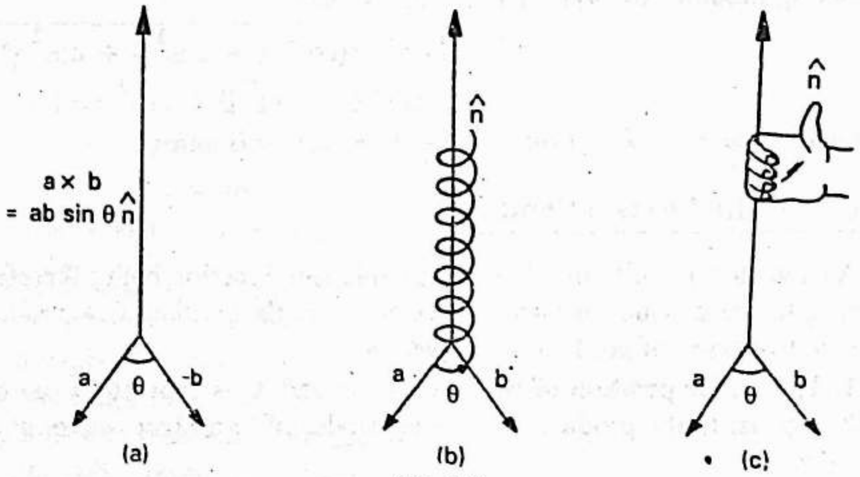


Fig. 1.6

If the fingers of the right hand are turned from the direction of **a** to **b** through a small angle, the thumb points along the correct directioin of $\hat{n}$ (or $\mathbf{a}\times\mathbf{b}$) [fig. 1·6 (c)]. The vector product is also called *cross-product.*

Characteristics of Vector Product of two Vectors :

(i) Commutative law fails (i.e. $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$).

From definition

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\hat{n} \qquad \text{[fig. 1·7 (a)]}$$

$$\mathbf{b}\times\mathbf{a} = ba\sin\theta\,(-\hat{n}) \qquad \text{[fig. 1·7 (b)]}$$

i.e. $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$

or $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}.$

Thus vector product of two vectors does not obey commutative law.

(ii) Condition for two collinear vectors. Two vectors are collinear if $\theta = 0$ or π, *i.e.* $\sin\theta = 0$. Thus for *two collinear vectors* **a** and **b**

$$\mathbf{a}\times\mathbf{b} = ab\sin 0^\circ\,\hat{n} = 0. \qquad \text{(Since } \sin 0^\circ = 0\text{).}$$

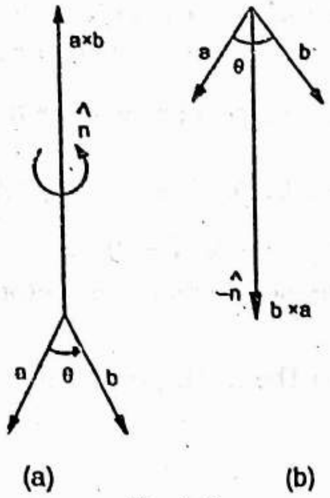


Fig. 1.7

Thus two vectors are collinear (parallel or anti-parallel) *if and only if*

$$\mathbf{a} \times \mathbf{b} = 0.$$

(iii) Vector product of two equal vectors. If $\mathbf{a} = \mathbf{b}$, $\theta = 0^\circ$, so that the vector product of two equal vectors is given by

$$\mathbf{a} \times \mathbf{a} = aa \sin 0 \, \hat{n} = 0$$

i.e., the vector product of two equal vectors is always zero. In particular if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors, then

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.$$

(iv) Vector product of two perpendicular vectors. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors.

$$\mathbf{a} \times \mathbf{b} = ab \sin \frac{\pi}{2} \, \hat{n} = ab\hat{n}$$

where $\hat{n}$ is a unit vector along the direction such that $\mathbf{a}, \mathbf{b}$ and $\hat{n}$ form a right handed system.

In particular if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the principal axes X, Y, Z respectively, then

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} = -\mathbf{j} \times \mathbf{i} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} = -\mathbf{k} \times \mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} = -\mathbf{i} \times \mathbf{k}. \end{aligned}$$

(v) Associative law. If m is scalar and $\mathbf{a}, \mathbf{b}$ are vectors, then

$$(m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) = m\,(\mathbf{a} \times \mathbf{b}) = mab \sin \theta \, \hat{n}$$

i.e., vector product of two vectors obeys associative law.

(vi) Distributive law. Vector product obeys distributive law of multiplication *i.e.*, $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

(vii) Vector product in terms of components.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) \times (\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z) \\ &= a_x b_y \mathbf{k} - a_x b_z \mathbf{j} - a_y b_x \mathbf{k} + a_y b_z \mathbf{i} + a_z b_x \mathbf{j} - a_z b_y \mathbf{i} \\ &= (a_y b_z - a_z b_y)\,\mathbf{i} + (a_z b_x - a_x b_z)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \end{aligned}$$

(viii) Determination of sin θ from vector product. We have $\mathbf{a} \times \mathbf{b} = ab \sin \theta \, \hat{n}$ so that $|\mathbf{a} \times \mathbf{b}| = ab \sin \theta$

$$\therefore \quad \sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{ab}.$$

1·1. (c) Product of three Vectors.

The vector product of two vectors $\mathbf{b}$ and $\mathbf{c}$ is a vector quantity, so this product $(\mathbf{b} \times \mathbf{c})$ may be multiplied scalarly or vectorially with a third vector $\mathbf{a}$ to give two triple products namely $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. The former being a scalar quantity is called the *scalar triple product* ; the latter being a vector quantity is called the *vector triple product.*

(a) Scalar triple product. The scalar product of three vectors is the scalar product of any vector with the vector product of remaining two vectors. For example $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ are scalar triple products of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The scalar triple product is usually denoted by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Let

$$\begin{aligned} \mathbf{a} &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \\ \mathbf{b} &= b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \\ \mathbf{c} &= c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}, \end{aligned}$$

we have
$$(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (b_y c_z - b_z c_y)\,\mathbf{i} + (b_z c_x - b_x c_z)\,\mathbf{j} + (b_x c_y - b_y c_x)\,\mathbf{k}$$

$$\therefore \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot [(b_y c_z - b_z c_y)\,\mathbf{i} + (b_z c_x - b_x c_z)\,\mathbf{j} + (b_x c_y - b_y c_x)\,\mathbf{k}]$$
$$= a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x)$$

i.e.
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [\mathbf{abc}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

We know that if any two rows of determinant are interchanged, a minus sign appears. Hence,

$$[\mathbf{abc}] = - \begin{vmatrix} b_x & b_y & b_z \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix} = -\,\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\,[\mathbf{bac}]$$
$$= + \begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = [\mathbf{bca}]$$

Thus we have $[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}]$. *i.e.*

In scalar triple product the positions of dot and cross may be interchanged without changing the value of product, provided the cyclic order is maintained.

Characteristics of Scalar Triple Prodcut :

(i) Distributive law holds for scalar triple product, *e.g.*,

$$[\mathbf{a}, \mathbf{b} + \mathbf{d}, \mathbf{c} + \mathbf{e}] = [\mathbf{abc}] + [\mathbf{abe}] + [\mathbf{adc}] + [\mathbf{ade}]$$

the cyclic order of factors must be retained.

(ii) **Condition for coplanar vectors.** If three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, their scalar triple product is zero, since then $\mathbf{b} \times \mathbf{c}$ is perpendicular to $\mathbf{a}$ and their scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ or $[\mathbf{abc}] = 0$. Thus the vanishing of $[\mathbf{abc}]$ is the condition that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ should be *coplanar.* If two of the vectors are parallel, this condition is satisfied. In particular if two of the vectors are *equal,* the scalar triple product is zero, *i.e.*, $[\mathbf{aab}] = [\mathbf{abb}] = [\mathbf{cbc}] = 0$.

(iii) For unit vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the scalar triple product is unity

i.e.,
$$[\mathbf{i\,j\,k}] = [\mathbf{j\,k\,i}] = [\mathbf{k\,i\,j}] = 1,$$
and
$$[\mathbf{i\,k\,j}] = [\mathbf{k\,j\,i}] = [\mathbf{j\,i\,k}] = -\,1.$$

(b) **Vector triple product.** The vector triple product of three vectors is a vector product of any vector with the vector product of remaining two vectors. For example, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is vector triple product of three vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and is expressed as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}.$$

Proof. Let $\mathbf{p} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

From the property of vector product of two vectors $\mathbf{b} \times \mathbf{c}$ is a vector perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$. Also $\mathbf{p} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a vector perpendicular to the plane containing $\mathbf{a}$ and $(\mathbf{b} \times \mathbf{c})$, *i.e.*, $\mathbf{p}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$, and therefore is expressible in terms of $\mathbf{b}$ and $\mathbf{c}$ in the form

$$\mathbf{p} = \lambda_1 \mathbf{b} + \lambda_2 \mathbf{c}, \qquad \text{...(ii)}$$

λ_1 and λ_2 being scalars.

Multiplying both sides scalarly by $\mathbf{a}$, we get

$$\mathbf{a}\cdot\mathbf{p} = \lambda_1(\mathbf{a}\cdot\mathbf{b}) + \lambda_2(\mathbf{a}\cdot\mathbf{c}).$$

But $\mathbf{a}\cdot\mathbf{p} = 0$ [since $\mathbf{p} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to **a**] therefore

$$\lambda_1(\mathbf{a}\cdot\mathbf{b}) + \lambda_2(\mathbf{a}\cdot\mathbf{c}) = 0.$$

or
$$\frac{\lambda_1}{\mathbf{a}\cdot\mathbf{c}} = -\frac{\lambda_2}{\mathbf{a}\cdot\mathbf{b}} = \lambda \text{ (say).} \qquad \text{...(iii)}$$

This gives $\lambda_1 = \lambda(\mathbf{a}\cdot\mathbf{c})$ and $\lambda_2 = -\lambda(\mathbf{a}\cdot\mathbf{b})$.

Substituting these values of λ_1 and λ_2 in (ii), we get

$$\mathbf{p} = \lambda[(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]. \qquad \text{...(iv)}$$

If is easy to show that λ is independent of the magnitudes and directions of vector **a, b, c.** To find the value of λ, let us consider a particular case. Consider a right handed co-ordinate system such that *X*-axis is along **b**, *Y*-axis is normal to **b** but in the plane of **b** and **c**, then *Z* axis will be along $(\mathbf{b}\times\mathbf{c})$ [fig. 1·8], then

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = b\mathbf{i}$$
$$\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j},$$

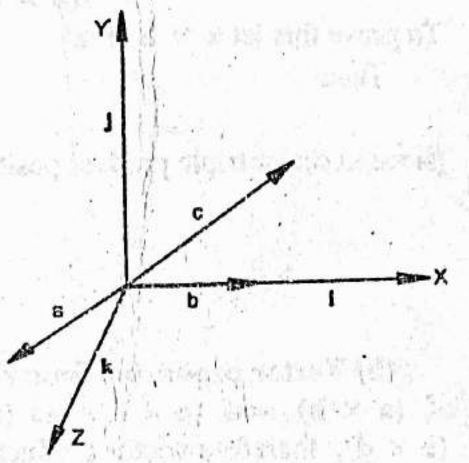


Fig. 1.8

so that

$$(\mathbf{b}\times\mathbf{c}) = (b\mathbf{i})\times(c_x\mathbf{i} + c_y\mathbf{j}) = bc_y\mathbf{k}$$

Therefore,

$$\begin{aligned}\mathbf{p} &= \mathbf{a}\times(\mathbf{b}\times\mathbf{c})\\ &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\times bc_y\mathbf{k}\\ &= a_ybc_y\mathbf{i} - a_xbc_y\mathbf{j}\end{aligned} \qquad \text{...(v)}$$

Also $\quad \mathbf{a}\cdot\mathbf{c} = a_xc_x + a_yc_y.$

$$\begin{aligned}\therefore \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} &= (a_xc_x + a_yc_y)\,(b\mathbf{i})\\ &= a_xc_x\,b\mathbf{i} + a_yc_y\,b\mathbf{i}\end{aligned} \qquad \text{...(vi)}$$

$$(\mathbf{a}\cdot\mathbf{b}) = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b\mathbf{i}) = a_xb.$$

$$\therefore \quad (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} = (a_xb)\,(c_x\mathbf{i} + c_y\mathbf{j}) = a_xbc_x\mathbf{i} + a_ybc_y\mathbf{j} \qquad \text{...(vii)}$$

Using (v), (vi), and (vii) equation (iv) yields

$$\begin{aligned}a_ybc_y\mathbf{i} - a_xbc_y\mathbf{j} &= \lambda[(a_xc_xb\mathbf{i} + a_yc_yb\mathbf{i}) - (a_xbc_x\mathbf{i} + a_xbc_y\mathbf{j})]\\ &= \lambda(a_ybc_y\mathbf{i} - a_xbc_y\mathbf{j})\end{aligned}$$

or
$$\lambda = \frac{a_ybc_y\mathbf{i} - a_xbc_y\mathbf{j}}{a_ybc_y\mathbf{i} - a_xbc_y\mathbf{j}} = 1.$$

Substituting this value of λ in (iv), we get

$$\mathbf{p} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \qquad \text{...(viii)}$$

Characteristics of vector triple product :

(i) Vector triple product is not commutative :

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{b}\times\mathbf{c})\times\mathbf{a}$$

Since
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -(\mathbf{b}\times\mathbf{c})\times\mathbf{a}.$$

(ii) Vector triple product is not associative ***i.e.***

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

Since
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}.$$

and
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\,\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$$

$$= -\,[(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}]$$
$$= (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$$

If follows that the product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ represents a vector coplanar with $\mathbf{b}$ and $\mathbf{c}$, while the product $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ represents a vector lying in the plane of $\mathbf{a}$ and $\mathbf{b}$. Hence the two products do not represent the same vector quantity.

1·1. (f) Product of four Vectors

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors.

(a) Scalar product of four Vectors. The scalar product of four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is defined as

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c})$$

To prove this let $\mathbf{a}\times\mathbf{b} = \mathbf{n}$

Then
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{n}\cdot(\mathbf{c}\times\mathbf{d})$$
$$= (\mathbf{n}\times\mathbf{c})\cdot\mathbf{d}$$

(since in scalar triple product positions of dot and cross may be interchanged)

$$= \{(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\}\cdot\mathbf{d}$$
$$= -\,\{\mathbf{c}\times(\mathbf{a}\times\mathbf{b})\}\cdot\mathbf{d}$$
$$= -\,\{(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}\}\cdot\mathbf{d}$$
$$= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c}).$$

(b) Vector product of four vectors. The vector product of four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, is the vector product of $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$. As $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is a vector perpendicular to vectors $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$, therefore vector product of four vectors is a vector coplanar with $\mathbf{a}, \mathbf{b}$ and also with $\mathbf{c}, \mathbf{d}$ and is defined as

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{acd}]\,\mathbf{b} - [\mathbf{bcd}]\,\mathbf{a} \qquad \text{...(i)}$$
$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d} \qquad \text{...(ii)}$$

The prove these relationships, let $\mathbf{c}\times\mathbf{d} = \mathbf{m}$

then
$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{m}$$
$$= (\mathbf{a}\cdot\mathbf{m})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{m})\,\mathbf{a}$$
$$= \{\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})\}\,\mathbf{b} - \{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})\}\,\mathbf{a}$$
$$= [\mathbf{acd}]\,\mathbf{b} - [\mathbf{bcd}]\,\mathbf{a} \qquad \text{...(i)}$$

Again let $\mathbf{a}\times\mathbf{b} = \mathbf{n}$, then

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{n}\times(\mathbf{c}\times\mathbf{d})$$
$$= (\mathbf{n}\cdot\mathbf{d})\,\mathbf{c} - (\mathbf{n}\cdot\mathbf{c})\,\mathbf{d}$$
$$= \{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}\}\,\mathbf{c} - \{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\}\,\mathbf{d}$$
$$= [\mathbf{a\,b\,d}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d} \qquad \text{...(ii)}$$

1·1. (g) Differentiation of Vectors.

If a vector $\mathbf{r}$ varies corresponding to the variation of a single scalar variable t in any interval, then $\mathbf{r}$ is said to be a vector function of scalar variable t and is expressed as $\mathbf{r} = \mathbf{f}(t)$. Let $\delta\mathbf{r}$ be the increase in $\mathbf{r}$ corresponding to the increase δt in the scalar t, then the derivative of vector function $\mathbf{r}(t)$ with respect to scalar t is defined as

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t}$$

provided the limit exists.

When this limit exists, the function $\mathbf{r}$ is said to be differentiable and the process of determining the derivative of the function is called *differentiation.*

As the vectorial properties of a vector are not changed by its division by a scalar, thus the derivative of a vector with respect to a scalar is itself a vector. If $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are vector functions of scalar variable t and c is a constant vector, then it is easy to see that

(i) $\frac{d\mathbf{c}}{dt} = 0$

(ii) $\frac{d}{dt}(\mathbf{r}_1 + \mathbf{r}_2) = \frac{d\mathbf{r}_1}{dt} + \frac{d\mathbf{r}_2}{dt}$

(iii) $\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt}$, s being a scalar function of t,

(iv) $\frac{d(s\,\mathbf{r})}{dt} = \frac{ds}{dt}\mathbf{r} + s\frac{d\mathbf{r}}{dt}$

(v) If $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is the position vector, then

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

(vi) $\frac{d}{dt}(\mathbf{r}_1 \cdot \mathbf{r}_2) = \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2 + \frac{d\mathbf{r}_2}{dt} \cdot \mathbf{r}_1$

If $\mathbf{r}_1 = \mathbf{r}_2$ (say), then

$$\frac{d}{dt}(\mathbf{r}^2) = \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}$$

(vii) $\frac{d}{dt}(\mathbf{r}_1 \times \mathbf{r}_2) = \frac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2 + \mathbf{r}_1 \times \frac{d\mathbf{r}_2}{dt}$.

In the differentiation of vector product, the order of vectors must be reserved.

(viii)
$$\frac{d}{dt}[\mathbf{r}_1\,\mathbf{r}_2\,\mathbf{r}_3] = \frac{d\mathbf{r}_1}{dt} \cdot (\mathbf{r}_2 \times \mathbf{r}_3) + \mathbf{r}_1 \cdot \left(\frac{d\mathbf{r}_2}{dt} \times \mathbf{r}_3\right) + \mathbf{r}_1 \cdot \left(\mathbf{r}_2 \times \frac{d\mathbf{r}_3}{dt}\right)$$
$$= \left[\frac{d\mathbf{r}_1}{dt}\;\mathbf{r}_2\;\mathbf{r}_3\right] + \left[\mathbf{r}_1\;\frac{d\mathbf{r}_2}{dt}\;\mathbf{r}_3\right] + \left[\mathbf{r}_1\;\mathbf{r}_2\;\frac{d\mathbf{r}_3}{dt}\right]$$

(ix)
$$\frac{d}{dt}\{\mathbf{r}_1 \times (\mathbf{r}_2 \times \mathbf{r}_3)\} = \frac{d\mathbf{r}_1}{dt} \times (\mathbf{r}_2 \times \mathbf{r}_3) + \mathbf{r}_1 \times \left(\frac{d\mathbf{r}_2}{dt} \times \mathbf{r}_3\right) + \mathbf{r}_1 \times \left(\mathbf{r}_2 \times \frac{d\mathbf{r}_3}{dt}\right)$$

Partial Derivaties. If $\mathbf{f}$ is any vector function depending upon more than one scalar variable say (x, y, z, t) *i.e.*

$$\mathbf{f} = \mathbf{f}(x, y, z, t)$$

then the partial derivative of $\mathbf{f}$ with respect to any of the variables, taking all other variables constant, is defined as

$$\frac{\partial \mathbf{f}}{\partial x} = \lim_{\delta x \to 0} \frac{\mathbf{f}(x + \delta x, y, z, t) - \mathbf{f}(x, y, z, t)}{\delta x} \qquad \text{...(1)}$$

$$\frac{\partial \mathbf{f}}{\partial y} = \lim_{\delta y \to 0} \frac{\mathbf{f}(x, y + \delta y, z, t) - \mathbf{f}(x, y, z, t)}{\delta y} \qquad \text{...(2)}$$

$$\frac{\partial \mathbf{f}}{\partial z} = \lim_{\delta z \to 0} \frac{\mathbf{f}(x, y, z + \delta z, t) - \mathbf{f}(x, y, z, t)}{\delta z} \qquad \text{...(3)}$$

$$\frac{\partial \mathbf{f}}{\partial t} = \lim_{\delta t \to 0} \frac{\mathbf{f}(x, y, z, t + \delta t) - \mathbf{f}(x, y, z, t)}{\delta t} \qquad \text{...(4)}$$

Above equations (1), (2), (3) and (4) represent the partial derivatives of $\mathbf{f}$ with respect to x, y, z, t respectively, if these limits exist.

The partial derivatives of higher order are expressed as

$$\frac{\partial^2 \mathbf{f}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{f}}{\partial x}\right); \frac{\partial^2 \mathbf{f}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{f}}{\partial y}\right), \frac{\partial^2 \mathbf{f}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{f}}{\partial y}\right) \text{ etc.}$$

Further if there exist continous partial derivatives of second order $\mathbf{f}$, then $\frac{\partial^2 \mathbf{f}}{\partial x\, \partial y} = \frac{\partial^2 \mathbf{f}}{\partial y\, \partial x}$ *i.e.* order of differentiation does not matter.

Moreover the partial derivatives of vectors obey the same rules as do the scalar functions, in ordinary calculus with the condition that the order of vectors must not be changed in cases involving vector product.

1·2. Gradient of a Scalar Field.

Let $\phi(x, y, z)$ be any continuously differentiable scalar function. The gradient of scalar function ϕ is mathematically defined as

$$\text{grad } \phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} \quad \text{...(1)}$$

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \nabla \phi$$

where $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is called 'del' or *nabla.*

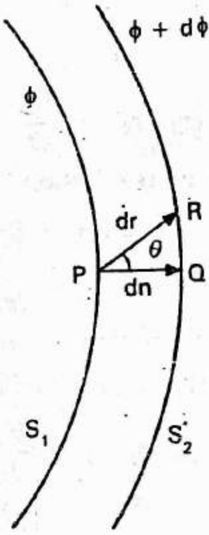


Fig. 1.9

Physical Interpretation. In the scalar field let there be two level surfaces S_1 and S_2 very close to each other characterised by scalar functions ϕ and $\phi + d\phi$ respectively. Consider two points P and R on the level surfaces* S_1 and S_2 respectively. Let $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ be the position vectors of P and R respectively relative to any arbitary origin, then $\overrightarrow{PR} = d\mathbf{r}$.

If co-ordinates of P and R are (x, y, z) $(x + dx,\ y + dy,\ z + dz)$ respectively, then

$$d\mathbf{r} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz \quad \text{...(2)}$$

As the values of scalar function at $P(x, y, z)$ and $R(x + dx,\ y + dy,\ z + dz)$ are ϕ and $\phi + d\phi$, we may write

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \quad \text{...(3)}$$

$$= \left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}\right)\cdot(\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz)$$

$$= (\nabla \phi)\cdot d\mathbf{r} \quad \text{...(4)}$$

using (1) and (2)

In particular if we consider that the point R (*i.e.* $d\mathbf{r}$) lies on the level surface S_1, then $d\phi = 0$.

so that

$$(\nabla \phi)\cdot d\mathbf{r} = 0 \quad \text{...(5)}$$

thereby showing that *the vector* $\nabla \phi$ *is normal to the level surface* S_1.

In $d\mathbf{n}$ represents the distance along the normal from point P to the surface S_2, then

---

*A level surface is one which has same value of scalar function at each point.

$$dn = PQ = dr\cos\theta = \hat{n}\cdot d\mathbf{r} \qquad ...(6)$$

where $\hat{n}$ is a unit vector normal to the surface S_1 at P. If we propagate from P to Q, the value of scalar function ϕ increases by an amount $d\phi$; therefore we may write

$$d\phi = \frac{\partial\phi}{\partial n}\,dn = \frac{\partial\phi}{\partial n}\,\hat{n}\cdot d\mathbf{r} \text{ using (6)} \qquad ...(7)$$

Comparing (4) and (7), we get

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial n}\,\hat{n} \qquad ...(8)$$

thus *the gradient of a scalar function ϕ at any point is a vector whose magnitude is equal to the rate of change of scalar function ϕ along the normal to the level surface and whose direction is normal to the level surface at the point.*

As $\frac{\partial\phi}{\partial n}\,\hat{n}$ gives the greatest rate of increase of ϕ with respect to space variables, therefore grad ϕ may be defined as follows :

The gradient of a scalar function ϕ is a vector whose magnitude is equal to maximum rate of change of scalar function ϕ with respect to space variables and whose direction is along that change.

If u and v are scalar differentiable functions, then it may be easily seen that

$$\text{grad}\,(u+v) = \text{grad}\,u + \text{grad}\,v \qquad ...(9)$$

$$\text{grad}\,(uv) = u\,\text{grad}\,v + v\,\text{grad}\,u. \qquad ...(10)$$

1·3. Line, Surface and Volume Integrals.

(i) Line Integral. The integral of a point function along a curve is called the *line integral.*

Let $\mathbf{r} = \mathbf{r}(t)$ be the equation of a curve. If ϕ and $\mathbf{A}$ are the scalar and vector fields respectively and $d\mathbf{r}$ is the vector increment of length, then we may encounter the integrals

$$\int_C \phi\, d\mathbf{r} \qquad ...(1)$$

$$\int_C \mathbf{A}\cdot d\mathbf{r} \qquad ...(2)$$

$$\int_C \mathbf{A}\times d\mathbf{r} \qquad ...(3)$$

each of which being as *line integral* along the curve C that may be open or closed The results of integration are respectively a vector, a scalar and a vector.

To compute any of the integrals, the method of attack will be to reduce the vector integrals, into scalar integrals with which one is assumed to be familiar.

As ϕ is a scalar function and $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$, integral (1) immediately reduces to

$$\int_C \phi\, d\mathbf{r} = \mathbf{i}\int_C \phi(x,y,z)\,dx + \mathbf{j}\int_C \phi(x,y,z)\,dy + \mathbf{k}\int_C \phi(x,y,z)\,dz \qquad ...(4)$$

the three integrals on R.H.S. of (4) are ordinary scalar integrals and to avoid complications we shall assume that they are Riemann integrals. The integral with respect to x can not be evaluated unless y and z are known in terms of x and similarly for the integrals with respect to y and z. This simply means that the path of integration C must be specified. Unles the integrand has special properties that lead the integral to depend only on the value of end points, the value will depend upon the particular choice of C.

The line integrals (2) and (3) may be interpreted in the similar fashion and like integral (1) they are dependent, in general, on the choice of the path.

Line integral (2) in most commonly used in vector analysis, it is called the *line interal of the tangential component of vector* **A** *along the curve* **C.** If **A** (vector functions of position) represents the force **F** acting on the particle, then the line integral *i.e.* $\int_C \mathbf{F}\cdot d\mathbf{r}$ represents the work done by the force.

An important special case arises in scalar calculus when the function to be integrated is an exact differential where the value of integral is independent of the path. In vector analysis let

$$\mathbf{A} = \text{grad}\,\phi = \nabla\phi$$

Then

$$\int_P^Q \mathbf{A}\cdot d\mathbf{r} = \int_P^Q \nabla\phi\cdot d\mathbf{r}$$

$$= \int_P^Q \left\{\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right\}$$

$$= \int_P^Q d\phi = \phi_Q - \phi_P$$

where P and Q are initial are final points of the curve with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2).

If the integration is taken around a closed curve, then $Q = P$, so that

$$\int_P^P \mathbf{A}\cdot d\mathbf{r} = \int_P^P \nabla\phi\cdot d\mathbf{r} = \oint \nabla\phi\cdot d\mathbf{r} = \phi_P - \phi_P = 0$$

where the symbol $\oint$ denotes the integral along the boundary of closed curve.

Thus if $\mathbf{A} = \text{grad}\,\phi$, then the line integral $\int_P^Q \mathbf{A}\cdot d\mathbf{r}$ depends only on initial and final values of ϕ and is independent of the path and also $\oint \mathbf{A}\cdot d\mathbf{r} = 0$.

Conversely if $\oint \mathbf{A}\cdot d\mathbf{r} = 0$, then there must exist a scalar point function ϕ such that $\mathbf{A} = \text{grad}\,\phi$.

If the line integral $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of path, then the vector field $\mathbf{A}(x, y, z)$ is said to be *conservative* or lamellar or noncurl vector field.

From above discussion it follows that :

A vector field **A** *is said to be conservative if and only if there exists a scalar point function* ϕ *such that* $\mathbf{A} = \text{grad}\,\phi$.

Conversely if a vector field **A** is derivable from a scalar point function ϕ according to relation $\mathbf{A} = \text{grad}\,\phi$, then vector field **A** is a conservative field.

2. Surface Integral. Let S be any surface divided into infinitesimal elements each of which may be considered as a vector $d\mathbf{S}$. Then if ϕ and **A** are scalar and vector fields repectively, the surface integrals may be encountered as

$$\iint_S \phi\, d\mathbf{S} \qquad ...(5)$$

$$\iint_S \mathbf{A}\cdot d\mathbf{S} \qquad ...(6)$$

$$\iint_S \mathbf{A}\times d\mathbf{S} \qquad ...(7)$$

The results of integration are respectively a vector, a scalar and a vector.

Often the area element $d\mathbf{S}$ may be written as $d\mathbf{S} = \hat{n}\, dS$ where $\hat{n}$ is a unit (normal) vector to indicate the positive direction. There are two conventions for chosing the positive direction. Firstly if the surface is a closed surface, we agree to take the outward normal as positive. Secondly if the surface is an open surface, the positive normal depends on the direction in which the perimeter of the open surface is traversed. If the right hand fingers are placed in the direction of travel around the perimeter, the positive normal is indicated by the thumb of the right hand.

The surface integral $\iint_S \mathbf{A} \cdot d\mathbf{S}$ is called the flow or flux of the vector function $\mathbf{A}$ through the given surface S. For example if $\mathbf{A} = \rho \mathbf{v}$ where ρ is the density and $\mathbf{v}$ the velocity of fluid, then the surface integral $\iint_S \rho \mathbf{v} \cdot d\mathbf{S}$ denotes the amount of fluid flowing through the given surface in unit time.

Volume Integrals. Let $dV = dx\,dy\,dz$ denote the element of volume. As volume element dV is a scalar, there are only two possible volume integrals

$$\iiint_V \phi \, dV \quad \text{...(8)}$$

$$\iiint_V \mathbf{A} \, dV \quad \text{...(9)}$$

the results of integration are respectively a scalar and a vector.

It is often convenient to convert multiple integrals into others with fewer integrals sign. This is done by two important theorems viz Gauss' divergence theorem and Stoke's theorem of vector analysis.

1·4. Divergence of a Vector Function

Let there exist a vector field $\mathbf{A}$ in a certain region of space. The divergence of the vector field $\mathbf{A}$ at certain point $P(x, y, z)$ is defined as the outward flux of the vector field $\mathbf{A}$ per unit volume enclosed, through an infinitesimal closed surface surrounding the point P.

If τ is the infinitesimal volume enclosed by an infinitely small closed surface S surrounding the point $P(x, y, z)$, then the divergence of the vector field $\mathbf{A}$ at point P is defined as

$$\text{div } \mathbf{A} = \lim_{\tau \to 0} \frac{\iint_S \mathbf{A} \cdot d\mathbf{S}}{\tau} \quad \text{..(1)}$$

The divergence of a vector function is a scalar quantity. It should be noted that divergence itself is simply an operator and has no physical meaning in itself. After operating on suitable physical vector functions, it represents various significant physical scalar quantities. To understand the physical meaning of the divergence of a vector field, we consider the following two examples :

1. The divergence of fluid velocity $\mathbf{v}$ at a point represents the quantity of fluid flowing out per second per unit volume enclosed by infinitesimal closed surface surrounding that point. If div $\mathbf{v}$ at a point is negative, it means that fluid is constantly flowing towards that point and thus there exists a sink for the fluid. If div $\mathbf{v}$ at a point is positive, it indicates the existence of the sources of fluid at that point.

2. The divergence of current density $\mathbf{J}$ at a point is numerically equal to the quantity of electric charge flowing out per second per unit volume through an infinitesimal closed surface surrounding that point. Taking the law of conservation of charge into considration, div $\mathbf{J}$ at any point is equal to the rate of decrease of charge density in the neighbourhood of that point, thus

$$\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

or

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad \text{...(2)}$$

This equation is known as equation of continuity and represents the law of *conservation of charge.*

If the divergence of any vector function in a region is zero ((*i.e.* div $\mathbf{A} = 0$), this means that the flux of the vector function entering any element of this region is equal to that leaving it. Any vector $\mathbf{A}$, satisfying the condition div $\mathbf{A} = 0$, is called a solenoidal vector.

The divergence in cartesian co-ordinates : Consider an infinitesimal rectangular box (fig. 1.10), with sides Δx, Δy and Δz and one corner at the point $P(x, y, z)$ in the region of any vector function $\mathbf{A}$ with rectangular faces perpendicular to co-ordinate axes. The rectangular box, in all, has six faces, two of them are perpendicular to X-axis having area $\Delta y\ \Delta z$ each, other two are perpendicular to Y-axis having area $\Delta x\ \Delta z$ each and the last two are perpendicular to Z-axis having area $\Delta x\ \Delta y$ each. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors and A_x, A_y, A_z are components of vector function $\mathbf{A}$ at the middle point of the box along X, Y and Z axes respectively, the vector function $\mathbf{A}$ at this point is expressed in cartesian co-ordinates as

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z .$$

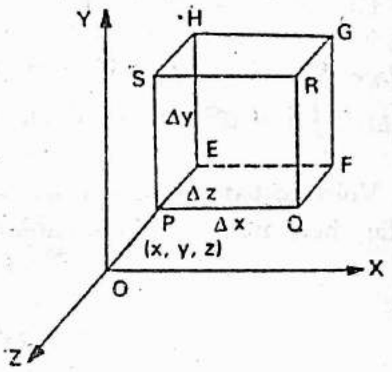


Fig. 1.10

To arrive at a cartesian expression for the divergence of $\mathbf{A}$ *i.e.* (div $\mathbf{A}$), consider two opposite faces *PEHS* and *QFGR* of this box, each of area $\Delta y\ \Delta z$ perpendicular to X-axis.

The flux emerging or outward flux through the face *QFGR*,

$$= \int_{QFGR} \overline{\mathbf{A}}_2 \cdot d\mathbf{S} = (\mathbf{i}\,\overline{A}_{2x} + \mathbf{j}\,\overline{A}_{2y} + \mathbf{k}\,\overline{A}_{2z})\cdot(\Delta y\ \Delta z\ \mathbf{i})$$

$$= \overline{A}_{2x} \cdot \Delta y\ \Delta z .$$

where $\overline{\mathbf{A}}_2$ is the average of the vector function over the surface *QFGR*, $\overline{A}_{2x}$ is the average of X-component of vector function over the face *QFGR*.

Similarly the outward flux through the face *PEHS*

$$= \int_{PEHS} \overline{\mathbf{A}}_1 \cdot d\mathbf{S} = (\mathbf{i}\,\overline{A}_{1x} + \mathbf{j}\,\overline{A}_{1y} + \mathbf{k}\,\overline{A}_{1z})\cdot(-\ \Delta y\ \Delta z\ \mathbf{i})$$

$$= -\overline{A}_{1x} \cdot \Delta y\ \Delta z .$$

where $\overline{A}_{1x}$ is the average of X-component of vector function over the face *PEHS*.

Thus the net outward flux of vector function $\mathbf{A}$ through the two faces perpendicular to X-axis

$$= (\overline{A}_{2x} - \overline{A}_{1x})\ \Delta y\ \Delta z . \qquad \text{...(1)}$$

Although A_x does not remain uniform over the two faces considered ; but if the box is small, the difference between averages of A_x over the two faces (*i.e.* $\overline{A}_{2x} - \overline{A}_{1x}$) is approximately equal to the difference between the values of A_x at P and Q. Thus

$$\overline{A}_{2x} - \overline{A}_{1x} = A_x(x + \Delta x, y, z) - A_x(x, y, z)$$

$$= \frac{\partial A_x}{\partial x}\ \Delta x$$

where $\frac{\partial A_x}{\partial x}$ is the rate of variation of A_x with distance along X-axis

Thus the net outward flux of vector function $\mathbf{A}$ through the two faces perpendicular to X-axis from (1) is,

$$= \frac{\partial A_x}{\partial x}\ \Delta x\ \Delta y\ \Delta z$$

Similarly the net outward flux of **A** through the two faces *PQFE* and *SRGH* perpendicular to *Y*-axis is,

$$= \frac{\partial A_y}{\partial y}\, \Delta x\ \Delta y\ \Delta z$$

and net outward flux of **A** thorugh two faces *PQRS* and *FEGH*, perpendicular to *Z*-axis is,

$$= \frac{\partial A_z}{\partial z}\, \Delta x\ \Delta y\ \Delta z$$

Therefore the net outward flux of vector field **A** through the whole of the box of infinitesimal volume $\Delta x\ \Delta y\ \Delta z$ is given by,

$$\delta\phi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \Delta x\ \Delta y\ \Delta z$$

Now div **A** at any point, which is the flux enclosed per unit infinitesimal volume surrounding that point, is given by

$$\text{div}\,\mathbf{A} = \lim_{\delta V (= \delta x \delta y \delta z) \to 0} \frac{\delta\phi}{\delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad ...(3)$$

This is the required expression for div **A** in cartesian co-ordintes, div **A** may also be written as

$$\begin{aligned} \text{div}\,\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}\,A_x + \mathbf{j}\,A_y + \mathbf{k}\,A_z) \\ &= \nabla \cdot \mathbf{A} \end{aligned}$$

where (∇) del is a vector operator defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Thus for divergence (div) we may write *'del-dot'* ($\nabla\,\cdot$).

1·5. Curl of a Vector Function.

To understand the meaning of the curl of a vector function, consider a small plane ara (say rectangular) in the region of *non-lamellar vector field.* (For non lamellar field **A**, div **A** $\neq 0$). Then the area is perpendicular to the vector field as in position 1, the vector field is normal to each side of the rectangle and hence the line integral along all sides is zero. In position 2, area is parallel to the field, the line integrals along *BC* and *DA* are zero, while the line integrals along *AB* and *CD* have finite values. Thus the line integral around the boundary of the area has finite value, since the value of the vector field along the upper and lower edges is assumed to be different. Thus the value of the line integral around a closed path depends upon the orientation of the small vector area considered in the region of the vector field. *There is a certain orientation of the area, for which the line integral of the vector field is maximum.*

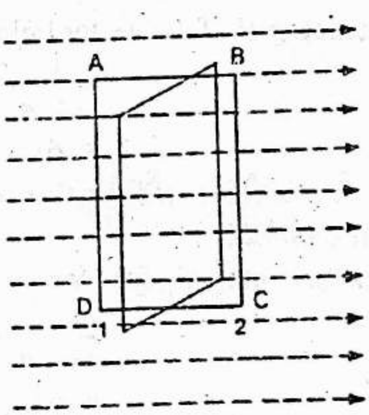


Fig. 1.11

The *curl of a vector field* at any point is defined as a vector quantity whose magntiude is equal to the maximum line integral per unit area along the boundary of an infinitesimal test area at that point and whose direction is perpendicular to the plane of the test area.

If **A** is any vector field at any point *P* and δ**S** an infinitesimal test area at point *P*, then curl **A** at point *P* is defined as

$$|\,\text{curl } \mathbf{A}\,| = \lim_{\delta S \to 0} \frac{\left[\oint \mathbf{A} \cdot d\mathbf{r}\right]_{\text{maximum}}}{\delta S}$$

or

$$\text{curl } \mathbf{A} = \lim_{\delta S \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{r}}{\delta S} \cdot \hat{n}. \qquad \text{...(i)}$$

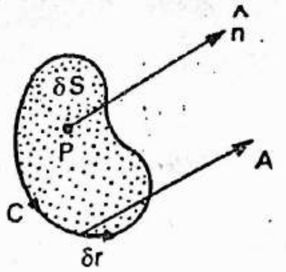


Fig. 1.12

where $\hat{\mathbf{n}}$ is the unit vector along the normal to the plane of test area (determined by the right hand screw rule).

The curl of a vector field is sometimes called *circulation* or *rotation* (or simply rot). If the fluid velocity vector $\mathbf{v}$ has a 'curl' somewhere, this means that the velocity fluid, over and above the general motion in a certain direction, has something like $\begin{smallmatrix}\leftarrow\uparrow\\ \downarrow\rightarrow\end{smallmatrix}$ or $\begin{smallmatrix}\rightarrow\downarrow\\ \uparrow\leftarrow\end{smallmatrix}$ superimposed on it in that region, because the existence of 'curl' of the vector field at a point of space indicates the circulation or vorticity there.

Since the line integral of a conservative field $\mathbf{A}$ around any closed path is zero *i.e.*

$$\oint \mathbf{A} \cdot d\mathbf{r} = 0,$$

therefore *the conservative vector fields have a zero curl at all points of space.* That is why such fields are known as *non-curl fields or lamellar vector fields.*

An example of conservative or lamellar vector field is the electrostatic field $\mathbf{E}$, therefore curl $\mathbf{E} = 0$.

The curl in cartesian co-ordinates : Let us assume an infinitesimal rectangular area $EFGH$ with sides Δx and Δy parallel to X–Y plane in the region of a vector function $\mathbf{A}$ (fig. 1.13). In order to find the curl $\mathbf{A}$ at point P within the rectangle, we have to calculate the line integral of the vector function $\mathbf{A}$ around the boundary of the rectangle.

We assign the positive direction according to right hand rule, when fingertips of right hand point in the direction of positive sense of $\delta\mathbf{r}$, the thumb points in the direction of positive normal to the surface (*i.e.*, the direction of the curl $\mathbf{A}$). According to this rule, the line interal of $\mathbf{A}$ when taken along the circulation $EFGH$ will give us z-component of curl $\mathbf{A}$. Let the co-ordinates of E be (x, y, z).

If A_x, A_y and A_z are cartesian components of $\mathbf{A}$ at P, then

$$\mathbf{A} = \mathbf{i}\,A_x + \mathbf{j}\,A_y + \mathbf{k}\,A_z\,.$$

The line integral of the vector field along the path EF

$$= \int_{EF} \overline{\mathbf{A}} \cdot d\mathbf{r} = (\mathbf{i}\,\overline{A}_{1x} + \mathbf{j}\,\overline{A}_{1y} + \mathbf{k}\,\overline{A}_{1z}) \cdot (\mathbf{i}\,\Delta x)$$

$$= \overline{A}_{1x}\ \Delta x$$

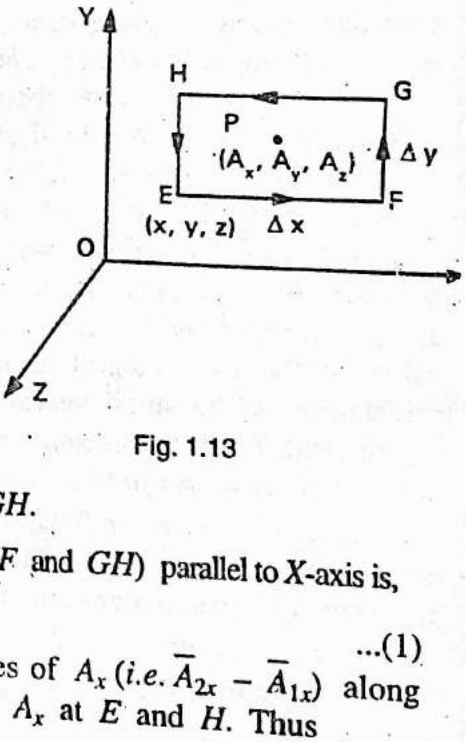


Fig. 1.13

where $\overline{A}_{1x}$ is the average value of X-component of the vector function over the path EF.

The line integral of the vector field along the path GH

$$= \int_{GH} \overline{\mathbf{A}}_2 \,.\, d\mathbf{r}$$

$$= (\mathbf{i}\overline{A}_{2x} + \mathbf{j}\,\overline{A}_{2y} + \mathbf{k}\overline{A}_{3z}) \,.\, (-\,\mathbf{i}\,\Delta x)$$

$$= -\ \overline{A}_{2x} \,.\, \Delta x$$

where $\overline{A}_{2x}$ is the average of X-component of vector function over the path GH.

Hence the contribution to the line integral $\oint_{EFGH} \mathbf{A} \cdot d\mathbf{r}$ from these two parts (EF and GH) parallel to X-axis is,

$$-\ (\overline{A}_{2x} - \overline{A}_{1x})\ \Delta x. \qquad \text{...(1)}$$

As the rectangle is infinitely small, the difference between the averages of A_x (*i.e.* $\overline{A}_{2x} - \overline{A}_{1x}$) along these two paths may be approximated to the difference between the values of A_x at E and H. Thus

$$\begin{aligned}\overline{A}_{2x} - \overline{A}_{1x} = A_{Hx} - A_{Ex} &= A_x\,(x, y+\Delta y, z) - A_x\,(x, y, z)\\ &= \frac{\partial A_x}{\partial y}\,\Delta y. \qquad \text{...(2)}\end{aligned}$$

Hence, from eqn. (1) the contribution to the line integral $\oint_{EFGH} \mathbf{A}\cdot d\mathbf{r}$ from the path EF and GH

$$= -\frac{\partial A_x}{\partial y}\,\Delta y\,\Delta x \qquad \text{...(3)}$$

Similarly the line integral of vector field along path $FG = +\overline{A}_{3y}\cdot\Delta y$, and the line integral of the vector field along path $HE = -\overline{A}_{4y}\,\Delta y$.

Hence contribution to the line integral $\oint_{EFGH} \mathbf{A}\cdot d\mathbf{r}$ from the paths FG and HE is

$$\begin{aligned}&= (\overline{A}_{3y} - \overline{A}_{4y})\,\Delta y = \{A_y\,(x+\Delta x, y, z) - A_y\,(x, y, z)\}\,\Delta y\\ &= \frac{\partial A_y}{\partial x}\,\Delta x\,\Delta y \qquad \text{...(4)}\end{aligned}$$

Therefore the line integral around the whole rectangle $EFGH$ from eqns. (3) and (4), is given by

$$T = \oint_{EFGH} \mathbf{A}\cdot d\mathbf{r} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\,\Delta y. \qquad \text{...(5)}$$

Now curl **A** at any point is defined as the line integral per unit infinitesimal area at that point along its boundary and is directed along positive normal to the area. Here the area of the rectangle is $\Delta x\ \Delta y$. whose positive normal is along z-direction. Hence from eqn. (5) we get z-component of curl **A** as

$$(\text{curl}\ \mathbf{A})_z = \underset{\delta S\,(\Delta x\ \Delta y)\to 0}{\text{limit}}\ \frac{\delta T}{\delta S} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad \text{...(6)}$$

If we rotate the rectangle in YZ plane with its positive normal along X-axis, we get X-component of line integral in the similar manner as,

$$(\text{curl}\ \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \qquad \text{...(7)}$$

Similarly if the rectangle lies in Z–X plane with its positive normal along Y-axis, we get Y-component of curl **A** as,

$$(\text{curl}\ \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad \text{...(8)}$$

Summing up the results given in eqns. (6), (7) and (8), we get

$$\begin{aligned}\text{curl}\ \mathbf{A} &= \mathbf{i}\ (\text{curl}\ \mathbf{A})_x + \mathbf{j}\ (\text{curl}\ \mathbf{A})_y + \mathbf{k}\ (\text{curl}\ \mathbf{A})_z\\ &= \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad \text{...(9)}\\ &= \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)\times(\mathbf{i}\,A_x + \mathbf{j}\,A_y + \mathbf{k}\,A_z)\\ &= \nabla\times\mathbf{A}. \qquad \text{...(10)}\end{aligned}$$

Thus for curl, we may write *'del-cross'* ($\nabla \times A$).

Eqn. (9) or (10) represents the curl **A** in terms of cartesian co-ordinates which may also be written in the determinant form as,

$$\text{curl}\ \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z \end{vmatrix} \qquad \text{...(11)}$$

Thus for curl we may write del-cross ($\nabla\times$).

1·6. Laplacian, Laplace's and Poisson's Equations.

The *Laplacian operator* ∇^2 is defined as

$$\nabla^2 = \text{Div} \cdot \text{grad} = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Laplace's equation. If ϕ is a scalar potential function, then in charge free space $\nabla^2 \phi = 0$. This is called **Laplace's equation.**

Poisson's equation. If ρ is the volume charge density in a region, then **Poisson's equation** is

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon}.$$

where $\varepsilon = \varepsilon_r \varepsilon_0$ (or $K_e \varepsilon_0$) is called absolute perimitivity of the medium [ε_r or K_e = Relative permitivity of medium and ε_0 is the absolute permitivity of free space.

1·7. Gauss Divergence Theorem.

Gauss divergence theorem enables us to transform volume integral of the divergence of the vector function into surface integral of the vector field and vice-versa. The theorem states.

The flux of a vector field A over any closed surface S is equal to the volume integral of the divergence of the vector field over the volume V enclosed by the surface S i.e.

$$\iint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V \text{div } \mathbf{A} \, dV.$$

Let us consider a finite volume V of any shape enclosed by the surface S (fig. 1.14a) in the region of any vector function **A**.

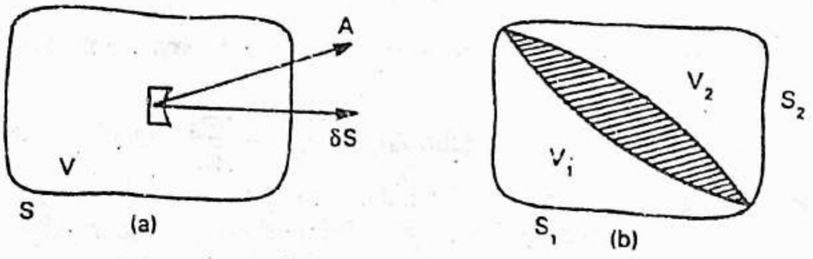


Fig. 1.14

The flux diverging from the surface S of volume V is,

$$\phi = \int_S \mathbf{A} \cdot d\mathbf{S}.$$

Let us now divide the volume V into two parts of volumes V_1 and V_2 enclosed by surfaces S_1 and S_2 respectively (fig. 1.14 b). The flux emerging out of surface $S_1 = \int_{S_1} \mathbf{A} \cdot d\mathbf{S_1}$

and the flux emerging out of surface $S_2 = \int_{S_2} \mathbf{A} \cdot d\mathbf{S_2}$

As the shaded surface in fig. (1.14b) is common to both the volumes, the flux emerging out of it for both the volumes being equal and in opposite directions cancel each other when considered together. The rest of the surfaces S_1 and S_2 is identical to the original surface S. Thus obviously

$$\phi = \int_S \mathbf{A} \cdot d\mathbf{S} = \int_{S_1} \mathbf{A} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{A} \cdot d\mathbf{S}_2 \qquad \text{...(1)}$$

Similarly if we divide the volume V into a large number of parts $V_1, V_2 \ldots V_i \ldots V_N$ enclosed by surfaces $S_1, S_2, \ldots S_i, \ldots S_N$ respectively, we must have

$$\phi = \int_S \mathbf{A} \cdot d\mathbf{S} = \int_{S_1} \mathbf{A} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{A} \cdot d\mathbf{S}_2 + \ldots + \int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i + \ldots + \int_{S_N} \mathbf{A} \cdot d\mathbf{S}_N$$

$$= \sum_{i=1}^{N} \int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i$$

We may also write

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \sum_{i=1}^{N} V_i \frac{\int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i}{V_i}$$

If N is sufficiently large, then volume V_i becomes infinitely small, *i.e.*, if N tends to infinity, V_i tends to zero and in the limit we may write

$$\underset{V_i \to 0}{\text{limit}} \frac{\int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i}{V_i} = \text{div } \mathbf{A} \qquad \text{...(3)}$$

and convert the summation into integration writing dV for infinitely small volume V_i; so that eqn. (3) in the limit, may be written as

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_V \text{div } \mathbf{A} \cdot dV \qquad \text{...(4)}$$

This equation is called *Gauss's divergence theorem and is used to convert the volume integral of the divergance of the vector field into surface integral of the vector field and vice-versa.*

1·8. Stoke's Theorem.

Stoke's theorem enables us to transform the surface integral of the curl of the vector field $\mathbf{A}$ into the line integral of that vector field $\mathbf{A}$ over the boundary C of that surface and vice-versa. The theorem states.

The flux of the curl of a vector function $\mathbf{A}$ over any surface S of any shape is equal to the line integral of the vector field $\mathbf{A}$ over the boundary C of that surface i.e.

$$\iint_S \text{curl } \mathbf{A} \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{r}.$$

To derive Stoke's theorem, let us consider a surface S with C its boundary. Let us calculate the line integral of the vector function $\mathbf{A}$ around the boundary C of the surface S.

The line integral of $\mathbf{A}$ around the boundary C of surface S

$$= \oint_C \mathbf{A} \cdot d\mathbf{r}.$$

Now let us divide the surface S into two parts of surfaces S_1 and S_2, having boundaries C_1 and C_2 respectively (fig. 1.15 b).

The line integral of $\mathbf{A}$ around the boundary of S_1

$$= \oint_{C_1} \mathbf{A} \cdot d\mathbf{r}_1$$

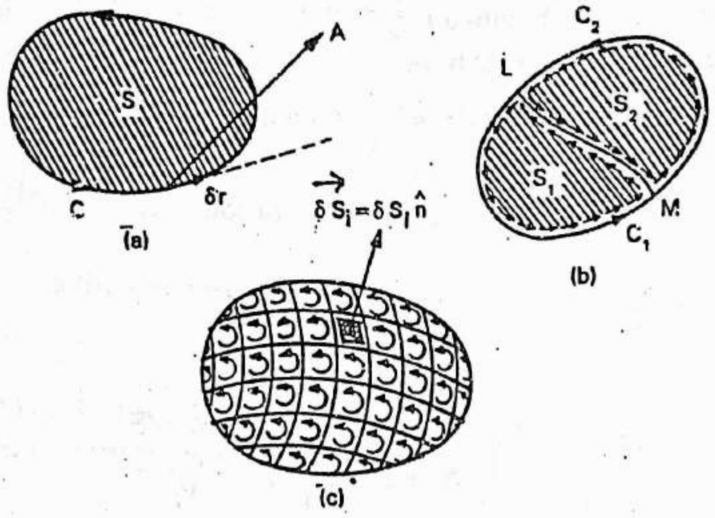


Fig. 1.15

and the line integral of **A** around the boundary of S_2

$$= \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}_2 .$$

As the thick line LM is common boundary to both the surfaces and is traversed in opposite directions in the two integrals (fig. 1.15 b), the integral of **A** along this common boundary, being equal and opposite, cancel each other when considered together. The rest of the boundaries C_1 and C_2 is identical to the original boundary C. Thus obviously

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}_2 . \qquad ...(1)$$

Simiarly if we divide the surface S into a large number of parts $S_1, S_2 ..., S_i ... S_N$ having boundaries $C_1, C_2, ..., C_i ..., C_N$ respectively (fig. 1.15 c), then

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}_2 + ...$$

$$... + \oint_{C_i} \mathbf{A} \cdot d\mathbf{r}_i + ... + \oint_{C_N} \mathbf{A} \cdot d\mathbf{r}_N$$

$$= \sum_{i=1}^{N} \oint_{C_i} \mathbf{A} \cdot d\mathbf{r}_i . \qquad ...(2)$$

This may also be written as

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \sum_{i=1}^{N} S_i \frac{\int_{C_i} \mathbf{A} \cdot d\mathbf{r}_i}{S_i} . \qquad ...(3)$$

If N is sufficiently large, then suface area S_i becomes infinitely small, *i.e.*, if N tends to infinity, S_i tends to zero and in the limit we may write definition of curl A as

$$\underset{S_i \to 0}{\text{limit}} \frac{\int_{C_i} \mathbf{A} \cdot d\mathbf{r}_i}{S_i} = \text{curl } \mathbf{A} \cdot \hat{n}$$

and convert the summation into integration writing dS for infinitely small area S_i, so that eqn. (3), in the limit, may be written as

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C \text{curl } \mathbf{A}\cdot \hat{n} d\mathbf{S}$$

or

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C \text{curl } \mathbf{A}\cdot d\mathbf{S} \qquad ...(4)$$

This equation is called **Stoke's theorem.** It is necessary for vector feild **A** to be well behavedc continuous function. Stoke's theorem ralates the line integral of a vector function to the surface integral of the curl of the vector, whereas Gauss's theorem relates the surface integral of the vector to the volume integral of the divergence of the vector.

1·9. Green's Theorem.

If Ψ_1 and Ψ_2 are the scalar point functions having continuous derivatives at least of second order, then Green's theorem states

$$\int\int\int (\Psi_1 \nabla^2 \Psi_2 - \Psi_2 \nabla^2 \Psi_1)\ dx\, dy\, dz = \int\int (\Psi_1 \text{ grad } \Psi_2 - \Psi_2 \text{ grad } \Psi_1) \cdot d\mathbf{S}$$

1·10. Some Important Formulas from Vector Analysis.

In various applications of vector analysis we shall have the occasion to use the following vector identities frequently. If ϕ, Ψ are scalar point functions and **A, B** are vector point functions in a certain region ; then,

1. $\nabla(\phi + \Psi) = \nabla\phi + \nabla\Psi$
2. $\nabla(\phi\Psi) = \phi\nabla\Psi + \Psi\nabla\phi$.
3. $\text{div}(\mathbf{A}+\mathbf{B}) = \text{div }\mathbf{A} + \text{div }\mathbf{B}$.
4. $\text{curl}(\mathbf{A}+\mathbf{B}) = \text{curl }\mathbf{A} + \text{curl }\mathbf{B}$
5. $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A}$
6. $\text{div}(\phi\mathbf{A}) = \phi\,\text{div }\mathbf{A} + \mathbf{A}\cdot\text{grad }\phi$.
7. $\text{curl}(\phi\mathbf{A}) = \phi\,\text{curl }\mathbf{A} + \text{grad }\phi\times\mathbf{A}$.
8. $\text{div curl }\mathbf{A} = 0$.
9. $\text{curl grad }\phi = 0$.
10. $\text{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\text{curl }\mathbf{A} - \mathbf{A}\cdot\text{curl }\mathbf{B}$.
11. $\text{curl}(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\,\text{div }\mathbf{B} - \mathbf{B}\,\text{div }\mathbf{A}$.
12. $\text{curl curl }\mathbf{A} = \text{grad div }\mathbf{A} - \nabla^2\mathbf{A}$.
13. $\oint_S \mathbf{A}\cdot\hat{n}\, d\mathbf{S} = \int_V \text{div }\mathbf{A}\, dV$.
14. $\oint_C \mathbf{A}\cdot\overrightarrow{dl} = \int_S \text{curl }\mathbf{A}\cdot\hat{n}\, dS$.
15. $\oint_C \phi\hat{n}\, dS = \int_V \nabla\phi\, dV$.
16. $\oint_S \mathbf{A}(\mathbf{B}\cdot\hat{n})\, dS = \int_V \mathbf{A}\,\text{div }\mathbf{B}\ dV + \int_V (\mathbf{B}\cdot\nabla)\mathbf{A}\, dV$.
17. $\oint_S \hat{n}\times\mathbf{A}\ dS = \int_V \text{curl }\mathbf{A}\ dV$.
18. $\oint_C \phi\,\overrightarrow{dl} = \int_S \hat{n}\times\nabla\phi\, dS$.

Ex. 1. *Show that any vector* $\overrightarrow{\mathbf{v}}$ *may be expressed as a sum of irrotational part and a solenoidal part :*

$$\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}_i} + \overrightarrow{\mathbf{v}_s}; \qquad \overrightarrow{\nabla}\times\overrightarrow{\mathbf{v}_i} = 0,\ \overrightarrow{\nabla}\cdot\overrightarrow{\mathbf{v}_s} = 0$$

(Delhi 1996)

Solution. An irrotational vector is one which has on curl, while a solenoidal vector is one which has no divergence. If s is scalar function and $\vec{A}$ is vector function, then we have two vector identities

$$\text{curl grad } s = 0 \text{ and div curl } \vec{A} = 0$$

Therefore $\vec{v}$ may be expressed as

$$\vec{v} = \text{grad } s + \text{curl } \vec{A} \qquad \ldots(1)$$

Taking div of both sides

$$\text{div } \vec{v} = \text{div grad } s + \text{div curl } \vec{A}$$

As

$$\text{div curl } \vec{A} = 0$$

we have

$$\text{div } \vec{v} = \text{div grad } s \neq 0$$

This represents that grad s is non-solenoidal vector while curl $\vec{A}$ is solenoidal vector. Taking curl of equation (i)

$$\text{curl } \vec{v} = \text{curl grad s} + \text{curl curl } \vec{A}$$

As curl grad $s = 0$, we have curl $\vec{v}$ = curl curl $\vec{A}$ = grad div $\vec{A} - \nabla^2 \vec{A} \neq 0$

i.e. This represents that curl $\vec{A}$ is rotational vector, while grad s is irrotational vector.

Hence we write grad $s = \vec{v}_i$ and curl $\vec{A} = \vec{v}_s$

$$\therefore \quad \vec{v} = \text{grad } s + \text{curl } \vec{A} = \vec{v}_i + \vec{v}_s$$

Hence a general vector may be expressed as the sum of two vectors, one being irrotational while other being solenoidal. This is well known Helmholtz theorem.

1·11. Orthogonal Curvilinear Co-ordinates.

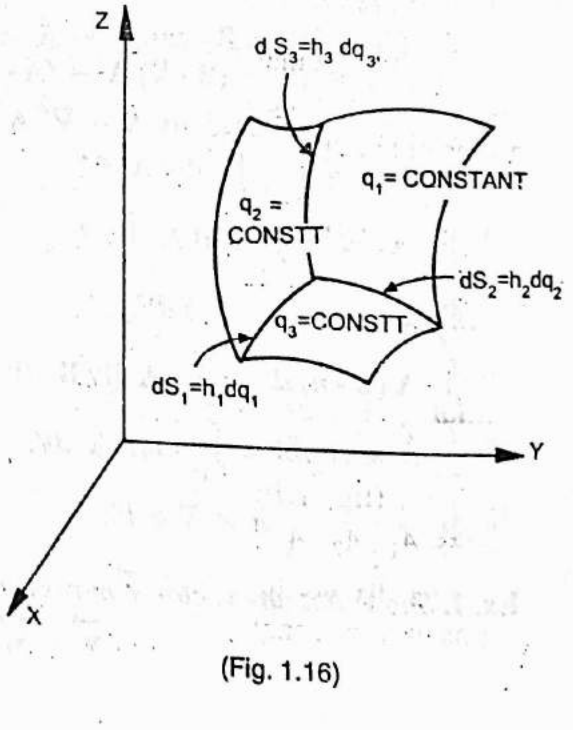
In many problems of physics and applied mathematics it is usually necessary to write vector equations in terms of suitable co-ordinates instead of cartesian co-ordinates before the final solution of a specific problem is obtained. Now we shall introduce curvilinear co-ordinates which are so general in nature that it is co-ordinate systems which have been found useful in physical problems.

In cartesian co-ordinates the position of a point $P(x, y, z)$ is determined by the intersection of 3 mutually perpendicular planes $x =$ constant, $y =$ constant and $z =$ constant. Imagine that we superimpose on this system three other families of surfaces. The surfaces of any family need not be parallel and they need not be planes. The three new families of surfaces need not be mutually perpendicular.

Let the three new families of surfaces, described by $q_1 =$ constant, $q_2 =$ constant, $q_3 =$ constant, interesct at point P. The value of q_1, q_2, q_3 for the three surfaces intersecting at P are called *curvillinear co-ordinates* or *curvilinear surfaces*. If the co-ordinate surfaces are mutually perpendicular at every point $P(x, y, z)$, then the curvilinear co-ordinates (q_1, q_2, q_3) are said to be orthogonal curvilinear *co-ordinate lines* (or *co-ordinate curves*). The co-ordinate axes are determined by the tangents to the co-ordinate lines at the intersection of the three surfaces. They are not in general fixed-directions is space, as is true for simple cartesian co-ordinates. Obviously, any given point P may be identified by curvilinear co-ordinates (q_1, q_2, q_3) as well as by cartesian co-ordiantes (x, y, z). This means that in principle we may write

$$\left.\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned}\right\} \qquad \ldots(1)$$

with inverses



(Fig. 1.16)

$$\left.\begin{aligned} q_1 &= q_1\,(x\,,y\,,z) \\ q_2 &= q_2\,(x\,,y\,,z) \\ q_3 &= q_3\,(x\,,y\,,z) \end{aligned}\right\} \qquad \text{...(2)}$$

with each family of surfaces q_i = constant, we can associate a unit vector $\hat{\mathbf{u}}_i$ normal to the surface q_i = constant and in the direction on increasing q_i.

The partial differentiation of equation (1) yields

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial q_1}\,dq_1 + \frac{\partial x}{\partial q_2}\,dq_2 + \frac{\partial x}{\partial q_3}\,dq_3 \\ dy &= \frac{\partial y}{\partial q_1}\,dq_1 + \frac{\partial y}{\partial q_2}\,dq_2 + \frac{\partial y}{\partial q_3}\,dq_3 \\ dz &= \frac{\partial z}{\partial q_1}\,dq_1 + \frac{\partial z}{\partial q_2}\,dq_2 + \frac{\partial z}{\partial q_3}\,dq_3 \end{aligned}\right\}$$

Hence the square of the distance between two neigbouring points is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \Sigma\, h_{ij}^2\; dq_i\; dq_j\;\; (i, j = 1, 2, 3) \qquad \text{...(4)}$$

where the coefficients h_{ij}^2 are given by

$$h_{ij}^2 = \frac{\partial x}{\partial q_i}\,\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\,\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\,\frac{\partial z}{\partial q_j} \qquad \text{...(5)}$$

The most useful coordinate systems are orthogonal ones *i.e.*, the systems in which surface always intersect at right angles. At this point we limit ourselves to *orthogonal coordinate systems* which means

$$h_{ij} = 0,\; i \neq j. \qquad \text{...(6)}$$

Now to specify the notation, we write $h_{ij} = h_i$ so that equation (4) yields

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2 \qquad \text{...(7)}$$

the specific coordinate systems are described by specifying the scale factors h_1, h_2, h_3.

The distance between any two points on a coordinate line is called the *line element*. When the variation is limited to any given q_i, holding the other q's constant, then the line element, from equation (4), is given by

$$ds_i = h_i\, dq_i\,. \qquad \text{...(8)}$$

From this equation it may be noted that the three curvilinear coordinates q_1, q_2, q_3 need not be length. The scale factors h_i may depend upon q's and they may have dimensions. The product $h_i\, dq_i$ must have the dimensions of length.

From equations (8) we may immediately develop the surface and volume elements.

The three possible *surface elements in orthogonal system* thus become

$$ds_{ij} = ds_i\; ds_j = h_i\, h_j\, dq_i\, dq_j\;\; (i, j = 1, 2, 3\,;\, i \neq j) \qquad \text{..(9)}$$

and the *volume element*

$$d\tau = ds_1\, ds_2\, ds_3 = h_1 h_3 h_3\;\, dq_1\, dq_2\, dq_3\,. \qquad \text{...(10)}$$

Differential operators in terms of orthogonal curvilinear co-ordinates.

Consider three mutually perpendicular co-ordinate surfaces described by q_1 constant, q_2 = constant, q_3 = constant, (fig. 1.16). Let $\psi\,(q_1\,,q_2\,,q_3)$ be a scalar function and **A** be a vector3 function with components A_1, A_2, A_3 in the three directions in which q_1, q_2, q_3 increase.

If $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$, $\hat{\mathbf{u}}_3$ are unit vectors along the directions of increasing q_1, q_2, q_3 respectively, then vector **A** in terms of orthogonal curvilinear co-ordinates may be written as

$$\mathbf{A} = A_1\hat{\mathbf{u}}_1 + A_2\hat{\mathbf{u}}_2 + A_3\hat{\mathbf{u}}_3 . \qquad ...(11)$$

1. Gradient. The gradient of a scalar function ψ is a vector whose magnitude and direction gives the maximum space rate of change of scalar function ψ. From this interpratation the component of $\nabla \psi (q_1, q_2, q_3)$ in the direction normal to the surface $q_1 =$ constant and hence in the direction of q_1 is

$$\nabla \psi \Big|_1 = \underset{\delta s \to 0}{\text{Lim}} \frac{\delta \psi}{\delta s_1} = \frac{\partial \psi}{\partial s_1} = \frac{\partial \psi}{h_1 \partial q_1} = \frac{1}{h_1} \frac{\partial \psi}{\partial q_1} \qquad ...(12)$$

where $\delta s_1 = h_1 \delta q_1$ is the differential length in the direction of increasing q_1 and $\delta\psi$ represents an increase in ψ on travelling a distance δs_1 in the limit $\delta s_1 \to 0$.

By repeating equation (12) for q_2 and again for q_3, we get

$$\nabla \psi \Big|_2 = \frac{1}{h_2} \frac{\partial \psi}{\partial q_2} \qquad ...(13)$$

$$\nabla \psi \Big|_3 = \frac{1}{h_3} \frac{\partial \psi}{\partial q_3} . \qquad ...(14)$$

Adding equations (12), (13) and (14) vectorially, the gradient of scalar function in *orthogonal curvilinear coordinates* becomes

$$\text{grad } \psi = \nabla \psi = \frac{\hat{\mathbf{u}}_1}{h_1} \frac{\partial \psi}{\partial q_1} + \frac{\hat{\mathbf{u}}_2}{h_2} \frac{\partial \psi}{\partial q_2} + \frac{\hat{\mathbf{u}}_3}{h_3} \frac{\partial \psi}{\partial q_3} . \qquad ...(15)$$

Thus the operator grad $(= \nabla)$ in *orthogonal curvilinear coordinates is*

$$\text{grad} = \nabla = \frac{\hat{\mathbf{u}}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\hat{\mathbf{u}}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\hat{\mathbf{u}}_3}{h_3} \frac{\partial}{\partial q_3} . \qquad ...(16)$$

2. Divergence. The divergence of a vector $\mathbf{A}$ is written as

$$\begin{aligned} \text{div } \mathbf{A} &= \nabla \cdot \mathbf{A} = \nabla \cdot (\hat{\mathbf{u}}_1 A_1 + \hat{\mathbf{u}}_2 A_2 + \hat{\mathbf{u}}_3 A_3) \\ &= \nabla \cdot (\hat{\mathbf{u}}_1 A_1) + \nabla \cdot (\hat{\mathbf{u}}_2 A_2) + \nabla \cdot (\hat{\mathbf{u}}_3 A_3). \qquad ...(17) \end{aligned}$$

Bu from section (1.10) we have

$$\text{div } (\hat{\mathbf{u}}_i A_i) = \nabla \cdot (\hat{\mathbf{u}}_i A_i) = A_i \nabla \cdot \hat{\mathbf{u}}_i + \hat{\mathbf{u}}_i \cdot \nabla A_i .$$

Thus in order to find $\nabla \cdot \mathbf{A}$, it is needed to obtain $\nabla \cdot \mathbf{u}_i$ which may be obtained as follows :

Remembering that $\frac{\hat{\mathbf{u}}_i}{h_i}$ is a product of a scalar and a vector, we may write in view of section 1.10.

$$\text{curl } \frac{\hat{\mathbf{u}}_i}{h_i} = \nabla \times \left(\frac{\hat{\mathbf{u}}_i}{h_i} \right) = \nabla \left(\frac{1}{h_i} \right) \times \hat{\mathbf{u}}_i + \frac{1}{h_i} (\nabla \times \hat{\mathbf{u}}_i) \qquad ...(19)$$

using equation (16) we see that

$$\nabla q_1 = \frac{\hat{\mathbf{u}}_1}{h_1} .$$

Since $\qquad \text{curl grad } q_1 = 0 \text{ } i.e. \ \nabla \times \frac{\hat{\mathbf{u}}_1}{h_1} = 0,$

therefore equation (19) yields

$$\hat{\mathbf{u}}_1 \times \nabla \left(\frac{1}{h_1} \right) = \frac{1}{h_1} (\nabla \times \hat{\mathbf{u}}_1) \qquad ...(20)$$

using equation (16) again and performing the differentiation, we obtain

$$\nabla\left(\frac{1}{h_1}\right) = -\frac{\hat{\mathbf{u}}_1}{h_1^3}\frac{\partial h_1}{\partial q_1} - \frac{\hat{\mathbf{u}}_2}{h_1^2 h_2}\frac{\partial h_1}{\partial q_2} - \frac{\hat{\mathbf{u}}_3}{h_1^2 h_3}\frac{\partial h_1}{\partial q_3} \qquad \text{...(21)}$$

using this equation, equation (20) gives

$$\nabla \times \hat{\mathbf{u}}_1 = -\frac{\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_1}{h_1^2}\frac{\partial h_1}{\partial q_1} - \frac{\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3}{h_1 h_3}\frac{\partial h_1}{\partial q_3} \qquad \text{...(22)}$$

Let us now choose the positive directions of $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$ so that (q_1, q_2, q_3) or $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$ form a right banded set. Then recalling that $\hat{\mathbf{u}}_i \times \hat{\mathbf{u}}_i = 0, \hat{\mathbf{u}}_i \times \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_k, \hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_i = 1, \hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_j = 0$.

In particular $\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_1 = 0, \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 = \hat{\mathbf{u}}_3$ and $\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3 = -\hat{\mathbf{u}}_2$.

Substituting these values in (22), we obtain

$$\nabla \times \hat{\mathbf{u}}_1 = \frac{\hat{\mathbf{u}}_2}{h_1 h_3}\frac{\partial h_1}{\partial q_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2}\frac{\partial h_1}{\partial q_2}. \qquad \text{...(23a)}$$

Now by cyclic permutation of co-ordinates, similar result for curl of $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$ may be written *i.e.*

$$\nabla \times \hat{\mathbf{u}}_2 = \frac{\hat{\mathbf{u}}_3}{h_2 h_1}\frac{\partial h_2}{\partial q_1} - \frac{\hat{\mathbf{u}}_1}{h_2 h_3}\frac{\partial h_3}{\partial q_3} \qquad \text{...(23b)}$$

$$\nabla \times \hat{\mathbf{u}}_3 = \frac{\hat{\mathbf{u}}_1}{h_3 h_2}\frac{\partial h_3}{\partial q_2} - \frac{\hat{\mathbf{u}}_2}{h_3 h_1}\frac{\partial h_3}{\partial q_1}. \qquad \text{...(23c)}$$

Keeping in mind the relation div $(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \text{curl}\, \mathbf{a} - \mathbf{a} \cdot \text{curl}\, \mathbf{b}$, we have

$$\nabla \cdot \hat{\mathbf{u}}_1 = \nabla \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) = \hat{\mathbf{u}}_3 \cdot (\nabla \times \hat{\mathbf{u}}_2) - \hat{\mathbf{u}}_2 \cdot (\nabla \times \hat{\mathbf{u}}_3). \qquad \text{...(24)}$$

Substituting the values of $\nabla \times \hat{\mathbf{u}}_2$ and $\nabla \times \hat{\mathbf{u}}_3$ from (23) and using the fact $\hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_i = 1, \hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_j = 0$, equation (24) yields

$$\nabla \cdot \hat{\mathbf{u}}_1 = \frac{1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3)}{\partial q_1}, \qquad \text{...(25)}$$

From (16), (18) and (25), we get

$$\begin{aligned}\nabla \cdot (\hat{\mathbf{u}}_1 A_1) &= A_1 \nabla \cdot \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_1 \cdot \nabla A_1 \\ &= \frac{A_1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3)}{\partial q_1} + \hat{\mathbf{u}}_1 \cdot \left(\frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial A_1}{\partial q_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial A_1}{\partial q_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial A_1}{\partial q_3}\right) \\ &= \frac{A_1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3)}{\partial q_1} + \frac{1}{h_1}\frac{\partial A_1}{\partial q_1} = \frac{1}{h_1 h_2 h_3}\frac{\partial(A_1 h_2 h_3)}{\partial q_1}\end{aligned} \qquad \text{...(26a)}$$

Again by cyclic permutaiton of coordinates

$$\nabla \cdot (\hat{\mathbf{u}}_2 A_2) = \frac{1}{h_1 h_2 h_3}\frac{\partial(A_2 h_3 h_1)}{\partial q_2} \qquad \text{...(26b)}$$

$$\nabla \cdot (\hat{\mathbf{u}}_3 A_3) = \frac{1}{h_1 h_2 h_3}\frac{\partial(A_3 h_1 h_2)}{\partial q_3} \qquad \text{...(26c)}$$

using equation (26), equation (17) reduces to

$$\text{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(A_1 h_2 h_3)}{\partial q_1} + \frac{\partial(A_2 h_3 h_1)}{\partial q_2} + \frac{\partial(A_3 h_1 h_2)}{\partial q_3}\right] \qquad \text{...(27)}$$

This equation represents div $\mathbf{A}$ in orthogonal curvilinear co-ordinates.

3. Laplacian. Laplacian may be obtained by combining equations (15) and (27). Substituting $\mathbf{A} = \nabla \psi\ (q_1, q_2, q_3)$ in equation (27), we get

$$\nabla \cdot \nabla \psi = \nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\{h_2 h_3 (\nabla \psi_1)\} + \frac{\partial}{\partial q_2}\{h_3 h_1 (\nabla \psi)_2\} + \frac{\partial}{\partial q_3}\{h_1 h_2 (\nabla \psi)_3\}\right]$$

Substituting components of $\nabla \psi$ from equation (15), we get

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right] \qquad \text{...(28)}$$

4. Curl. The curl of a vector **A** is written as

$$\begin{aligned}\text{curl } \mathbf{A} = \nabla \times \mathbf{A} &= \nabla \times (\hat{\mathbf{u}}_1 A_1 + \hat{\mathbf{u}}_2 A_2 + \hat{\mathbf{u}}_3 A_3)\\ &= \nabla \times (\hat{\mathbf{u}}_1 A_1) + \nabla \times (\hat{\mathbf{u}}_2 A_2) + \nabla \times (\hat{\mathbf{u}}_3 A_3). \qquad \text{...(29)}\end{aligned}$$

Keeping in mind the relation curl $(\phi \mathbf{A}) = \phi$ curl $\mathbf{A} - \mathbf{A} \times$ grad ϕ, we have

$$\text{Curl } (\hat{\mathbf{u}}_1 \mathbf{A}_1) = \nabla \times (\hat{\mathbf{u}}_1 \mathbf{A}_1) = A_1 (\nabla \times \hat{\mathbf{u}}_1) - \hat{\mathbf{u}}_1 \times \nabla A_1$$

Substituting values of $\nabla \times \hat{\mathbf{u}}_1$ and ∇A_1 from (23a) and (15), we get

$$\begin{aligned}\nabla \times (\hat{\mathbf{u}}_1 A_1) &= A_1\left[\frac{\hat{\mathbf{u}}_2}{h_1 h_3}\frac{\partial h_1}{\partial q_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2}\frac{\partial h_1}{\partial q_2}\right] - \hat{\mathbf{u}}_1 \times \left[\frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial A_1}{\partial q_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial A_1}{\partial q_1} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial A_1}{\partial q_3}\right]\\ &= \frac{\hat{\mathbf{u}}_2 A_1}{h_1 h_3}\frac{\partial h_1}{\partial q_3} - \frac{\hat{\mathbf{u}}_3 A_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{\hat{\mathbf{u}}_3}{h_2}\frac{\partial A_1}{\partial q_2} + \frac{\hat{\mathbf{u}}_2}{h_3}\frac{\partial A_1}{\partial q_3}\\ &= \hat{\mathbf{u}}_2\left(\frac{A_1}{h_3 h_1}\frac{\partial h_1}{\partial q_3} + \frac{1}{h_3}\frac{\partial A_1}{\partial q_3}\right) - \hat{\mathbf{u}}_3\left(\frac{A_1}{h_1 h_3}\frac{\partial h_1}{\partial q_2} + \frac{1}{h_2}\frac{\partial A_1}{\partial q_2}\right)\\ &= \frac{\hat{\mathbf{u}}_2}{h_3 h_1}\frac{\partial (A_1 h_1)}{\partial q_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2}\frac{\partial (A_1 h_1)}{\partial q_2}. \qquad \text{...[30 (a)]}\end{aligned}$$

Either by similar treatment or by cyclic permutation of co-ordinates, we have

$$\nabla \times (\hat{\mathbf{u}}_2 A_2) = \frac{\hat{\mathbf{u}}_3}{h_3 h_2}\frac{\partial (A_2 h_2)}{\partial q_1} - \frac{\hat{\mathbf{u}}_2}{h_2 h_3}\frac{\partial (A_2 h_2)}{\partial q_3} \qquad \text{...[30 (b)]}$$

$$\nabla \times (\hat{\mathbf{u}}_3 A_3) = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}\frac{\partial (A_3 h_3)}{\partial q_2} - \frac{\hat{\mathbf{u}}_2}{h_3 h_1}\frac{\partial (A_3 h_3)}{\partial q_1}. \qquad \text{...[30 (c)]}$$

Substituting these values (29), we get

$$\begin{aligned}\text{curl } \mathbf{A} &= \nabla \times \mathbf{A}\\ &= \frac{\hat{\mathbf{u}}_1}{h_2 h_3}\left[\frac{\partial (A_3 h_3)}{\partial q_2} - \frac{\partial (A_2 h_2)}{\partial q_3}\right] + \frac{\hat{\mathbf{u}}_2}{h_3 h_1}\left[\frac{\partial (A_1 h_1)}{\partial q_3} - \frac{\partial (A_3 h_3)}{\partial q_1}\right]\\ &\quad + \frac{\hat{\mathbf{u}}_3}{h_1 h_2}\left[\frac{\partial (A_2 h_2)}{\partial q_1} - \frac{\partial (A_1 h_1)}{\partial q_2}\right] \qquad \text{...[31 (a)]}\end{aligned}$$

In determinant form this may be written as

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_2 \hat{\mathbf{u}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \qquad \text{...[31 (b)]}$$

Cor. In cartesian co-ordinates $ds^2 = dx^2 + dy^2 + dz^2$, therefore

$$q_1 = x,\ q_2 = y,\ q_3 = z \text{ and } h_1 = h_2 = h_3 = 1.$$

Also $\hat{\mathbf{u}}_1 = \mathbf{i}$, $\hat{\mathbf{u}}_2 = \mathbf{j}$ and $\hat{\mathbf{u}}_3 = \mathbf{k}$.

Then

$$\text{grad } \psi = \mathbf{i}\frac{\partial\psi}{\partial x} + \mathbf{j}\frac{\partial\psi}{\partial y} + \mathbf{k}\frac{\partial\psi}{\partial z},$$

$$\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

and

$$\text{curl } \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}.$$

1·12. Differential operators in Terms of Spherical Polar Co-ordinates and Cylindrical Co-ordinates :

1. Spherical polar co-ordinates and differential operators. The spherical polar co-ordinate system consists of :

(i) Concentric spheres about the origin O ($r =$ constant).

(ii) Right circular cones about Z-axis with the veritces at origin O ($\theta =$ constnat).

(iii) Half planes through the Z-axis.

Let P be a point in space such that its cartesian co-ordinates are (x, y, z). Then the spherical polar co-ordinates of point P are specified by (r, θ, ϕ) where $r = (OP)$ is the radius of the sphere on which point P lies ; θ is the co-ordinate *i.e.* the angle between OP and z-axis ; ϕ is the longitudinal or azimuthal angle *i.e.* the angle included between the X–Z plane and the plane OPZ.

From fig. 1.17 the transformations between the rectangular coordinates (x, y, z) and spherical co-ordinates (r, θ, ϕ) are given by

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \\ \text{and} \quad z &= r\cos\theta, \end{aligned}\right\} \quad ...(1)$$

we have

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi$$

$$= \sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta + r\sin\theta\sin\phi\,d\phi. \quad ...(2)$$

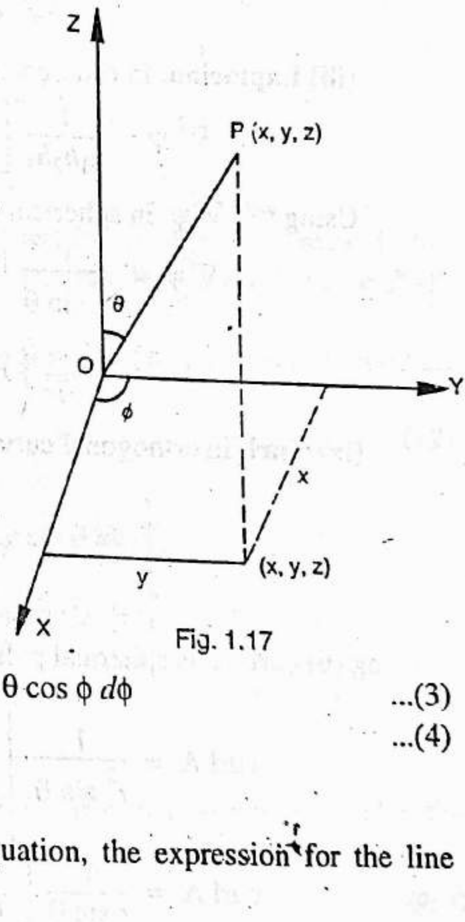


Fig. 1.17

Similarly,

$$dy = \sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi \quad ...(3)$$

and

$$dz = \cos\theta\,dr - r\sin\theta\,d\theta. \quad ...(4)$$

The line element ds in cartesian co-ordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Substituting values of dx, dy, dz from (2), (3) and (4), in above equation, the expression for the line element is spherical polar co-ordinates becomes

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2. \quad ...(5)$$

Comparing this equation with equation (7) of section, 1.11, we have

$$\left.\begin{aligned} q_1 &= r,\ q_2 = \theta,\ q_3 = \phi \\ \text{and} \quad h_1 &= 1,\ h_2 = r,\ h_3 = r\sin\theta \end{aligned}\right\} \quad ...(6)$$

Now we write *differential operators in spherical polar co-ordinates.*

(i) Gradient. In curvilinear co-ordinates grad ψ is

$$\text{grad } \psi = \nabla\psi = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial\psi}{\partial q_3}.$$

If $\hat{\mathbf{u}}_r$, $\hat{\mathbf{u}}_\theta$ and $\hat{\mathbf{u}}_\phi$ are unit vectors along r, θ and ϕ axes respectively, then using (6), grad ψ in spherical co-ordinates may be expressed as ...(7)

$$\text{grad}\ \psi = \frac{\partial \psi}{\partial r}\ \hat{\mathbf{u}}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\ \hat{\mathbf{u}}_\theta + \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial \phi}\ \hat{\mathbf{u}}_\phi \qquad ...(8)$$

or

$$\text{grad} = \nabla = \hat{\mathbf{u}}_r \frac{\partial}{\partial r} + \hat{\mathbf{u}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{u}}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

(ii) Divergence. In orthogonal curvilinear co-ordinates div **A** is

$$\text{div}\ \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_3 h_1) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right]$$

using (6), div **A** in spherical co-ordinates becomes

$$\text{div}\ \mathbf{A} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, A_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, A_\theta) + \frac{\partial}{\partial \phi}(r A_\phi)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad ...(9)$$

(iii) Laplacian. In orthogonal curvilinear co-ordinates $\nabla^2 \psi$ is

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right]$$

Using (6) $\nabla^2\psi$ in spherical polar co-ordinates becomes

$$\nabla^2 \psi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} \qquad ...(10)$$

(iv) Curl. In orthogonal curvilinear co-ordinates curl **A** is

$$\text{curl}\ \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_2 \hat{\mathbf{u}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Using (6) curl **A** is spherical polar co-ordinates becomes

$$\text{curl}\ \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & r\sin\theta\ \hat{\mathbf{u}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix} \qquad ...[11\,(a)]$$

or

$$\text{curl}\ \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right]\hat{\mathbf{u}}_r$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r}\right]\hat{\mathbf{u}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\mathbf{u}}_\phi \qquad ...[11(b)]$$

2. Cylindrical co-ordinates and differential operators. The cylindrical co-ordinate system consists of :

(i) Right circular cylinders which form families of concentric circles about the origin O in X–Y plane.

$$r = (x^2 + y^2)^{1/2} = \text{constant}.$$

(ii) Half planes through Z-axis, $\theta = \tan^{-1}(y/x) = \text{constant}$.

(iii) Planes parallel to X–Y plane, $Z = \text{constant}$.

Let P be a point in space such that its cartesian co-ordinates are (x, y, z). Then to find the position co-ordinates of P in cyclindrical co-ordinates draw a radius vector $\mathbf{r}$ from O' as shown in fig. 1.18 (b). Then draw PM parallel to z-axis and finally OM parallel to $O'P$. The angle MOX thus formed is θ.

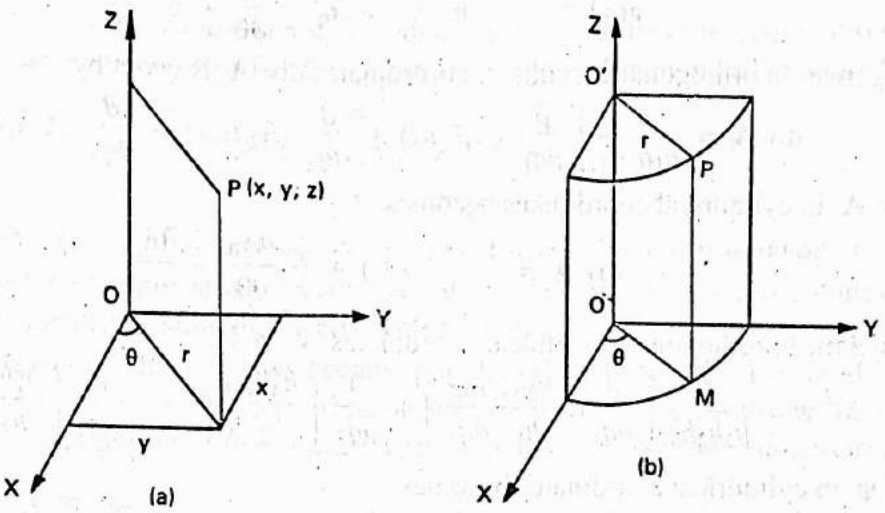


Fig. 1.18

Thus the position of point P in cylindrical co-ordinates is specified by (r, θ, z), where r is the distance is the X–Y plane from the origin to cylinder on which the point P lies ; θ is $\angle MOX$ in the X–Y plane ; and z is the distance from the X–Y plane to the point P.

From fig. 1.18 the transformations between rectangular co-ordinates (x, y, z) and cylindrical co-ordiantes (r, θ, z) are given by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z, \qquad \text{...(12)}$$

Therefore, we have

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta + \frac{\partial x}{\partial z}\,dz = \cos\theta\,dr - r\sin\theta\,d\theta \\ dy &= \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta + \frac{\partial y}{\partial z}\,dz = \sin\theta\,dr + r\cos\theta\,d\theta \\ dz &= dz \end{aligned}\right\} \qquad \text{...(13)}$$

As the line element in cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

Substituting values of dx, dy and dz from (13), the *line element ds in cylindrical coordinates* takes the form

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 \qquad \text{....(14)}$$

Comparing this equation with equation (7) of section 1.11, we get

$$\left.\begin{aligned} & q_1 = r,\ q_2 = \theta,\ q_3 = z \\ \text{and} \qquad & h_1 = 1,\ h_2 = r,\ h_3 = 1 \end{aligned}\right\} \qquad \text{...(15)}$$

Now we shall write differential operators in cylindrical co-ordinates.

(i) Gradient. In orthogonal curvilinear coordinates grad ψ is

$$\text{grad}\,\psi = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial\psi}{\partial q_3}$$

If $\hat{\mathbf{u}}_r, \hat{\mathbf{u}}_\theta$ and $\hat{\mathbf{u}}_z$ are unit vectors along r, θ and z axes respectively, then using (15) grad ψ in *cylindrical coordinates*.

$$\text{grad } \phi = \frac{\partial \phi}{\partial r} \hat{\mathbf{u}}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\mathbf{u}}_\theta + \frac{\partial \psi}{\partial z} \hat{\mathbf{u}}_z \qquad \text{...(16a)}$$

or

$$\text{grad } = \nabla = \hat{\mathbf{u}}_r \frac{\partial}{\partial r} + \hat{\mathbf{u}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{u}}_z \frac{\partial}{\partial z} \qquad \text{...(16b)}$$

(ii) Divergence. In orthogonal curvilinear co-ordinates div $\mathbf{A}$ is given by

$$\text{div } \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_3 h_1) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right]$$

using (15), div $\mathbf{A}$ in cylindrical coordinates becomes

$$\text{div } \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \qquad \text{...(17)}$$

(iii) Laplacian. In orthogonal curvilinear coordinates $\nabla^2 \psi$ is

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right]$$

using (15), $\nabla^2 \psi$ in cylindrical coordinates becomes

$$\begin{aligned} \nabla^2 \psi &= \frac{1}{r} \left[\left\{ \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) \right\} + \left\{ \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) \right\} + \left\{ \frac{\partial}{\partial z} \left(r \frac{\partial \psi}{\partial z} \right) \right\} \right] \\ &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad \text{...(18)} \end{aligned}$$

(iv) Curl. In orthogonal curvilinear coordinates curl $\mathbf{A}$ is

$$\text{curl } \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_2 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

using (15), curl $\mathbf{A}$ is cylindrical coordinates becomes

$$\begin{aligned} \text{curl } \mathbf{A} &= \frac{1}{r} \begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & r A_\theta & A_z \end{vmatrix} \\ &= \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{\mathbf{u}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\mathbf{u}}_\theta + \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{\mathbf{u}}_z. \qquad \text{...(19)} \end{aligned}$$

1·13. Tensors

Introduction. In many physical problems vector notation is not sufficiently general to express the relations between quantities involved. For example, let us consider the flow of electric current in a conducting medium. If the medium is *isotropic,* then relation between current density vector $\mathbf{J}$ and the electric field vector $\mathbf{E}$ is given by the vector equation

$$\mathbf{J} = \sigma \mathbf{E} \qquad \text{...(1)}$$

where σ is electrical conductivity of the medium. For such a medium the electrical conductivity σ is scalar and for x-component we write

$$J_1 = \sigma E_1 \qquad \text{...(2)}$$

However, if the medium is anisotropic as in many crystals, the current density in x-direction may depend on the electric fields in Y and Z directions as well as on the field in X-direction. Assuming the linear relationship, we may replace equation (2) with

$$\mathrm{J}_1 = \sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3 \qquad \text{...(3)}$$

and in general

$$\mathrm{J} = \Sigma\, \sigma E. \qquad \text{...(4)}$$

Thus for ordinary three dimensional space the scalar quantity σ must be replaced by a more general quantity σ_{ik} (a set of nine elements) to effect the requried change in direction and magnitude

$$\sigma_{ik} \rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

This general quantity σ_{ik} (array of nine elements) is called a *tensor. In tensor language a vector is called a tensor of rank first and a scalar a tensor of rank* zero. The description of elastic, optical, electrical and magnetic properties of anisotropic solid may well involve tensors.

The tensor formulation of physical laws became popular when Albert Einstein used it as an excellent tool for the presentation of his general theory of relativity. Now a days it is invaluable in its applications to many branches of theoretical physics : mechanics elasticity, plasticity, electrodynamics etc.

1·14. Fundamentals of Tensors.

1·14 a. n-dimensional space. In the three dimensional space a point is determined by a set of three numbers called the coordinates of that point; for example (x, y, z) are the coordinates of a point in rectangular cartesian coordinate system. By analogy if a point is represented by a set of n real variables $(x_1, x_2, x_3 ..., x_n)$ or more conveniently $(x^1, x^2, x^3 ..., x^n)$. [Here the suffixes (1, 2, 3, ...n) denote variables and not the powers of variables involved], then all the points corresponding to all values of coordinates are said to form an n-dimensional space, denoted by V_n.

A curve in n-dimensional space V_n is defined as the collection of points which satisfy n-equations.

$$x^\mu = x^\mu (u), \;(\mu = 1, 2, 3, ...n).$$

where u is a parameter and $x^\mu (u)$ are n functions of u, which satisfy certain continuity equations.

A *sub-space* V_m $(m < n)$ of V_n is defined as the collection of points which satisfy the n-equations.

$$x^\mu = x^\mu (u^1, u^2, u^3 \dots u^m), \;(\mu = 1, 2, 3, ...n)$$

where $u^1, u^2, u^3 \dots u^m$ are m parameters and $x^\mu (u^1, u^2, \dots u^m)$ are n functions of $u^1, u^2 \dots u^m$ which satisfy certain continuity conditions.

1·14 b. Superscripts and subscripts. The suffixes μ and ν in A^μ_ν are called *superscript* and *subscript* respectively. The upper position of suffix always denotes superscript, while the lower position always denotes subscript. It is always to be kept in mind that the suffixes μ in the coordinates x^μ do not have the character of power indices. Usually power will be indicated by the use of brackets. Thus $(x^\mu)^2$ means square of x^μ. The reason for using superscripts and subscripts will be indicated in due course.

1·14 c. Coordinate Transformation. Tensor analysis is immediately connected with the subject of coordinate transformation.

Consider two sets of variables $(x^1, x^2, x^3 \dots x^n)$ and $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \dots \bar{x}^n)$ which determine the coordinates of a point in n-dimensional space in two different frames of reference.

Let the two sets of variables be related to each other by the transformation equations.

$$\begin{aligned}
\bar{x}^1 &= \phi^1 (x^1, x^2, x^3, \ldots x^n) \\
\bar{x}^2 &= \phi^2 (x^1, x^2, x^3, \ldots x^n) \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\bar{x}^n &= \phi^n (x^1, x^2, x^3, \ldots x^n)
\end{aligned} \qquad \ldots(1)$$

or briefly $x^{\mu} = \phi^{\mu} (x^1, x^2, x^3, \ldots x^n) \; (\mu = 1, 2, 3, \ldots n)$

where functions ϕ^{μ} are single valued, continuous differentiable functions of coordinates. It is essential that the n-functions ϕ^{μ} be independent. Equations (1) can be solved for coordinate x^{μ} as functions of $\bar{x}^{\mu}$ to yield

$$x^{\mu} = \phi_{\mu} (\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n).$$

Equations (1) and (2) are said so define coordinate transformations.

From equation (1) the differentials $d\bar{x}^{\mu}$ are transformed as

$$\begin{aligned}
d\bar{x}^{\mu} &= \frac{\partial \bar{x}^{\mu}}{\partial x^1} dx^1 + \frac{\partial \bar{x}^{\mu}}{\partial x^2} dx^2 + \ldots + \frac{\partial \bar{x}^{\mu}}{\partial x^n} dx^n \\
&= \sum_{\alpha = 1}^{n} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha} \; (\mu = 1, 2, 3, \ldots n).
\end{aligned} \qquad \ldots(3)$$

1·14 e. Indicial and Summation Conventions. Let us now introduce the following two conventions.

1. Indicial Convention. Any index, used either as subscript or superscript will take all values from 1 to n-unless contrary is specified. Thus equation (1) can be briefly written as

$$\bar{x}^{\mu} = \phi^{\mu} (x^{\alpha}). \qquad \ldots(4)$$

The convention reminds us that there are n equations with $\mu = 1, 2, \ldots n$ and ϕ^{μ} are the functions of n coordinates x^{α} where $\alpha = 1, 2, 3, \ldots n$.

2. Einstien's summation convention. If any index is repeated in a term, then a summation with respect to that index over the range 1, 2, 3, ...n is implied. This convention is called Einstien's summation convention.

According to this convention instead of expression $\sum_{\mu = 1}^{n} a_{\mu} x^{\mu}$ we merely write $a_{\mu} x^{\mu}$.

Using these two conventions equations (3) may be written as

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha}. \qquad \ldots(5)$$

Thus the summation convention means the drop of sigma sign for the index appearing twice in a given term. In other words *the summation convention implies the sum of the terms for the index appearing twice in that term over defined range.*

1·14 f. Dummy and real indices. *Any index which is repeated in a given term, so that the summation convention applies, is called a* **dummy index** and it may be replaced freely by any other index not already used in that term. For example α is the dummy index in equation (5), so that equation (5) is equally written as

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}}\, dx^{\beta} = \frac{\partial \bar{x}^{\mu}}{\partial x^{k}}\, dx^{k}.$$

Also two or more dummy indices can be interchanged. In order to avoid confusion the same index must not be used more than twice in any single term. For example $(\sum_{\mu} a_{\mu} x^{\mu})$ will not be written as $a_{\mu} x^{\mu} a_{\mu} x^{\mu}$ but rather $a_{\mu} a_{\nu} x^{\mu} x^{\nu}$, *Any index, which is not repeated in a given term, is called is* **real index.** For example μ is a real index in $a_{\alpha}^{\mu} x^{\mu}$. A real index can not be replaced by another real index.

$$a_{\alpha} x^{\alpha} \neq a_{\alpha}^{\mu} x^{\alpha}.$$

1·14 g. Kronecker delta symbol. Kronecker delta symbol δ_{ν}^{μ} is defined as

$$\delta_{\nu}^{\mu} = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \text{if} \quad \mu \neq \nu \end{cases} \qquad \text{...(6)}$$

Some properties of Kronecker delta.

(i) If $x^{1}, x^{2}, x^{3}, \ldots x^{n}$ are independent variables, then

$$\frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta_{\nu}^{\mu} \qquad \text{...(7)}$$

(ii) An obvious property of Kronecker delta symbol is

$$\delta_{\nu}^{\mu} A^{\mu} = A^{\nu} \qquad \text{...(8)}$$

(iii If we are dealing with n-dimensions, then

$$\delta_{\mu}^{\mu} = \delta_{\nu}^{\nu} = n \qquad \text{...(9)}$$

Since by summation convention

$$\begin{aligned} \delta_{\mu}^{\mu} &= \delta_{1}^{1} + \delta_{2}^{2} + \delta_{3}^{3} + \ldots + \delta_{n}^{n} \\ &= 1 + 1 + 1 + \ldots + 1 = n \end{aligned}$$

(iv) $$\delta_{\nu}^{\mu}\, \delta_{\sigma}^{\nu} = \delta_{\sigma}^{\mu} \qquad \text{...(10)}$$

By summation convention

$$\begin{aligned} \delta_{\nu}^{\mu}\, \delta_{\sigma}^{\nu} &= \delta_{1}^{\mu}\, \delta_{\sigma}^{1} + \delta_{2}^{\mu}\, \delta_{\sigma}^{2} + \delta_{3}^{\mu}\, \delta_{\sigma}^{3} + \ldots + \delta_{\mu}^{\mu}\, \delta_{\sigma}^{\mu} + \ldots + \delta_{n}^{\mu}\, \delta_{\sigma}^{n} \\ &= 0 + 0 + 0 + \ldots + 1.\delta_{\sigma}^{\mu} + \ldots + 0 \\ &= \delta_{\sigma}^{\mu}. \end{aligned}$$

(v) $$\frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}} \frac{\partial \bar{x}_{\nu}}{\partial x^{\sigma}} = \frac{\partial x^{\mu}}{\partial x^{\sigma}} = \delta_{\sigma}^{\mu} \qquad \text{...(11)}$$

1·14 h. Scalars; contravariant vectors and covariant vectors.

(a) Scalars. Consider a function ϕ in a system of variables x^{μ} and let this function have the value $\bar{\phi}$ in another system of variable $\bar{x}^{\mu}$. Then if $\bar{\phi} = \phi$; the function ϕ is said to be *scalar* or *invariant* or a tensor of *rank zero.* The quantity

$$\delta_{\mu}^{\mu} = \delta_{1}^{1} + \delta_{2}^{2} + \ldots + \delta_{n}^{n} = n$$

(b) Contravariant vector. Consider a set of n quantities $A^1, A^2, A^3 \ldots A^n$ in a system of variables x^μ and let these quantities have values $\overline{A}^1, \overline{A}^2, \overline{A}^3 \ldots \overline{A}^n$ in another system of variables $\overline{x}^\mu$. If these quantities obey the transformation relation

$$\overline{A}^\mu = \frac{\partial \overline{x}^\mu}{\partial x^\alpha} A^\alpha \qquad \text{...(12)}$$

then the quantities A^α are said to be *the components of a contravariant vector or a contravariant tensor of rank one.*

Any n functions can be chosen as the components of a contravariant vector in a system of variables $\overline{x}^\mu$.

Multiplying equation (12) by $\dfrac{\partial x^\beta}{\partial \overline{x}^\mu}$ and taking the sum over index μ from 1 to n, we get

$$\frac{\partial x^\beta}{\partial \overline{x}^\mu} \overline{A}^\mu = \frac{\partial x^\beta}{\partial \overline{x}^\mu} \frac{\partial \overline{x}^\mu}{\partial x^\alpha} A^\alpha = \frac{\partial x^\beta}{\partial x^\alpha} A^\alpha = \delta^\beta_\alpha A^\alpha = A^\beta$$

or

$$A^\beta = \frac{\partial x^\beta}{\partial \overline{x}^\mu} \overline{A}^\mu \qquad \text{...(13)}$$

Equations (13) represent the solution of equation (12).

The transformation of differentials dx^μ and $d\overline{x}^\mu$ in systems of variables x^μ and $\overline{x}^\mu$ respectively, from (5a) is given by

$$d\overline{x}^\mu = \frac{\partial \overline{x}^\mu}{\partial x^\alpha} dx^\alpha \qquad \text{...(14)}$$

As equations (12) and (14) are similar transformation equations, we can say that the differentials dx^μ form the components of contravariant vector ; whose components in any other system of variables $\overline{x}^\mu$ are the differentials $d\overline{x}^\mu$. Also we conclude that the components of a contravariant vector are actually the *components of a contravariant tensor of rank one :*

Note : A single superscript is always used to indicate a contravariant vector unless the contrary is explicitly stated.

(c) Covariant vector. Consider a set of quantities $A_1, A_2, A_3, \ldots, A_n$ in a system of variables x^μ and let these quantities have values $\overline{A}_1, \overline{A}_2, \overline{A}_3 \ldots \overline{A}_n$ in another system of variables $\overline{x}^\mu$. If these quantities obey the transformation equations

$$\overline{A}_\mu = \frac{\partial x^\alpha}{\partial \overline{x}^\mu} A_\alpha \qquad \text{...(15)}$$

then the quantities A_α are said to be *the components of a covariant vector or a covariant tensor of rank one.*

Any n functions can be chosen as the components of a covariant vector in a system of variables x^μ and equations (15) determine the n-components in the new system of variables $\overline{x}^\mu$. Multiplying equation (15) by $\partial \overline{x}^\mu / \partial x^\beta$ and taking the sum over the index μ from 1 to n, we get

$$\frac{\partial \overline{x}^\mu}{\partial x^\beta} \overline{A}_\mu = \frac{\partial \overline{x}^\mu}{\partial x^\beta} \frac{\partial x^\alpha}{\partial \overline{x}^\mu} A_\alpha = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha = \delta^\alpha_\beta A_\alpha = A_\beta$$

Thus
$$A_\beta = \frac{\partial \bar{x}^\mu}{\partial x^\beta} \bar{A}_\mu \qquad \text{...(16)}$$

Equations (16) represent the solution of equation (15)

As
$$\frac{\partial \psi}{\partial \bar{x}^\mu} = \frac{\partial \psi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial \psi}{\partial x^\alpha} \qquad \text{...(17)}$$

It follows from (15) that $\frac{\partial \psi}{\partial x^\alpha}$ form the components of a convariant vector, whose components in any other system are the corresponding derivatives $\frac{\partial \psi}{\partial x^\mu}$. This covariant vector is called grad ψ.

Note : A single subscript is always used to indicate covariant vector unless contrary is explicity stated ; but exception occurs in the notation of coordinates.

1·14 I. Tensors of higher ranks.

The laws of transformation of vectors are

contravariant.
$$\bar{A}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} A^\alpha \qquad \text{...(12)}$$

covariant
$$\bar{A}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} A_\alpha \qquad \text{...(15)}$$

(a) Contravariant tensor of second rank. Let us consider quantities $A^{\mu\nu}$ (here μ and ν take the values from 1 to n independently) in a system of variable x^μ and let these quantities have values $\bar{A}^{\mu\nu}$ in another system of variables $\bar{x}^\mu$. If these quantities obey the transformation.

$$\bar{A}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\beta} A^{\alpha\beta} \qquad \text{...(18)}$$

then the quantities $A^{\mu\nu}$ are said to be the *components of a contravariant tensor of second rank.*

The transformation law represented by (18) is the generalisation of the transformation law (12). Any set of $(n)^2$ quantities can be chosen as the components of a contravariant tensor of second rank in a system of variables x^μ and then equations (18) determine $(n)^2$ components in any other system of variables $\bar{x}^\mu$.

(b) Covariant tensor of second rank. If $(n)^2$ quantities $A_{\mu\nu}$ in a system of variables x^μ are related to other $(n)^2$ quantities in another system of variables $\bar{x}^\mu$ by the transformation equations

$$\bar{A}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} A_{\alpha\beta} \qquad \text{...(19)}$$

then the quantities $A_{\mu\nu}$ are said to be the components of a covariant tensor of second rank.

The transformation law represented by (19) is the generalisation of the transformation law (15). Any set of $(n)^2$ quantities can be chosen as the components of a covariant tensor of second rank in a system of variables x^μ and then equations (19) determine $(n)^2$ components in any other system of variables $\bar{x}^\mu$.

(c) Mixed tensor of second rank. If $(n)^2$ quantities A^μ_ν in a system of variables x^μ are related to other $(n)^2$ quantities $\bar{A}^\mu_\nu$ in another system of variables $\bar{x}^\mu$ by the transformation equations

$$\bar{A}^{\mu}_{\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} A^{\alpha}_{\beta} \qquad \text{...(20)}$$

then the quantities A^{μ}_{ν} are said to *be the components of a mixed tensor of second rank.*

An important example of mixed tensor of second rank is kronecker delta δ^{μ}_{ν}.

Note : In contravariant tensors the indices are placed superscripts, while in covariant tensors the indices are placed as subscripts. The mixed tensor A^{μ}_{ν} transforms like a contravariant vector with respect to index μ and like a covariant vector with respect to index ν.

(D) Tensors of higher rank ; Rank of a tensor. The tensors of higher ranks are defined by similar laws. *The rank of a tensor only indicates the number of indices attached to its per component.* For example $A^{\mu\nu\sigma}_{\lambda}$ are the components of mixed tensor of a rank 4 (contravariant of rank 3 and covariant of rank 1) : if they transform according to the equations

$$\bar{A}^{\mu\nu\sigma}_{\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A^{\alpha\beta\gamma}_{\delta} \qquad \text{...(21)}$$

The rank of a tensor when raised as power to the number of dimensions gives the number of components of that tensor. For example a tensor of rank r in n-dimensional space has $(n)^r$ components. Thus the rank of a tensor gives the number of mode of changes of a physical quantity when passing from one system to other which is in rotation relative to the first. Obviously *a quantity that remains unchanged when axes are rotated is a tensor of zero rank.* The tensors of zero rank are scalars or invariant. Similarly, the tensors of rank one are vectors. It may be easily seen that (i) The transformation of tensors form a group and (ii) There is no distinction between contravariant and covariant vectors (or tensors) when we restrict ourselves to rectangular cartesian transformation of coordinates.

1·15. Algebraic Operations of Tensors.

(a) Addition and Subtraction. The addition and subtraction of tensors is defined only in the case of ensors of *same rank and same type.* Same type means the same number of contravariant and covariant indices. The addition or subtraction of two tensors, like vectors, involves the individual elements. To Add or subtract two tensors the corresponding elements are added or subtracted.

The sum or difference of two tensors of the same rank and same type is also a tensor of the same rank and same type.

For example if there are two tensors $A^{\mu\nu}_{\sigma}$ and $B^{\mu\nu}_{\sigma}$ of the same rank and same type, then the laws of ddition and subtraction are given by

$$A^{\mu\nu}_{\sigma} + B^{\mu\nu}_{\sigma} = C^{\mu\nu}_{\sigma} \qquad \text{(addition)} \qquad \text{...(1)}$$

$$A^{\mu\nu}_{\sigma} - B^{\mu\nu}_{\sigma} = D^{\mu\nu}_{\sigma} \qquad \text{(subtraction)} \qquad \text{...(2)}$$

where $C^{\mu\nu}_{\sigma}$ and $D^{\mu\nu}_{\sigma}$ are the tensors of the same rank and same type as given tensors.

The transformation laws for given tensors are

$$\bar{A}^{\mu\nu}_{\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} A^{\alpha\beta}_{\gamma} \qquad \text{...(3)}$$

$$\overline{B}_{\sigma}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} B_{\gamma}^{\alpha\beta} \qquad ...(4)$$

Adding (3) and (4), we get

$$\left(\bar{A}_{\sigma}^{\mu\nu} + \bar{B}_{\sigma}^{\mu\nu}\right) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\gamma}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} (A_{\gamma}^{\alpha\beta} + B_{\gamma}^{\alpha\beta})$$

or

$$\overline{C}_{\sigma}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} C_{\gamma}^{\alpha\beta} \qquad ...(5)$$

which is a transformation law for the sum and it similar to transformation laws for $A_{\sigma}^{\mu\nu}$ and $B_{\sigma}^{\mu\nu}$ given by (3) and (4). Hence *the sum* $C_{\sigma}^{\mu\nu} = (A_{\sigma}^{\mu\nu} + B_{\sigma}^{\mu\nu})$ *is itself a tensor of the same rank and same type as the given tensors.*

Subtracting equations (4) from (3) we get

$$\overline{A}_{\sigma}^{\mu\nu} - \overline{B}_{\sigma}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} (A_{\gamma}^{\alpha\beta} - B_{\gamma}^{\alpha\beta})$$

or

$$\overline{D}_{\sigma}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} D_{\gamma}^{\alpha\beta}. \qquad ...(6)$$

which is a transformation law for the differnce and is again similar to the transformation laws for $A_{\sigma}^{\mu\nu}$ and $B_{\sigma}^{\mu\nu}$. Hence *the difference* $D_{\sigma}^{\mu\nu} = (A_{\sigma}^{\mu\nu} - B_{\sigma}^{\mu\nu})$ *is itself a tensor of a same rank and same type as the given tensors.*

The addition of tensors is commutative and associative.

(b) Equality of tensors. The two tensors of the same rank same type are said to be equal if their components are one to one equal, *i.e.*, if

$$A_{\sigma\lambda}^{\mu\nu} = B_{\sigma\lambda}^{\mu\nu} \text{ for all values of indices.} \qquad ...(7)$$

If two tensors are equal in one co-ordinate system, they will be equal in any other co-ordinate system. Thus if a particular equation is expressed in tensorial form, it will remain invariant under coordinate transformations.

(c) Outer product. *The outer product of two tensors is a tensor whose rank is the sum of the ranks of given tensors.* Thus if r and r' are the ranks of two tensors, their outer product will be tensor of rank $(r + r')$ for example, if $A_{\sigma}^{\mu\nu}$ and B_{ρ}^{λ} are tensors of ranks 3 and 2 respectivley, then

$$A_{\sigma}^{\mu\nu} B_{\rho}^{\lambda} = C_{\sigma\rho}^{\mu\nu\lambda} \text{ (say)} \qquad ...(8)$$

is a tensor of rank $(3 + 2 = 5)$.

For proof let us write the transformation equations of given tensors as

$$\overline{A}_{\sigma}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} A_{\gamma}^{\alpha\beta} \qquad ...(9)$$

$$\overline{B}_{\rho}^{\lambda} = \frac{\partial \bar{x}^{\lambda}}{\partial x^{\delta}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\rho}} B_{\tau}^{\delta} \qquad ...(10)$$

Multiplying (9) and (10), we get

$$\overline{A}_{\sigma}^{\mu\nu}\,\overline{B}_{\rho}^{\lambda} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \overline{x}^{\lambda}}{\partial x^{\delta}}\frac{\partial x^{\gamma}}{\partial \overline{x}^{\sigma}}\frac{\partial x^{\tau}}{\partial \overline{x}^{\rho}}A_{\gamma}^{\alpha\beta}\,B_{\tau}^{\delta}$$

or

$$\overline{C}_{\sigma\rho}^{\mu\gamma\lambda} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \overline{x}^{\lambda}}{\partial x^{\delta}}\frac{\partial x^{\gamma}}{\partial \overline{x}^{\sigma}}\frac{\partial x^{\tau}}{\partial \overline{x}^{\rho}}C_{\lambda\tau}^{\alpha\beta\delta} \qquad \text{[using (8)]} \qquad \text{...(11)}$$

which is a transformation law for a tensor of ranks 5. Hence the outer product of two tensors : one $A_{\sigma}^{\mu\nu}$ of rank 3 and other B_{ρ}^{λ} of rank 2, is a tensor $\overline{C}_{\sigma\rho}^{\mu\nu\lambda}$ of rank $5 = (3 + 2)$. The outer product of tensors is commutative and associative.

(d) Contraction of tensors. *The algebraic operation by which the rank of a mixed tensor is lowered by 2 is known as contraction.* In the process of contraction one contravarian index and one covariant index are set equal and the repeated index summed over, the result is a tensor of rank lower by 2 than the original tensor.

For example, consider a mixed tensor $A_{\lambda\rho}^{\mu\nu\sigma}$ of rank 5 with contravariant inddices μ, ν, σ and covariant indices λ, ρ.

The transformation law for the given tensor is

$$\overline{A}_{\lambda\rho}^{\mu\nu\sigma} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \overline{x}^{\sigma}}{\partial x^{\gamma}}\frac{\partial x^{\delta}}{\partial \overline{x}^{\lambda}}\frac{\partial x^{\tau}}{\partial \overline{x}^{\rho}}A_{\delta\tau}^{\alpha\beta\gamma} \qquad \text{...(12)}$$

To apply the process of contraction, we put $\rho = \sigma$ and obtain

$$\begin{aligned}
\overline{A}_{\lambda\sigma}^{\mu\nu\sigma} &= \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \overline{x}^{\sigma}}{\partial x^{\gamma}}\frac{\partial x^{\delta}}{\partial \overline{x}^{\lambda}}\frac{\partial x^{\tau}}{\partial \overline{x}^{\sigma}}A_{\delta\tau}^{\alpha\beta\gamma} \\
&= \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\delta}}{\partial \overline{x}^{\lambda}}\frac{\partial x^{\tau}}{\partial \overline{x}^{\sigma}}\frac{\partial \overline{x}^{\sigma}}{\partial x^{\gamma}}A_{\delta\tau}^{\alpha\beta\gamma} \\
&= \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\delta}}{\partial \overline{x}^{\lambda}}\delta_{\gamma}^{\tau}A_{\delta\tau}^{\alpha\beta\gamma} \qquad \left(\text{since } \frac{\partial x^{\tau}}{\partial \overline{x}^{\sigma}}\frac{\partial \overline{x}^{\sigma}}{\partial x^{\gamma}} = \frac{\partial x^{\tau}}{\partial x^{\gamma}} = \delta_{\gamma}^{\tau}\right) \\
&= \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \overline{x}^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\delta}}{\partial \overline{x}^{\gamma}}A_{\delta\gamma}^{\alpha\beta\gamma} \qquad \text{...(13)}
\end{aligned}$$

which is a transformation law for a mixed tensor of rank 3.

Hence $A_{\lambda\sigma}^{\mu\nu\sigma}$ is a mixed tensor of rank 3 and may be denoted by $A_{\lambda}^{\mu\nu}$ or $A_{\delta}^{\alpha\beta}$. In this example, we can further apply the contraction process and obtain the contravariant vector $A_{\nu\sigma}^{\mu\nu\sigma}$ or A^{μ}. Thus the process of contraction enables us to obtain a tensor of rank $(r - 2)$ from a mixed tensor of rank r.

As another example, consider the contraction of the mixed tensor A_{ν}^{μ} of rank 2, whose trnasformation law is

$$\overline{A}_{\nu}^{\mu} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}}A_{\beta}^{\alpha}$$

To apply process we put $\nu = \mu$ and obtain

$$\overline{A}_{\mu}^{\mu} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\mu}}A_{\beta}^{\alpha} = \delta_{\alpha}^{\beta}A_{\beta}^{\alpha} = A_{\alpha}^{\alpha}$$

which is a transformation law of a scalar or an invariant.

As by contraction process the rank 2 of mixed tensor A_ν^μ has been lowered to zero, it is therefore justified to call a scalar, a tensor of rank zero.

Note. Contraction process cannot be applied with respect to two indices of the same nature *i.e.* both contravariant or both covariant.

(e) Inner product. *The outer product of two tensors followed by a contraction results a new tensor called an 'inner product' of two tensors and the proces is called the 'inner multiplication of two tensors'.*

Examples. (i) Consider two tensors $A_\sigma^{\mu\nu}$ and B_ρ^λ.

The outer product of these two tensors is

$$A_\sigma^{\mu\nu} B_\rho^\lambda = B_{\sigma\rho}^{\mu\nu\lambda} \text{ (say)}$$

Applying contraction process by setting $\rho = \mu$

$$A_\sigma^{\mu\nu} B_\mu^\lambda = C_{\sigma\mu}^{\mu\nu\lambda} = D_\sigma^{\nu\lambda} \text{ (say)}.$$

The new tensors $D_\sigma^{\nu\lambda}$ is the inner product of two tensors $A_\sigma^{\mu\nu}$ and B_ρ^λ.

(ii) As an other example consider two tensors A^μ and B_ν, each of rank 1. The outer product of A^μ and B_ν is

$$A^\mu B_\nu = C_\nu^\mu \text{ (say)}.$$

Applying contraction process by setting $\nu = \mu$, we get

$$A^\mu B_\mu = C_\mu^\mu \text{ (scalar or a tensor of rank zero)}.$$

Thus the inner product of two tensors each of rank one is a tensor of rank zero (*i.e.* invariant).

(f) Extension of a rank. The rank of a tensor can be extended by differentiating its each component with respect to variables x^μ.

As an example consider a simple case in which the original tensor of rank zero *i.e.* a scalar $S(x^\mu)$ whose derivatives relative to variable x^μ are $\frac{\partial S}{\partial x^\mu}$. In the other system of variable $\bar{x}^\mu$, the scalar is $\bar{S}(\bar{x}^\mu)$, such that

$$\frac{\partial \bar{S}}{\partial \bar{x}^\mu} = \frac{\partial S}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \bar{x}^\mu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial S}{\partial x^\alpha}. \qquad \text{...(14)}$$

This shows that $\frac{\partial S}{\partial x^\alpha}$ transforms like the components of a tensor of rank 1. Thus the differential of tensor of rank zero gives a tensor of rank one. In general we may say that the *differentiation of a tensor with respect to variables x^μ yields a new tensor of rank one greater than the original tensor.*

The rank of a tensor can also be extended when a tensor depends upon another tensor and the differentiation with respect to that tensor is performed. As an example, consider a tensor S of rank zero (*i.e.* a scalar) depending upon another tensor $A_{\mu\nu}$, then

$$\frac{\partial S}{\partial A_{\mu\nu}} = B_{\mu\nu} = a \text{ tensor of rank 2.} \qquad \text{...(15)}$$

Thus the rank of the tensor of rank zero has been extended by 2.

(g) Quotient law. In tensor analysis it is often necessary to ascertain whether a given entity is a tensor or not. The direct method requires us to find out if the given entity obeys the tensor transformation law or not. In practice this is troublesome and a simpler test is provided by a law known as *quotient law* which states.

An entity whose inner product with an arbitrary tensor (contravariant or covariant) is a tensor, is itself a tensor.

Example. Let $A(\mu, \nu, \sigma)$ be the given entity to be tested whether it is a tensor or not. To apply quotient law, let us consider an arbitrary tensor $B^{\lambda}_{\nu\sigma}$ whose inner product with $A(\mu, \nu, \sigma)$ is a tensor, *i.e.*

$$A(\mu, \nu, \sigma)\, B^{\lambda}_{\nu\sigma} = C^{\lambda}_{\mu}. \qquad \text{...(16)}$$

We have to show that $A(\mu, \nu, \sigma)$ is a tensor. In the other system of variables $\bar{x}^{\mu}$, we must have

$$\bar{A}(\mu, \nu, \sigma)\, \bar{B}^{\lambda}_{\nu\sigma} = \bar{C}^{\lambda}_{\mu}.$$

Hence,

$$\bar{A}(\mu, \nu, \sigma) \frac{\partial \bar{x}^{\lambda}}{\partial x^{\delta}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}\,\sigma} B^{\delta}_{\beta\gamma} = \frac{\delta \bar{x}^{\lambda}}{\partial x^{\delta}} \frac{\partial x^{\alpha}}{\partial \bar{x}\,\mu} C^{\delta}_{\alpha}$$

Using (16), we get

$$\bar{A}(\mu, \nu, \sigma) \frac{\partial \bar{x}^{\lambda}}{\partial x^{\delta}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} B^{\delta}_{\beta\gamma} = \frac{\partial \bar{x}^{\lambda}}{\partial x^{\delta}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha, \beta, \gamma)\, B^{\delta}_{\beta\gamma}$$

or

$$\frac{\partial \bar{x}^{\lambda}}{\partial x^{\delta}} \left[\bar{A}(\mu, \nu, \sigma) \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} - \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha, \beta, \gamma) \right] B^{\delta}_{\beta\gamma} = 0.$$

On inner multiplication with $\frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}}$, we obtain

$$\frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{\delta}} \left[\bar{A}(\mu, \nu. \sigma) \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} - \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha, \beta, \gamma) \right] B^{\delta}_{\beta\gamma} = 0$$

or

$$\delta^{\tau}_{\delta} \left[\bar{A}(\mu, \nu, \sigma) \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} - \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha, \beta, \gamma) \right] B^{\delta}_{\beta\gamma} = 0$$

or

$$\left[\bar{A}(\mu, \nu, \sigma) \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} - \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha, \beta, \gamma) \right] B^{\tau}_{\beta\gamma} = 0$$

Since $B^{\tau}_{\beta\gamma}$ is an arbitrary tensor, we can arrange that only one of its component is non-zero. Now each component $B^{\tau}_{\beta\gamma}$ may be chosen in turn as that one which does not vanish. Therefore the expression in bracket is identically zero.

i.e.

$$\bar{A}(\mu, \nu ; \sigma) \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} - \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha ; \beta, \gamma) = 0.$$

Again on innermultiplying with $\frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}}$, we get

$$\frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}} \left[\bar{A}(\mu, \nu, \sigma) \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} - \frac{\partial \bar{x}^{\alpha}}{\partial \bar{x}^{\mu}} A(\alpha, \beta, \gamma) = 0. \right]$$

Rearranging, we get

$$\frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \cdot \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \left(\frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}} \cdot \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} \right) \cdot \frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} \bar{A}\,(\mu, \nu, \sigma) - \frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \cdot \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} A\,(\alpha, \beta, \gamma) = 0$$

or

$$\delta^{\rho}_{\nu}\, \delta^{\varepsilon}_{\sigma}\, \bar{A}\,(\mu, \nu, \sigma) = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}} A\,(\alpha, \beta, \gamma)$$

or

$$\bar{A}\,(\mu, \rho, \varepsilon) = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}} A\,(\alpha, \beta, \gamma)$$

Equivalently

$$\bar{A}\,(\mu, \nu, \sigma) = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} A\,(\alpha, \beta, \gamma). \qquad \ldots(17)$$

This equation represents the transformation law for the entity $A\,(\mu, \nu, \sigma)$ which is a tensor transformation law. Hence $A\,(\mu, \nu, \sigma)$ is a tensor. Equation (17) represents the transformation law for a mixed tensor with μ as covariant index and ν, σ as contravariant indices. Hence the tensor $A\,(\mu, \nu, \sigma)$ justifies the notation $A^{\nu\sigma}_{\mu}$. Thus *quotient* law not only tells us that the entity $A\,(\mu, \nu, \sigma)$ is a tensor but also tells its type *i.e.* number of contravariant and covariant indices.

1.16. Symmetric and Anti-symmetric Tensors.

(a) Symmetric tensors. *If two contravariant or covariant indices can be interchanged without altering the tensor, then the tensor is said to be symmetric with respect to these two indices.*

For example, if $A^{\mu\nu} = A^{\nu\mu}$
or $A_{\mu\nu} = A_{\nu\mu}$ ...(1)

then the contravariant tensor of second rank $A^{\mu\nu}$ or covariant tensor of second rank $A_{\mu\nu}$ is said to be symmetric.

For a tensor of higher rank $A^{\mu\nu\sigma}_{\lambda}$, if

$$A^{\mu\nu\sigma}_{\lambda} = A^{\nu\mu\sigma}_{\lambda},$$

then the tensor $A^{\mu\nu\sigma}_{\lambda}$ is said be symmetric with respect to indices μ and ν.

The symmetry property of a tensor is independent of coordinate system used. So *if a tensor is symmetric with respect to two indices in one co-ordinate system, it remains symmetric with respect to these two indices in any co-ordinate system.* This can be seen as follows :

If tensor $A^{\mu\nu\sigma}_{\lambda}$ is symmetric with respect to first two indices μ and ν, we have

$$A^{\mu\nu\sigma}_{\lambda} = A^{\nu\mu\sigma}_{\lambda}. \qquad \ldots(2)$$

According to tensor transformation law

$$\bar{A}^{\mu\nu\sigma}_{\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A^{\alpha\beta\gamma}_{\delta}$$

$$= \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A^{\beta\alpha\gamma}_{\delta} \qquad \text{[using (2)]}$$

Now interchanging dummy indices α and β, we get

$$\bar{A}^{\mu\nu\sigma}_{\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A^{\alpha\beta\gamma}_{\delta}$$

$$= \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A_{\delta}^{\alpha\beta\gamma} = \bar{B}_{\lambda}^{\nu\mu\sigma}$$

i.e. given tensor is again symmetric with respect to first two indices in new co-ordinate system. Thus result can similarly be proved for two covariant indices. Thus the symmetry property of a tensor is independent of co-ordinate system. We cannot usually define symmetry property with respect to two indices of which one is contravariant and the other covariant, because these symmetry properties cannot be preserved after a co-ordinate transformation. It can be confirmed as follows :

Let $A_{\lambda}^{\mu\nu\sigma}$ be symmetric with respect to two indices, one contravariant μ and the other covariant λ, then we have

$$A_{\lambda}^{\mu\nu\sigma} = \bar{A}_{\mu}^{\lambda\nu\sigma}. \qquad \text{...(3)}$$

By tensor transformation law

$$\bar{A}_{\lambda}^{\mu\nu\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A_{\delta}^{\alpha\beta\gamma}$$

$$= \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A_{\alpha}^{\delta\beta\gamma} \qquad \text{[using (3)]}$$

Now interchanging dummy indices α and δ, we get

$$\bar{A}_{\lambda}^{\mu\nu\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\delta}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}} A_{\delta}^{\alpha\beta\gamma} \qquad \text{...(4)}$$

According to tensor transformation law

$$\bar{A}_{\mu}^{\lambda\nu\sigma} = \frac{\partial \bar{x}^{\lambda}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\mu}} A_{\delta}^{\alpha\beta\gamma} \qquad \text{...(5)}$$

Comparing equations (4) and (5), we note that

$$A_{\lambda}^{\mu\nu\sigma} \neq A_{\mu}^{\lambda\nu\sigma}$$

i.e. symmetry property is not preserved after a change of coordinate system if we consider the symmetry with respect to one contravariant and one covariant indices. But kronecker delta which is a mixed tensor is symmetric with respect to its indices.

When all the indices of either a contravariant or a covariant tensor can be interchanged without altering the tensor, then the tensor is said to be symmetric. A symmetric tensor of rank in n-dimensional space has at most $\frac{n(n+1)}{2}$ independent components. This may be proved as follows :

The total number of components in the array will be n^2, out of which all the n-diagonal terms will be different and the rest $(n^2 - n)$ will be equal in pairs. The number of pairs will be $\frac{n^2 - n}{2}$.

Therefore total number of independent components

$$= n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}.$$

(b) Anti-symmetric tensor. *A tensor, whose each component alters in sign but not in magnitude, when two contravariant or two covariant indices are interchanged is said to be skew-symmetric or antisymmetric with respect to these two indices.*

For example, if

$$\left.\begin{aligned} A^{\mu\nu} &= -A^{\nu\mu} \\ A_{\mu\nu} &= -A_{\nu\mu} \end{aligned}\right\} \qquad ...(6)$$

the contravariant tensor $A^{\mu\nu}$ or the covariant tensor $A_{\mu\nu}$ of second rank is anti-symmetric.

For a tensor of higher rank $A_\lambda^{\mu\nu\sigma}$, if

$$A_\lambda^{\mu\nu\sigma} = -A_\lambda^{\mu\sigma\nu},$$

then tensor $A_\lambda^{\mu\nu\sigma}$ is anti-symmetric with respect to indices ν and σ. The anti-symmety property of a tensor is also independent of the choice of coordinate system. So *if a tensor is anti-symmetric with respect to two indices in any co-ordinate system, it remains anti-symmetric with respect to these two indices in any other co-ordinate system.* This can be seen as follows :

If a tensor $A_\lambda^{\mu\nu\sigma}$ is anti-symmetric with respect to first two indices μ and ν, then we have

$$A_\lambda^{\mu\nu\sigma} = -A_\lambda^{\nu\mu\sigma}, \qquad ...(7)$$

Using tensor transformaion law, we get

$$\begin{aligned} \overline{A}_\lambda^{\mu\nu\sigma} &= \frac{\partial\overline{x}^\mu}{\partial x^\alpha}\frac{\partial\overline{x}^\nu}{\partial x^\beta}\frac{\partial\overline{x}^\sigma}{\partial x^\gamma}\frac{\partial x^\delta}{\partial\overline{x}^\lambda}A_\delta^{\alpha\beta\gamma} \\ &= \frac{\partial\overline{x}^\mu}{\partial x^\alpha}\frac{\partial\overline{x}^\nu}{\partial x^\beta}\frac{\partial\overline{x}^\sigma}{\partial x^\gamma}\frac{\partial x^\delta}{\partial\overline{x}^\lambda}A_\delta^{\beta\alpha\gamma} \qquad \text{[using (7)]} \; ...(8) \end{aligned}$$

Now interchanging dummy indices α and β, we get

$$\overline{A}_\lambda^{\mu\nu\sigma} = -\frac{\partial\overline{x}^\mu}{\partial x^\beta}\frac{\partial\overline{x}^\nu}{\partial x^\alpha}\frac{\partial\overline{x}^\sigma}{\partial x^\gamma}\frac{\partial x^\delta}{\partial\overline{x}^\lambda}A_\delta^{\alpha\beta\gamma} = -\overline{A}_\lambda^{\nu\mu\sigma}$$

i.e. given tensor is again anti-symmetric with respect to first two indices in new co-ordinate system. Thus anti-symmetry property is retained under co-ordinate transformations.

Anti-symmetry property, like symmetry property, cannot be defined with respect to two indices of which one is contravariant and the other covariant.

If all the indices of a contravariant or a covariant tensor can be interchanged so that its components change sign at each interchange of a pair of indices, then the tensor is said to be anti-symmetric *i.e.* if $A^{\mu\nu\sigma}$ is anti-symmetric tensor, then

$$A^{\mu\nu\sigma} = -A^{\nu\mu\sigma} = +A^{\nu\sigma\mu}.$$

Thus, we may say that a contravariant or a convariant tensor is anti-symmetric if its components change sign under an odd permutation of its indices and do not change sign under an even permutation of its indices. Anti-symmetric tensor of rank 2 in n-dimensional space has $\frac{n(n-1)}{2}$ independent components. This may be seen as follows :

The total number of components in array is $(n)^2$, out of which all diagonal terms of the array will be zero since all the quantities $A^{\mu\mu}$ (no summation) are zero. The rest $(n^2 - n)$ will be the equal in pairs. The number of such pairs will be $\frac{n^2-n}{2}$. Therefore the total number of independent components $= \frac{n^2-n}{2} = \frac{1}{2}n(n-1)$.

In general, an anti-symmetric tensor of rank r in n-dimensional space will have at most ${}^nC_r = \frac{n!}{r!\,(n-r)!}$ independent components. If $n = r$, then ${}^nC_r = {}^nC_n = 1$.

Thus, *any anti-symmetric tensor of rank equal to the number of dimensions of the space has only one independent component.* Antisymmetric tensors of rank higher than the number of dimensions of the space are identically zero.

Theorem. *Any tensor having either two contravariant or two covariant indices can be expressed as a sum of two tensors one symmetric and the other anti-symmetric.* Thus

$$A^{\mu\nu} = \tfrac{1}{2}(A^{\mu\nu}+A^{\nu\mu}) + \tfrac{1}{2}(A^{\mu\nu}-A^{\nu\mu}).$$

the first term on the right is symmetric part and the second is anti-symmetric part. The symmetric part is a symmetric tensor and the anti-symmetric part is an anti-symmetric tensor.

The process of writing a tensor as a sum of symmetric and anti-symmetric tensor not only holds for tensors of rank 2; but is quite general. For example, a tensor $A^{\mu\nu}_{\sigma\lambda}$ can be written as

$$A^{\mu\nu}_{\sigma\lambda} = \underset{\text{Symmetric part}}{\tfrac{1}{2}\left[A^{\mu\nu}_{\sigma\lambda}+A^{\nu\mu}_{\sigma\lambda}\right]} \quad + \quad \underset{\text{Anti-symmetric part}}{\tfrac{1}{2}\left[A^{\mu\nu}_{\sigma\lambda}-A^{\nu\mu}_{\sigma\lambda}\right]}$$

or

$$A^{\mu\nu}_{\sigma\lambda} = \underset{\text{Symmetric part}}{\tfrac{1}{2}\left[A^{\mu\nu}_{\sigma\lambda}+A^{\mu\nu}_{\lambda\sigma}\right]} \quad + \quad \underset{\text{Anti-symmetric part}}{\tfrac{1}{2}\left[A^{\mu\nu}_{\sigma\lambda}-A^{\mu\nu}_{\lambda\sigma}\right]}$$

1.17. Invariant Tensor.

The tensor which has the same components is all coordinate systems is said be invariant tensor. Kronecker delta symbol is an important example of invariant tensor.

Kronecker delta. The Kronecker delta is an defined as

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

We shall now show that kronecker delta is an invariant mixed tensor of rank two

Let A^ν by any arbitrary contravariant vector, then evidently

$$\delta^\mu_\nu A^\nu A^\mu = \text{contravariant tensor of rank one.}$$

As the inner product of δ^μ_ν with arbitrary vector (tensor of rank one) is a tensor, hence by quotient law δ^μ_ν is a tensor. Again δ^μ_ν has two distinct indices one of which is contravariant and the other is covariant. Hence δ^μ_ν a mixed tensor of rank 2.

As kronecker delta is a mixed tensor of rank 2, it must transform according to rule

$$\bar{\delta}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\delta^\alpha_\beta$$

$$= \frac{\delta \bar{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial x^\alpha}{\partial x^\beta} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \bar{x}^\nu}. \qquad \text{...(1)}$$

Since new variables $\bar{x}^\mu$ are the functions of old variables x^μ which in turn are the function of new variables $\bar{x}^\mu$, hence by chain rule

$$\frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\nu}} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\nu}} \quad ...(2)$$

This gives the change $\delta \bar{x}^{\mu}$ consequent upon a change $\delta \bar{x}^{\nu}$. Since $\bar{x}^{\mu}$ and $\bar{x}^{\nu}$ the coordinates of the same system, hence their variations are independent of each other unless $\mu = \nu$ in which case $\delta \bar{x}^{\mu} = \delta \bar{x}^{\nu}$. therefore

$$\frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\nu}} = \left\{ \begin{matrix} 1 \text{ for } \mu = \nu \\ 0 \text{ for } \mu \neq \nu \end{matrix} \right\}. = \delta^{\mu}_{\nu}.$$

Therefore from (1), (2) and (3), we have

$$\bar{\delta}^{\mu}_{\nu} = \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\nu}} = \delta^{\mu}_{\nu}$$

i.e. kronecker delta has same components in every co-ordinate system. Hence *kronecker delta is an invariant mixed tensor of rank* 2.

QUESTIONS AND PROBLEMS

1. (a) Find the value of c if the vector $\mathbf{A} = (\mathbf{i} + c\,\mathbf{j} + 5\,\mathbf{k})$ is perpendicular to vector $\mathbf{B} = (2\,\mathbf{i} + \mathbf{j} - \mathbf{k})$. [Ans. $c = 3$]

   (b) Show that the components of a vector **b** along and perpendicular to vector **a** are

   $$\left(\frac{\mathbf{a}\,.\,\mathbf{b}}{a^2}\right)\mathbf{a} \text{ and } \frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{a})}{a^2}$$

   respectively, where a is the magnitude of **a**. **(Rohilkhand 1985)**

2. Given two vectors $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. Calculate by vector methods.

   (i) the length of each (ii) **A . B**, (iii) **A × B**.

   [Ans. (i) $\sqrt{50}, \sqrt{41}$ (ii) – 25, (iii) $34\mathbf{i} - 13\mathbf{j} + 10\mathbf{k}$]

3. Show that the vectors

   $$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},\ \mathbf{B} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k},\ \mathbf{C} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$$

   form the sides of a right angle.

4. Given three vectors

   $$\mathbf{A} = 5\mathbf{i} - 7\mathbf{j} + 3\mathbf{k},\ \mathbf{B} = -4\mathbf{i} + 7\mathbf{j} - 8\mathbf{k},\ \mathbf{C} = 2\mathbf{i} - 3\mathbf{j}.$$

   Deduce the values of (i) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ (ii) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ (iii) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ **(Rohilkhand 1985)**

5. Define divergence of a vector field **F**. hence obtain div **F** in cartesian, cylindrical and spherical polar coordinates.
6. Define curl of a vector field **F**. Hence obtain curl **F** in cartesian, cylindrical and spherical polar coordinates.
7. Explain clearly the physical significance of divergence and curl and express them in spherical polar and circular cylindrical coordinates.
8. If $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, then prove that

   div $\mathbf{r} = 3$, curl $\mathbf{r} = 0$ and $(\mathbf{u} \cdot \text{grad})\,\mathbf{r} = \mathbf{u}$

   where **u** is any vector.

9. What is Laplace's equation ? Obtain Laplace's equaiton in spherical polar cordinates.
10. If r is the magnitude of position vector **r** and $f(r)$ is an arbitrary function of r, prove that

    $$\text{grad } f(r) = \frac{df}{dr}\frac{\mathbf{r}}{r}$$

    (Rohilkhand 1998, 86)

11. If a vector function **F** depends on both space coordinates (x, y, z) and time t, show that

    $$d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\,\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t}\,dt.$$

    (Rohilkhand 1983, 76)

12. (a) Given $\phi = 2x^3y^2z^4$, find $\nabla \cdot \nabla\,\phi$.

    (b) Prove that $\nabla \cdot (\phi\,\mathbf{A}) = (\nabla\,\phi) \cdot \mathbf{A} + \phi\,(\nabla \cdot \mathbf{A})$.

13. If div $\mathbf{A} = 0$, show that

    $$\text{curl curl } \mathbf{A} = -\nabla^2\,\mathbf{A}.$$

14. Define a tensor and obtain the law of transformation of a mixed tensor of given rank.

15. (a) State and prove quotient law of tensors, giving its use. **(Kanpur 1977; Agra 97; Rohilkhand 98)**
    (b) Show that the contraction of two suffixes in a mixed tensor reduces its rank by two. **(Agra 1986; Dibrugarh 88)**
16. Explain what is meant by covariant and mixed tensors. If $B_{\nu\sigma}$ is an arbitrary covariant tensor and $B_{(\mu\partial)} B_{\mu\sigma} = C_{\mu\sigma}$ where $C_{\mu\sigma}$ is a tensor, prove that $B_{(\mu\partial)}$ is a tensor.
17. Explain what is meant by the rank of a tensor. Show that the multiplication of tensors result in addition of their ranks and the contraction reduces the ranks by 2.

$$\text{Given } T = \frac{1}{2} M g_{mn} \bar{x}^m \bar{x}^n$$

$$\text{Prove that } x^j = \frac{1}{M} g^{jn} \left(\frac{\partial T}{\partial x^n} \right)$$

g_{mn} being the metric tensor.

18. (a) What is a tensor ? Distinguish between a symmetric and antisymmetric tensor. **(Agra 1972)**
    (b) If a contravariant tensor is symmetric in one coordinate system, show that it is symmetric in every other coordinate system. **(Rohilkhand 1991; Meerut 1999)**
19. (a) Define contravariant and covariant tensors and show that any inner product of two tensors A_p^r and B_t^{sq} is a tensor of rank three. **(Meerut 1992)**
    (b) Show that every tensor can be expressed as the sum of tensors. one of which s symmetric and the other skew symmetric in a pair of contravariant or covariant indices. **(Rohilkhand 1999; Meerut 1982, 81)**
20. Show that there is no distinction between contravariant and covariant vectors (or tensors) when we restrict ourselves to rectangular cartesian transformation of coordinates. **(Rohilkhand 1991; Banaras 93)**
21. (a) Show that Kronecker delta is an invariant mixed tensor of rank 2.
    (b) Show that a tensor of rank 2 can be expressed as a sum of a symmetric and an antisymmetric tensor both of rank 2. **(Agra 1995; Meerut 89)**

2

Electrostatic Fields

2·1. Coulomb's law

It was found experimentally that their is a force of attraction between two oppositely charged bodies and the force of repulsion between two similarly charged bodies. Coulomb in 1887, gave a law for the force of attraction or repulsion between two electrically charged bodies separated from each other by a definitie distance which is known as Coulomb's law after his name and is given below :

The force of attraction of repulsion between two electric point charges at a fixed distance apart is directly proportional to the product of two point charges and inversely proportional to the square of the distance between them. The direction of the force is always along a straight line joining the point charges.

If q_i and q_j are the magnitudes of the point charges separated from each other by a distance r_{ij}, then the force acting on charge q_j due to charge q_i is given by

$$\mathbf{F}_{ij} = \lambda \frac{q_i q_j}{r_{ij}^2} \hat{r}_{ij} \qquad \text{...(1)}$$

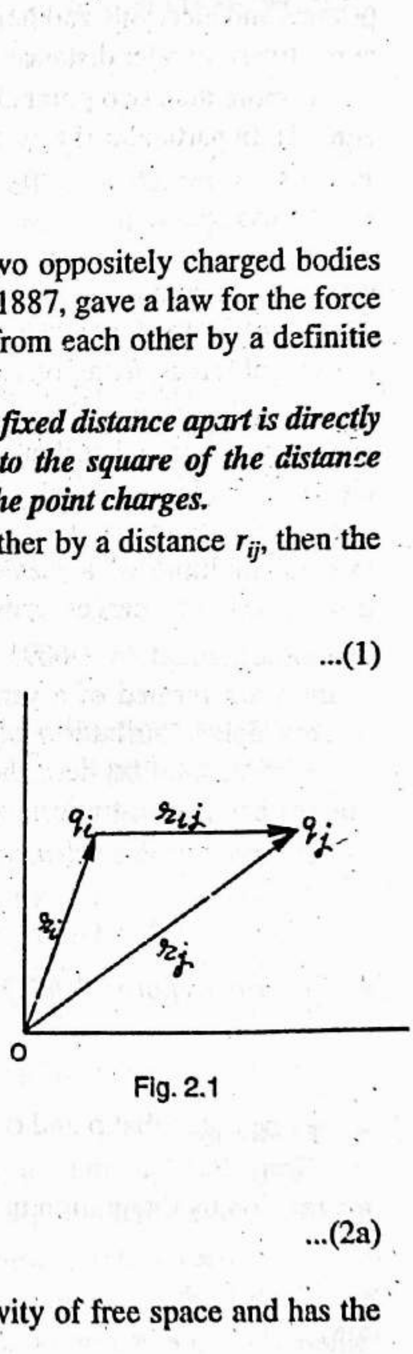


Fig. 2.1

where $\hat{r}_{ij}$ is the unit vector directed from q_i to q_j and λ is a constant of proportionality whose numerical value depends upon the medium of separation between the charges and the units in which various physical quantities are expressed. In M.K.S. system force F_{ij} is in Newton, charge q is in Coulomb and constant *lambda* is written as

$$\lambda = \frac{1}{4\pi\varepsilon}$$

where ε is known as the (absolute) *permittivity* of the medium between the charges. Thus in M.K.S. system Coulomb's law may be expressed as

$$\mathbf{F}_{ij} = \frac{1}{4\pi\varepsilon} \frac{q_i q_j}{r_{ij}^2} \hat{r}_{ij}$$

$$= \frac{1}{4\pi\varepsilon} \frac{q_1 q_j}{r_{ij}^3} \mathbf{r}_{ij} \qquad \text{...(2a)}$$

If the medium btween the charges is air or vacuum, then $\varepsilon = \varepsilon_0$, the permittivity of free space and has the value

$$\varepsilon_0 = 8{\cdot}854 \times 10^{-12}\ \text{Coul}^2/\text{N-m}^2.$$

So Coulomb's law for force on point charge q_j due to point charge q_i in free space or air is expressed as

$$\mathbf{F}_{ij} = \frac{1}{4\pi\varepsilon} \frac{q_i q_j}{r_{ij}^3} \mathbf{r}_{ij}. \qquad ...(2b)$$

If the force on q_i is to be found, it is only necessary to change every subscript i and j and every j to i. It is important to understand this notation, since in future work it will provide a technique for keeping track of field and source variables.

It may be noted that Coulomb's law applies to *point charges.* In the macroscopic sense a point charge is one whose spatial dimensions are very small as compared with any other length occuring in the problem under consideration. Coulomb's law is also applicable to the interaction of elementary particles such as protons and electrons and holds for electrostatic repulsion between nuclei at distances greater than 10^{-14} m; at relatively smaller distances the short-ranged powerful nuclear forces dominate the picture.

If more than two point charges are present, the mutual forces are determined by repeated application of eqn. (2). In particular if a system of n charges is considered, the force on j^{th} charge is given by

$$\mathbf{F}_j = \Sigma\, \mathbf{F}_{ij} = \frac{1}{4\pi\varepsilon_0} q_j \sum_{i \neq j}^{n} \frac{q_i \mathbf{r}_{ij}}{r_{ij}^3} \qquad ...(3)$$

where the summation is extended over all the charges except the jth one. This is just the principle of superposition for forces which simply states that the force acting on a point charge is the vector sum of the individual forces acting on it.

The idea of n interacting point charges may be extended to the interaction of a point charge with a continuous charge distribution. To understand the meaning of a continuous charge distribution we recall that electric charge is always an interaction of a point charge with a continuous charge distribution. To understand the meaning of a continuous charge distribution we recall that electric charge is always an integral multiple of a *elemeniary (basic) charge,* that of electron. For macroscopic point of view this discreteness of charges causes no difficulty simply because the electronic charge is extremely small in magnitude equal to $1{\cdot}6091 \times 10^{-19}$ coul. The smallness of elementary charge means that macroscopic charges are formed of a very large number of elementary charges; hence any small volume element in a macroscopic distribution contains a large number of elementary (electronic) charges. Then a charge distribution may be described in terms of a charge density function as the limit of the charge per unit volume becomes infinitesimal.

A *volume charge density* is defined as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} \qquad ...(4)$$

and a *surface charge density* is defined as

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} \qquad ...(5)$$

It may be noted that ρ and σ are *net* (excess) *charge densities.*

Thus force acting on charge q due to a continuous charge distribution is obtained by replacing summation by integration in (3) viz.

$$F_q = \frac{q}{4\pi\varepsilon_0} \int \frac{dq}{r^3} \mathbf{r} \qquad ...(6)$$

where dq is the charge on a small element of the charge distribution at distance $\mathbf{r}$ from the point charge q and the integration is to be performed over all volume and surface charge distributions. For example if a charge is distributed through a volume V with a density ρ and on the surface S which bounds V with a density σ, then the force exerted by the charge distribution on a point charge q located at the origin may be expressed as (raplacing dq by ρ dv or with σ da and proceeding to the limit)

$$F_q = \frac{q}{4\pi\varepsilon_0}\int_V \frac{\mathbf{r}}{r^3}\rho(r)\,dv + \frac{q}{4\pi\varepsilon_0}\int_S \frac{\mathbf{r}}{r^3}\sigma(r)\,da \qquad \text{...(7a)}$$

But if the point charge is at point **r** instead of origin, then eqn. (7a) taken the form

$$F_q = \frac{q}{4\pi\varepsilon_0}\int_V \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(r')\,dv' + \frac{q}{4\pi\,\varepsilon_0}\int_S \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(r')\,da'. \qquad \text{...(7b)}$$

Here the variable $\mathbf{r}'$ is used to locate a point within the charge distribution *i.e.* it plays the role of source point $\mathbf{r}_i$ in (3).

It is clear that the force on q given by (7) is proportional to q. This observation leads us to introduce a vector field which is independent of q namely the force per unit charge. This vector field is known as the *electric field.*

2·2. The Electric Field Strength

The region, surrounding any charge or group of charges, in which the effect of the presence of this charge is felt is known as the *electric field.* Let us suppose that an additional infinitesimal test charge q_0 is brought into the electric field and at a certain point in it, it experiences an electrostatic force **F**. The electric *field strength.* designated by **E** at the point in question is a vector defined by the equation $\mathbf{E} = \frac{\mathbf{F}}{q_0}$. ...(1)

The direction of the field strength **E** is the direction of force **F** *i.e.* it is in the direction in which a positive test charge placed at the point would tend to move.

The definition given by equation (1) assumes that the test charge is small enough to ensure that its presence in the region does not disturb the existing charge configuration and the very quantiy (electric field strength) that we are trying to measure remains unchanged. The importance of this provision may be represented by rewriting equation (1) in the form

$$\mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0} \qquad \text{...(2)}$$

Thus although the field strength has units of force per unit charge, we deliberately have not defined the electric field strength at any point in an electric field as the force experienced by unit positive charge placed at the point. The reason is that a unit charge (1 coulomb) is equivalent to $6{\cdot}25 \times 10^{18}$ elementary charges. So in many cases the presence of this much test charge (1 coulomb) would give rise to induced charge whose effects could mask completely those of the original charge configuration. In the way, a value for the field strength could be found, but it would be the value with the large test charge present and not as desired, its value prior to the importation of the test charge. For example if a positive charge is distributed on the surface of a conductor; then bringing a finite test charge q_0 into the vicinity of the conductor will cause the charge on the conductor to redistribute itself. If the electric field were calculated by the relation $\mathbf{E} = \frac{\mathbf{F}}{q_0}$ the field obtained would be that due to the redistributed charge rather than that due to the original charge distribution. In the special case where one of the charges of the charge distribution can be used as a test charge, the limiting process ($\lim q_0 \to 0$) is unnecessary. In this case the electric field at the location of the test charge will be that produced by all of the rest of the charge distribution; infact there will be no redistribution of the charge since the proper charge distribution is equivalent to that under the influence obtained by the entire charge distribution, including the charge being used as test charge. In certain other cases specially those in which the charge distribution is specified the force will be proportional to the size of the test charge. In these cases too the limit is unnecessary; howover if there exists any doubt, the limiting process becomes essential.

The force on infinitesimal test charge q_0 placed at position $\mathbf{r}$ relative to a point charge q is

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^3}\mathbf{r} \qquad \text{...(3)}$$

Therefore the electric field at point $\mathbf{r}$ due to point charge q is

$$\mathbf{E} = \lim_{q_0 \to 0}\frac{\mathbf{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r} \qquad \text{...(4)}$$

The force on a test charge q_0 located at point $\mathbf{r}$ due to n point charges $q_1, q_2, q_3, \dots q_n$ located at point $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots \mathbf{r}_n$ is given by

$$\mathbf{F} = \frac{q_0}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|(\mathbf{r}-\mathbf{r}_i)|^3} \qquad \text{...(5)}$$

therefore the *electric field* at $\mathbf{r}$ due to such a system of n point charges is

$$\mathbf{E} = \lim_{q_0 \to 0}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|(\mathbf{r}-\mathbf{r}_i)|^3} \qquad \text{...(6)}$$

Let us now consider a charge distribution with volume charge density $\rho(\mathbf{r}')$ in volume V and surface charge density $\sigma(\mathbf{r}')$ on the surface S (which bounds V). If a test charge q_0 is located at point $\mathbf{r}$, then it experiences a force given by

$$\mathbf{F} = \frac{q_0}{4\pi\varepsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dv' + \frac{q_0}{4\pi\varepsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(\mathbf{r}')\,da' \qquad \text{...(7)}$$

Hence *the electric field strength at* $\mathbf{r}$ *due to such a charge distribution is given by*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,v' + \frac{1}{4\pi\varepsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(\mathbf{r}')\,da' \qquad \text{...(8)}$$

Let us now write the most general expression for electric field strength. Let the charge distribution consists of n point charges $q_1, q_2, q_3, \dots q_n$ located at the points $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots \mathbf{r}_n$respectively and *a* volume distribution of charge specified by the charge density $\rho(\mathbf{r}')$ in the volume V and a surface distribution specified by the surface charge density $\sigma(\mathbf{r}')$ on the surface S, then keeping in mind (6) and (8), the electric field strength at point $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} + \frac{1}{4\pi\varepsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')}{|(\mathbf{r}-\mathbf{r}')|^3}\rho(\mathbf{r}')\,dv' + \frac{1}{4\pi\varepsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(\mathbf{r}')\,da' \qquad \text{...(9)}$$

In most of the cases one or more of the terms of above equation will be needed. Thus the electric field may be calculated at each point in space in the vicinity of a system of point charges or of a charge distribution. Thus $\mathbf{E} = \mathbf{E}(\mathbf{r})$ is a *vector point function* or a *vector field.*

The concept of electric field as a vector was not appreciated by Michael Faraday, who always thought in terms of *lines of force*. The lines of force still form a convenient way of visualising electric field patterns. The relationship between the (imaginary) lines of force and the electric field strength is given below :

The number of lines of force per unit cross-sectional area is proportional to the magnitude of the electric field strength **E**. The electric field strength **E** is large where the lines of force are closer and small where the lines of force are farther. The tangent to a line of force at any point gives the direction of electric field **E** at that point.

However, we shall not employ the concept of lines of force for quantitative description; but simply use it for qualitative description of electric field.

Ex. 1. *A charge* + 10^{-9} C *is located at the origin in free space and another charge Q at (2, 0, 0). If the X-component of the electric field at (3, 1,1) is zero, calculate the value of Q. Is the Y-component zero at (3, 1, 1).*

Solution. The electric field strength at a point having position vector $\vec{\mathbf{r}}$ relative to charge Q is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q\,\vec{\mathbf{r}}}{r^3}$$

The position vector of P relative to origin is

$$\vec{\mathbf{r}_1} = (3\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\;m$$

Modulus of $\vec{\mathbf{r}_1}$ is $r_1 = \sqrt{3^2+1^2+1^2} = \sqrt{11}\,m$

The position vector of P relative to point A (2, 0, 0) is given by

$$\vec{\mathbf{r}_2} = (3\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}) - 2\,\hat{\mathbf{i}} = \hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

Modulus of $\vec{\mathbf{r}_2}$, $|\vec{\mathbf{r}_2}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}\;m$

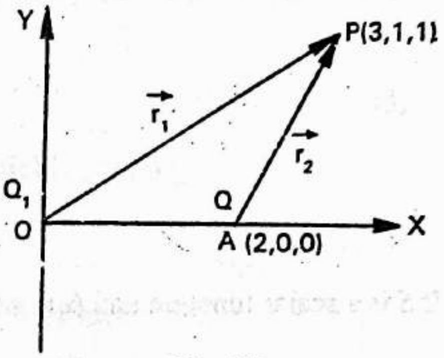


Fig. 2.2

The electric field at P due to charge Q_1 at O is

$$\vec{\mathbf{E}_1} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1\,(3\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})}{(11)^{3/2}} \qquad \text{...(1)}$$

The electric field at P due to charge at Q at A is

$$\vec{\mathbf{E}_2} = \frac{1}{4\pi\varepsilon_0}\frac{Q\,(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})}{(3)^{3/2}}$$

Net electric field at P

$$\vec{\mathbf{E}} = \vec{\mathbf{E}_1} + \vec{\mathbf{E}_2} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1\,(3\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})}{(11)^{3/2}} + \frac{Q\,(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})}{(3)^{3/2}}\right]$$

- X-component of electric field at P is

$$E_x = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1\,(3)}{(11)^{3/2}} + \frac{Q\,1}{(3)^{3/2}}\right] = 0$$

This gives

$$Q = -\left(\frac{3}{11}\right)^{3/2}(3\,Q_1)$$

$$= -\frac{3^{5/2}}{11^{3/2}} \times 10^{-9}\,\text{C}$$

Now Y-component of electric field

$$E_y = \frac{1}{4\pi\varepsilon_0}\left[\frac{10^{-9}}{(11)^{3/2}} - \frac{(3)^{5/2}}{(11)^{3/2}} \times 10^{-9}\right] \neq 0$$

Clearly Y-component of electric field at (3, 1, 1) is not equal to zero.

2·3. The Electrostatic Potential

If a vector field is irroational or the curl of the vector field vanishes, then the vector field may be expressed as the gradient of a scalar function since curl grad $S = 0$, where S is a scalar function. The electric field **E**, satisfies this criterion, hence it is defined as the negative gradient of a scalar function ϕ, *electrostatic potential, i.e.*

$$\mathbf{E} = -\operatorname{grad} \phi.$$

The electric field **E** due to a set of point charges is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \qquad \text{...(1)}$$

$$\therefore \qquad \operatorname{curl} \mathbf{E} = \operatorname{curl}\left[\frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \operatorname{curl}\left\{\frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}\right\} \qquad \text{...(2)}$$

If S is a scalar function and **A** is a vector, then we have the vector identity

$$\operatorname{curl} (S\,\mathbf{A}) = S \operatorname{curl} \mathbf{A} + (\operatorname{grad} S) \times \mathbf{A}; \qquad \text{...(3)}$$

using (3); we have

$$\operatorname{curl} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{|\mathbf{r} - \mathbf{r}_i|^3} \operatorname{curl} (\mathbf{r} - \mathbf{r}') + \left[\operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}_i|^3}\right] \times (\mathbf{r} - \mathbf{r}'). \qquad \text{...(4)}$$

A direct calculation shows that

$$\operatorname{curl} (\mathbf{r} - \mathbf{r}') = 0, \qquad \text{...(5)}$$

and

$$\operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}_i|^3} = -\frac{3 (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \qquad \text{...(6)}$$

using (5), (6) equation (4) becomes

$$\operatorname{curl} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} = 0 - \frac{3}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i) \times (\mathbf{r} - \mathbf{r}_i).$$

Since the vector product of a vector with a parallel vector is zero, therefore the second term on right hand side is also zero. Thus, we get

$$\operatorname{curl} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} = 0. \qquad \text{...(7)}$$

Using (7) equation (2), implies that

$$\operatorname{curl} \mathbf{E} = 0, \qquad \text{...(8)}$$

which means that i.e. *electric field vector* **E** *is irrotational and hence may be expressed as a gradient of a scalar function.*

Let ϕ be the function which satisfies

$$E(\mathbf{r}) = -\operatorname{grad} \phi (\mathbf{r}). \qquad \text{...(9)}$$

It still remains to find the form of the function ϕ. It may be noted that it is conventional to include the minus sign in (9) and to call ϕ the electrostatic potential.

For convenience let us find the electrostatic potential due to a point charge q_i placed at point $\mathbf{r}_i$. the electric field at point **r** is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3}. \qquad ...(10)$$

But
$$\text{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|^3}\right) = -\frac{(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} \qquad ...(11)$$

using (11), equation (10) may be written as

$$\mathbf{E} = \frac{q_i}{4\pi\varepsilon_0}\left[-\text{grad}\frac{1}{|\mathbf{r}-\mathbf{r}_i|}\right] = -\text{grad}\left(\frac{1}{4\pi\varepsilon_0}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}\right) \qquad ...(12)$$

Comparing (9) and (12); we see that the electrostatic potential at point **r** due to a point charge q_i placed at $\mathbf{r}_i$ is just

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}. \qquad ...(13)$$

With this a clue it is easy to guess that the *potential* which gives the electric field of (1) is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}. \qquad ...(14)$$

Finally, the most general electrostatic potential which gives electric field of (9) of section 2.2 is

$$\phi(\mathbf{r}) = \sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv' + \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da'. \qquad ...(15)$$

Equations (13), (14) and (15) may be easily verified by direct differentiation. The only condition that $\phi(\mathbf{r})$ represents electrostatic potential at **r** is that it must satisfy equation (9). It is immaterial how $\phi(\mathbf{r})$ is obtained. Once the existence of electrostatic potential is established, it may be obtained directly as follows :

Since $\phi(\mathbf{r})$ is known to exist, in the light of equation (9), we may write

$$\int_{ref}^{r}\mathbf{E}(\mathbf{r}')\cdot d\mathbf{r}' = -\int_{ref}^{r}\text{grad}\,\phi(\mathbf{r}')\cdot d\mathbf{r}' \qquad ...(16)$$

where *ref* stand for a reference point at which electrostatic potential ϕ is zero.

From the definition of gradient, it follows that

$$\text{grad}\,\phi(\mathbf{r}')\cdot d\mathbf{r}' = d\phi. \qquad ...(17)$$

So that equation (16) yields

$$\int_{ref}^{r}\mathbf{E}(\mathbf{r}')\cdot d\mathbf{r}' = -\int_{ref}^{r}d\phi$$

or
$$\phi(\mathbf{r}') = -\int_{ref}^{r}\mathbf{E}(\mathbf{r}')\cdot d\mathbf{r}'.$$

This equation states that *the electrostatic potential at a point may be defined as the work done by external source against the direction of the field in carrying per unit infinitesimal positive test charge from some reference point to the point under consideration.* The unit of electrostatic potential in M.K.S. system is *joule/coul* or *volt.*

If the reference point is taken at infinity where the potential is assumed to be zero and the electric field is assumed to be due to a point charge q (at origin), then equation (18) gives

$$\phi(\mathbf{r}) = -\int_{\infty}^{r}\left(\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\right)\cdot d\mathbf{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad \text{...(19)}$$

The derivation may be extended to obtain equations (14) and (15).

As curl $\mathbf{E} = 0$ everywhere therefore Stoke's theorem for electric field **E** viz,

$$\oint_S \text{curl } \mathbf{E} \cdot d\,\mathbf{a} = \oint \mathbf{E} \cdot \overrightarrow{dl}$$

implies that

$$\oint \mathbf{E} \cdot \overrightarrow{dl} = 0 \qquad \text{...(20)}$$

The shows that *the electrostatic field is a conservative field, therefore no work is done on a test charge if it is moved around a closed path in the field.*

Another interesting and useful aspect of the electrostatic potential is its relationship with the potential energy associated with the conservative electrostatic force. The potential energy associated with an arbitrary conservative force is

$$V(\mathbf{r}) = -\int_{ref}^{r} \mathbf{F}(\mathbf{r}') \cdot d\,\mathbf{r}' \qquad \text{...(21)}$$

where $V(\mathbf{r})$ is the potential energy at position **r** relative to the reference point at which potential energy is assumed to be zero. Since in electrostatic case $\mathbf{F} = q\,\mathbf{E}$, therefore

$$\mathrm{V}(\mathbf{r}) = -\int_{ref}^{r} q\,\mathbf{E}(\mathbf{r}') \cdot d\,\mathbf{r}'$$

or

$$\phi(\mathbf{r}) = \frac{V(\mathbf{r})}{q} = -\int_{ref}^{r} \mathbf{E}(\mathbf{r}') \cdot dr'.$$

Therefore if the same reference point is chosen for the electrostatic potential and potential energy, then the *electrostatic potential is just the potential energy per unit charge.* This idea may be used in several cases to introduce the electrostatic potential which has its importance in determining the electrostatic field [refer to equation (9)].

Thus we have two definitions of electric potential

1. The electrostatic potential at any point is defined as the work done by some external agency against the direction of field to bring per unit infinitesimal positive test charge from infinity to given point under consideration *i.e.*

Electrostatic potential $\phi\,(r) = \lim_{q_0 \to 0} \frac{W}{q_0}$ where W is the work done in bringing positive test charge q_0 from infinity to present position $(\overrightarrow{\mathbf{r}})$.

2. The electrosatic potential at any point is defined as the negative of the line integral of electric field from infinity to present position $(\overrightarrow{\mathbf{r}})$ *i.e.*

$$\phi\,(\overrightarrow{\mathbf{r}}) = -\int_{\infty}^{r} \overrightarrow{\mathbf{E}} \cdot d\,\overrightarrow{\mathbf{r}}$$

Ex. 2. *A non-conducting ring of radius* 0·5 *m carries a total charge of* $1{\cdot}11 \times 10^{-10}$ C *distributed non-uniformly as its circumference producing an electric field* $\overrightarrow{\mathrm{E}}$ *everywhere in space. Calculate the value of the line integral*

$$\int_{l=\infty}^{l=0} -\overrightarrow{\mathbf{E}} \cdot d\,\overrightarrow{l} \quad (l = 0 \text{ being centre of the ring})$$

Solution. Potential difference is equal to the negative of line integral

$$\int_{l=0}^{l=\infty} -\vec{\mathbf{E}}\cdot d\vec{l} = (\phi)_{l=0} - (\phi)_{l=\infty}$$

But potential at ∞ is zero, $V_{l=\infty} = 0$

$$\therefore \quad \int_{l=0}^{l=\infty} -\vec{\mathbf{E}}\cdot d\vec{l} = (\phi)_{l=0}$$

Potential due to ring at centre *i.e.* $(\phi)_{l=0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

$$\therefore \quad \int_{l=0}^{l=\infty} -\vec{\mathbf{E}}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$= 9\times10^9 \times \frac{1{\cdot}11\times10^{-10}}{0{\cdot}5} = +2\ \mathbf{V}.$$

2·4. Gauss's Theorem

Karl Friedrich Gauss (1777–1855) gave a theorem, known after his name *Gauss's theorem,* that relates the integral of the normal component of the electric field (*i.e.* net outward electric) flux over any hypothetical closed surface (called a Gaussian surface) to the net charge enclosed by the surface. The theorem states that *the normal component of the electric field (i.e. the net outward electric flux) over any closed surface of any shape drawn in an electric field is equal to* $\frac{1}{\varepsilon_0}$ *times the net charge enclosed by the surface, i.e.*

$$\oint_S \mathbf{E}\bullet\mathbf{n}\,da = \frac{1}{\varepsilon_0}(q)$$

where $\oint_S$ indicates the surface integral over whole of the closed surface and q is net charge enclosed by the surface.

Proof : Case (i) For an internal point. Let a point charge q coulomb be placed at origin O within the closed surface. Let $\mathbf{E}$ be the electric field strength at the point P on the surface due to charge q. Let $\overrightarrow{\mathrm{OP}} = \mathbf{r}$ and electric field strength vector E make an angle θ with the unit vector $\mathbf{n}$ drawn normal to surface element da surrounding point P.

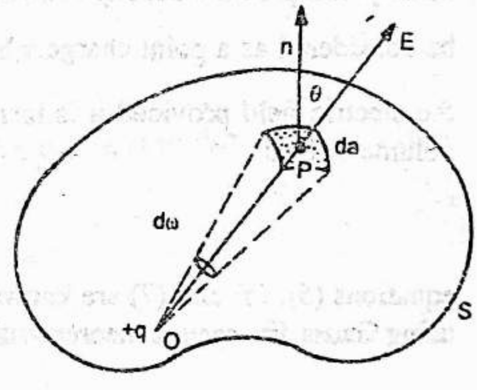


Fig. 2.3

The surface integral of the normal component of electric field E over the closed surface S is given by

$$\oint_S \mathbf{E}\bullet\mathbf{n}\,da. \qquad \text{...(1)}$$

But electric field strength at P,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r} \qquad \text{...(2)}$$

$$\therefore \quad \oint_S \mathbf{E}\bullet\mathbf{n}\,da = \frac{1}{4\pi\varepsilon_0}\,q\oint_S \frac{\mathbf{r}\bullet\mathbf{n}}{r^3}\,da \qquad \text{...(3)}$$

Now it can be seen that the quantity $\left(\dfrac{\mathbf{r}}{r}\right)\cdot\mathbf{n}\,da$ gives the projection of area da on a plane perpendicular to **r.** This projected area divided by r^2 $\left(\text{i.e. } \dfrac{\mathbf{r}\cdot\mathbf{n}\,da}{r^3}\right)$ is the solid angle subtended by da at q, which is written as $d\omega$. Thus

$$\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,da = d\omega$$

Hence equation (3) may be written as

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0}\oint_S d\omega \qquad \text{...(4)}$$

But $\int d\omega$ = soild angle subtended by entire closed surface at an internal point = 4π

Hence equation (4) gives

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0} \qquad \text{...(5)}$$

This result is known as Gauss law for a single point charge enclosed by the surface.

If several point charge q_1, q_2, q_n are enclosed by the surface S, then total electric field is given by first term of equation (9) of section 2.2. Each charge subtends a full solid angle 4π, hence equation (5) becomes

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i \qquad \text{...(6)}$$

This result can be generalised in the case of a continuous distribution of charge characterised by a charge density. Let ρ be the density at a point within an infinitesimal volume element dV; then the charge ρdV may be considered as a point charge which contributes $\dfrac{\rho dV}{\varepsilon_0}$ to the surface integral fo the normal component of the electric field provided it is inside the surface over which we integrate. Thus if surface S encloses the volume V, then

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho\,dV \qquad \text{...(7)}$$

equations (5), (6) and (7) are known as Gauss's law, Gauss's law may be expressed in yet another form by using Gauss divergence theorem viz.

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \int_V \operatorname{div}\mathbf{E}\,dV \qquad \text{...(8)}$$

using (8), equation (7) becomes

$$\int_V \operatorname{div}\mathbf{E}\,dV = \frac{1}{\varepsilon_0}\int_V \rho\,dV \qquad \text{or} \qquad \int_V\left(\operatorname{div}\mathbf{E} - \frac{\rho}{\varepsilon_0}\right)dV = 0$$

As volume is arbitrary therefore the integrand must be equal to zero *i.e.*

$$\operatorname{div}\mathbf{E} - \frac{\rho}{\varepsilon_0} = 0 \qquad \text{or} \qquad \operatorname{div}\mathbf{E} = \frac{\rho}{\varepsilon_0}. \qquad \text{...(9)}$$

This result is known as the *differential form of Gauss's law.*

Case (ii) For an external point. If the charge q is outside the surface, then it is clear from fig. 2·4, that the surface S can be divided into areas S_1, S_2, S_3, S_4 each of which subtends the same solid angle at the charge q. But at S_1 and S_2 the directions of the outward drawn normal are away from q while at S_2 and S_4

they are towards q. Therfore the contributions of two pairs (S_1, S_3) and (S_2, S_4) to the surface integral are equal and opposite. Consequently the net surface integral of the normal component of the electric field **E** vanishes *i.e.*

$$\oint \mathbf{E} \cdot \mathbf{n}\, da = 0$$

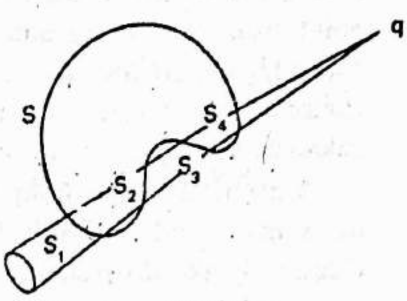


Fig. 2.4

It is interesting to note that Gauss's law remains valid as such even for charges in motion.

Ex. 3. *Five thousand electric lines of force enter a given volume and three thousand leave it. Find the total charge contained in it.*

(Kanpur 1997)

Solution. According to Gauss' theorem

Net electric flux $(\phi) = \frac{1}{\varepsilon_0} \times$ charge enclosed (Q)

Net flux, $\phi =$ net number of electric lines of force diverging from given surface

$$= -5000 + 3000 = -2000$$

Charge enclosed $\quad Q = \varepsilon_0 \phi = 8{\cdot}86 \times 10^{-12} \times (-2000)$

$$= -17{\cdot}72 \times 10^{-9} \text{ coulomb} = \mathbf{-1{\cdot}77 \times 10^{-8} \text{ coulomb}}$$

Ex. 4. *A charge 1 μC is placed at the centre of a hollow cube. Calculate the electric flux diverging (i) through the centre (ii) through each face.*

Solution. Given $\quad Q = 1\,\mu\text{C} = 10^{-6}\,\text{C}$

(i) Net electric flux diverging through the cube

$$\phi = \frac{1}{\varepsilon_0} Q = \frac{1}{8{\cdot}86 \times 10^{-12}} \times 10^{-6} \text{ volt–m}$$

$$= \frac{1}{8{\cdot}86} \times 10^{6} = 1{\cdot}12 \times 10^{5} \text{ volt–m}$$

(ii) As charge is placed symmetrically to all six faces of cube; hence the electric flux through each of six faces is divided equally.

$\therefore$ Electric flux through each face, $\phi_1 = \frac{1}{6}\left(\frac{Q}{\varepsilon_0}\right)$

$$= \frac{1}{6} \times 1{\cdot}12 \times 10^{5} = 5{\cdot}6 \times 10^{4} \text{ volt-m}$$

Ex. 5. *An electric field in a region is given by* $\vec{E} = 3\hat{i} + 4\hat{j} - 5\hat{k}$.

Calculate the electric flux through the surface $\vec{S} = 2{\cdot}0 \times 10^{-5}\,\hat{k}\,\text{m}^2$

Solution. $\quad \phi = \vec{E} \cdot \vec{S} = (3\hat{i} + 4\hat{j} - 5\hat{k}) \cdot (2{\cdot}0 \times 10^{-5}\,\hat{k})$

$$= 0 + 0 - 10 \times 10^{-5} = \mathbf{1{\cdot}0 \times 10^{-4}\ V\text{-}m}$$

2.5. Applications of Gauss's Law

Gauss's law is useful in calculating the electric field in the problems in which it is possible to choose a closed surface such that the electric field has a normal component which is either zero or a single fixed value at every point on the surface. The use of Gauss's law makes the calculations easier as compared to the use of Coulomb's law. Here we shall consider some of the important electrostatic problems where Gauss's law is applicable.

(a) Field due to spherically symmetric charge distribution. A spherically symmetric distribution of charge means the distribution of charge where the charge density ρ depends only on the distance of the point from the centre and not on the direction. Let the spherically symmetric charge distribution be characterised by the charge density function ρ (**r**) which varies in certain manner with distance from the centre of the spherical surface. Let the charge q be uniformly distributed over a sphere of radius R. Let us calculate the electric field strength at any point distant r from the centre.

Case (i) Electric field strength at an external point. Let P be a point outside the charge distribution of radius R and centre O. Let $OP = r$. Consider a sphere of radius $OP = r$ concentric with the spherical charge. As the distribution of the charge is uniform, by symmetry, the electric field strength $\mathbf{E}_0$ at every point of this spherical charge has the same magnitude and is directed along the outward normal to the surface.

The net surface integral of the normal component of the electric field *i.e.* net electric flux through the whole surface.

$$\int_S \mathbf{E}_0 \cdot \mathbf{n}\, da = \int_S \mathrm{E}_0\, da = E_0 \cdot 4\pi r^2$$

But according to Gauss's law total electric flux must be equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface *i.e.*

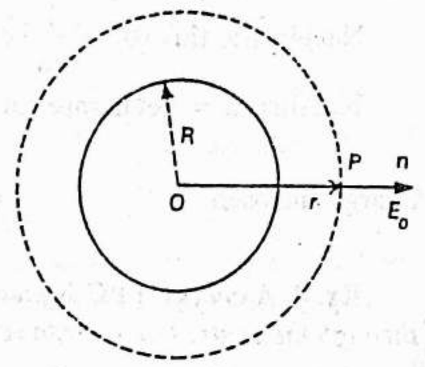


Fig. 2.5

$$E_0 \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$

$$= \frac{1}{\varepsilon_0}\int_0^R \rho\, 4\pi x^2\, dx$$

$$\therefore \qquad E_0 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^R 4\pi x^2 \rho(x)\, dx$$

Here $4\pi x^2 \rho(x)\, dx$ represents the charge on the shell of radius x and thickness dx, so that

$\int_0^R 4\pi x^2 \rho(x)\, dx$ is the total charge enclosed by the surface $= q$.

$$\therefore \qquad E_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{...(1)}$$

which is the same if the charge q were placed at the centre O.

Hence *the electric field strength at any point outside a spherical charge distribution is the same as through the whole charge were concentrated at the centre.*

Case. (ii) Electric field strength at the surface of spherical charge distribution. In this case point P lies on the surface of the spherical charge distribution *i.e.* $r = R$: therefore the electric field strength on the surface of the spherical charge distribution is

$$E_S = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \qquad \text{...(2)}$$

Case (iii) Electric field strength at an internal point. Let P be an internal point at a distance r from the centre of the charge distribution. Consider a sphere of radius $(OP = r)$ concentric with the spherical charge. Let ρ be the volume charge density (*i.e.* charge per unit volume) *i.e.*

$$\rho = \frac{\text{charge}}{\text{volume}} = \frac{q}{\frac{4}{3}\pi R^3}.$$

Let the whole surface be divided into thin spherical shells. The electric field strength E_i at P is the combined effect of shells outside spherical surface of radius r as well as those inside it.

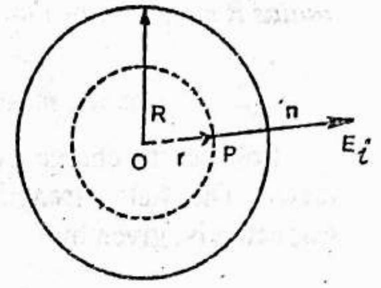


Fig. 2.6

But the electric field strength due to outer spherical shells is zero which may be seen as follows :

Consider any point P inside a charged thin shell of radius x. We have to find the electric field strength at P due to this shell. Consider a spherical surface of radius $OP = r$ concentric with spherical shell. By symmetry the electric field strength E at every point of this spherical surface has the same magnitude and directed along the outward drawn normal to this surface.

The electric flux through the whole surface $= \int_S \mathbf{E} \cdot \mathbf{n}\, da$

$$= \int_S E\, da = E \,.\, 4\pi r^2.$$

According to Gauss' theorem

$$E \,.\, 4\pi r^2 = \frac{1}{\varepsilon_0} \times \text{charge enclosed by the surface.}$$

$$= 0 \text{ since charge enclosed by the surface is zero.}$$

$$\therefore \qquad E = 0$$

Thus *electric field strength due to a charged spherical shell at an internal point is zero.*

So the electric field strength E_i at P is due to the inner shells only which may be found as follows :

By symmetry the electric field strength $\mathbf{E}_i$ at every point of the spherical surface of radius r has the same magnitude and directed along the outward drawn normal to the surface.

The total electric flux through the whole surface $= \int_S E_i\, da = E_i \,.\, 4\pi r^2.$

According to Gauss's law

$$E_i \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \times \text{charge enclosed by the Gaussian surface}$$

$$= \frac{1}{\varepsilon_0} \int_0^r 4\pi x^2\, dx\, \rho$$

$$= \frac{1}{\varepsilon_0} \rho \cdot \frac{4}{3} \pi r^3$$

$$E_i = \frac{1}{4\pi\varepsilon_0} \left(\frac{4}{3} \pi r \rho \right)$$

or

$$= \frac{1}{4\pi\varepsilon_0} \frac{4}{3} \pi r \cdot \frac{q}{\frac{4}{3}\pi R^3} \left(\text{since } \rho = \frac{q}{\frac{4}{3}\pi R^2} \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3} \qquad \text{...(3)}$$

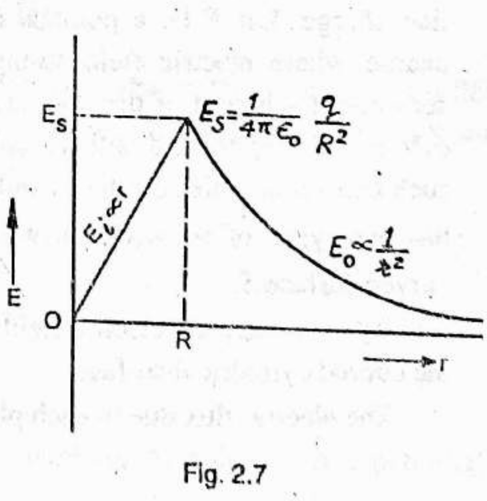


Fig. 2.7

Thus the electric field strength at point P inside a spherically symmetric charge distribution is directly proportional to the distance of the point P from the centre of the spherical charge.

The variation of magnitude of electric field strength with distance from the centre of spherically symmetric charge distribution is represented by the curve in fig. 2.7.

Ex. 6. *A charge q coulomb is distributed uniformiy throughout a non-conducting spherical volume of radius R metre. Show that the potential at a distance r from the centre $r \le R$, is given by*

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{q\,(3\,R^2 - r^2)}{2R^3}$$

(Rohil. 1997; Meerut 81; Agra 93,87)

Sol. Let the charge q coulomb be distributed uniformly throughout a non-conducting sphere of radius R meter. The field strength at points distant $r > R$ and $r < R$ from the centre of spherical charge are, respectively, given by

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{q\,\mathbf{r}}{r^3} \text{ and } \mathbf{E}_i = \frac{1}{4\pi\varepsilon_0}\frac{q\,\mathbf{r}}{R^3}$$

The potential at distance $r < R$ from the centre is given by

$$\phi = -\int_\infty^r \mathbf{E}\cdot dr = -\left[\int_\infty^R \mathbf{E}_0\cdot d\,\mathbf{r} + \int_R^r \mathbf{E}_i\cdot d\,\mathbf{r}\right]$$

$$= -\left[\int_\infty^R \frac{1}{4\pi\varepsilon_0}\frac{q\,\mathbf{r}}{r^3}\cdot d\,\mathbf{r} + \int_R^r \frac{1}{4\pi\varepsilon_0}\frac{q\,\mathbf{r}}{R^3}\cdot dr\right]$$

$$= -\left[\frac{1}{4\pi\varepsilon_0}\,q\int_\infty^R \mathbf{r}^{-2}\,dr + \frac{1}{4\pi\varepsilon_0}\int_r^r \frac{q\,r}{R^3}\cdot d\,r\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\,q\left[\left\{-\frac{1}{r}\right\}_\infty^R + \left\{\frac{r^2}{2R^3}\right\}_R^r\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\,q\left[-\frac{1}{R} + \frac{r^2}{2R^3} - \frac{1}{2R}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\,q\left[\frac{r^2}{2R^3} - \frac{3}{2R}\right] = \frac{1}{4\pi\varepsilon_0}\,q\left[\frac{3R^2 - r^2}{2R^3}\right]$$

(b) Electric field strength due to an infinite line charge. Let λ be the linear charge density (*i.e.* charge per unit length) uniformly distributed along an infinite straight line. Fig. 2.8 shows that section of the infinite long line charge. Let P be a point at a distance r from the line charge where electric field strength $\mathbf{E}$ is required. Let us surround a segment of the line charge with a closed circular cylinder of length l and radius r co-axial with the line charge, such that point P lies on the cylindrical surface. The cylinder has two types of surfaces (i) two plane faces S_1 and S_2 (ii) curved surface S.

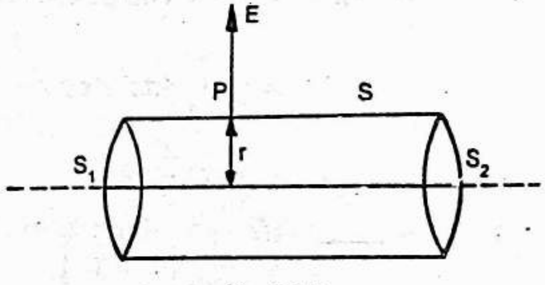


Fig. 22.8

By symmetry the electric field strength $\mathbf{E}$ is equal in magnitude and directed normally at every point of the curved cylindrical surface.

The electric flux due to each plane face

$$\int_{S_1} \mathbf{E}\cdot\mathbf{n}\,da = \int_{S_2} \mathbf{E}\cdot\mathbf{n}\,da = 0$$

since angle between normal to the plane face and the direction of electric field is 90°.

The electric flux due to curved surface

$$= \int_S \mathbf{E} \cdot \mathbf{n}\, da = \int_S E\, da = E\, 2\pi r l$$

Thus the *total electrtc flux is due to curved surface alone.* According to Gauss's law

$$E.\ 2\pi r l = \frac{1}{\varepsilon_0} \times \text{ (charge enclosed by the cylinder)}$$

$$= \frac{1}{\varepsilon_0} (\lambda l)$$

$$\therefore \quad E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{r} \qquad \text{...(1)}$$

Thus the electric field strength at a distance r from an infinite linear charge, with linear charge density λ, is directed radially away from the line charge and has magnitude $\left(\frac{\lambda}{2\pi\varepsilon_0\, r}\right)$ which is inversely proportional to r.

Ex.7. *Show that the potential difference between two points dtstant r_1 and r_2 from an infinitely long line charge of linear charge density (i.e. charge per unit length) λ is given by*

$$\nabla\phi = \frac{1}{4\pi\varepsilon_0} 2\lambda \log_e \frac{r_2}{r_1}.$$

(Rohilkhand 1995; Agra 1981)

Sol. The field strength at a distance r from an infinite linear charge of density λ is given by

$$E = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{r}$$

By definition electric potential at a distance r from the axis is

$$\phi_r = -\int_{r_{ref}}^{r} \mathbf{E} \cdot d\,\mathbf{r}$$

Hence r_{ref} denotes the reference distance for zero potential. Here reference distance can not be taken as infinity since the wire itself extends to infinity.

In this case we have to find the potential difference between two points distant r_1 and r_2 from the wire, which is given by

$$\therefore \Delta\phi = -\int_{r_2}^{r_1} \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{r} dr = \frac{1}{4\pi\varepsilon_0} \cdot 2\lambda \log_e \frac{r_2}{r_1}$$

Ex. 8. *Show that the potential at a point distance r_1 and r_2 respectively from centres of a long parallel pair of wires of negligible cross-section and having equal and opposite linear charge density λ coulomb per metre is given by*

$$\phi = \frac{\lambda}{4\pi\varepsilon_0} \log_e \frac{r_2}{r_1}.$$

(Meerut 1980)

Sol. The field strength at distance r from the line charge is

$$E = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{r}$$

By definition, electric potential, $\phi_r = -\int_{r_{ref}}^{r} \mathbf{E} \cdot d\mathbf{r}$ where r_{ref} denotes the reference point for zero potential. Here reference point r_{ref} cannot be taken to be infinity as charge itself extends to infinity. Therefore choosing reference point at a distance r_0 where potential is ϕ_0; we must have

Potential difference between $\mathbf{r}_0$ and $\mathbf{r}$

$$\phi - \phi_0 = -\int_{r_0}^{r} \mathbf{E}\cdot d\mathbf{r} = -\int_{r_0}^{r} \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}\,dr$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot 2\lambda \log_e \frac{r_0}{r} = \frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{r_0}{r}$$

i.e. **potential at distance *r* is**

$$\phi = \frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{r_0}{r} + \phi_0 \qquad ...(1)$$

In case of two lines of charge of linear charge densities λ and $(-\lambda)$ if the point under consideration (say P) lies at distances r_1 and r_2 from wires then potential (ϕ_1) at P due to one line charge of charge density $(+\lambda)$ will be

$$\phi_1 = \frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{r_0}{r_1} + \phi_0 \qquad ...(2)$$

while that due to other charge of definity $(-\lambda)$ will be

$$\phi_2 = \frac{(-\lambda)}{2\pi\varepsilon_0}\log_e\frac{r_0}{r_2} + \phi_0 \qquad ...(3)$$

Therefore the net potential due to both the wires will be

$$\phi = \phi_1 + \phi_2 = \frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{r_2}{r_1} + 2\phi_0 \qquad ...(4)$$

If the potential at reference point r_0 is zero; this is possible in this case if the reference point r_0 is at equal distance from both wires *i.e.* choosing $\phi_0 = 0$, we have

$$\phi = \frac{\lambda}{2\pi\,\varepsilon_0}\log_e\frac{r_2}{r_1} \qquad ...(5)$$

(ii) Electric field in inside and just outside charged conductor..

Let a conductor of any shape AB be placed in vacuum (permitivity ε_0.) Let a charge q be given to the conductor. Since charge is free to move in a conductor, even under influence of very small electric fields, the charge carriers (electron or ions) move until they find positions in which they experience no net force. When they come to rest, the interior of the conductor must be devoid of an electric field; this must be so because the charge carrier population in the interior is by no mean depleted and if the field existed, the carriers would continue to move. *Thus under static conditions the electric field in a conductor vanishes* *i.e.* $\mathbf{E}_{in} = 0$. As $\mathbf{E} = -\text{grad}\,\phi$, hence the electric potential is the same at all points in the conducting material. In other words, *under static conditions each conductor forms an equipotential region of space.* From $\mathbf{E} = -\text{grad}\,\phi$, it also follows that *electric field just outside a charged conductor must be normal to the surface of the conductor.*

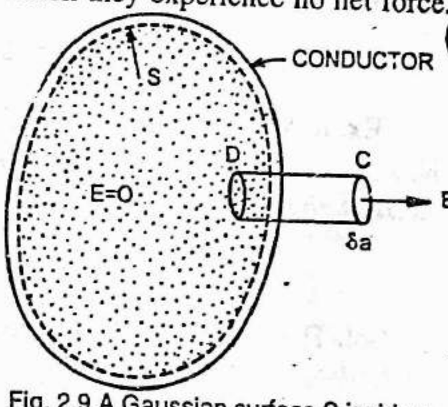


Fig. 2.9 A Gaussian surface S inside a charged conductor.

Let us consider a point P just outside the surface of the conductor. Let the surface charge density on the surface of the conductor in the neighbourhood of P be σ coul/m^2. Now consider a small cylindrical box CD having one base C at P; the other base D lying inside the conductor and the curved surface being perpendicular to the surface of the conductor.

Let the area of each case be δa. As the surface of the conductor is equipotential surface, the electric field $\mathbf{E}$ at P is perpendicular to the surface of the conductor in the neighbourhood of P.

The flux of electric field through the curved surface of the box is zero, since there is no component of electric field $\mathbf{E}$ normal to the curved surface. Also the flux of the electric field through the base D is zero as electric field $\mathbf{E}$ inside the conductor is zero. Therefore the resultant flux of electric field through the box chosen is that through the face C which is equal to

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = E\, \delta a$$

Hence by Gauss's law

$$E\,\delta a = \frac{1}{\varepsilon_0} \times \text{ Charge enclosed by the cylindrical box}$$

$$= \frac{1}{\varepsilon_0} \sigma \delta a$$

i.e.

$$E = \frac{\sigma}{\varepsilon_0} \qquad ...(5)$$

Thus the electric field strength at any point close to the surface of the charged conductor of any shape is equal to $1/\varepsilon_0$ times the surface charge density σ. This is known as 'Coulomb's law'. The electric field strength is directed normally away from the conductor if σ is positive and towards the conductor if σ is negative

2·6. Electric Field and Potential due to an Electric Dipole.

Two equal and opposite charges separated by a finite (small) distance form an electric dipole. Let charges $-q$ and $+q$ be located at positions $\mathbf{r}'$ and $\mathbf{r}' + \boldsymbol{l}$ with respect to origin O as shown in fig. 2.10.

Then the electric field at P whose position vector relative to origin O is give by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q\,(\mathbf{r} - \mathbf{r}' - \boldsymbol{l})}{|(\mathbf{r} - \mathbf{r}' - \boldsymbol{l})|^3} - \frac{1}{4\pi\varepsilon_0} \frac{q\,(\mathbf{r} - \mathbf{r}')}{|(\mathbf{r} - \mathbf{r}')|^3}$$

$$= \frac{q}{4\pi\varepsilon_0} \left\{ \frac{(\mathbf{r} - \mathbf{r}' - \boldsymbol{l})}{|(\mathbf{r} - \mathbf{r}' - \boldsymbol{l})|^3} \right\} - \left\{ \frac{(\mathbf{r} - \mathbf{r}')}{|(\mathbf{r} - \mathbf{r}')|^3} \right\}$$

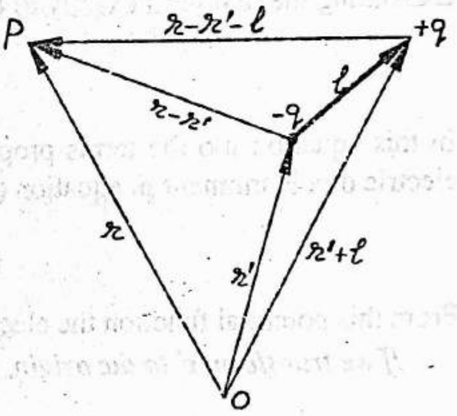


Fig. 2.10

As the dipole separation l is small compared with $(\mathbf{r} - \mathbf{r}')$; hence we may expand equation (1), keeping only the first nonvanishing term.

For detailed consideration let us first expand denominator of first term. The reciprocal of this denominator is written as

$$|\mathbf{r} - \mathbf{r}' - \boldsymbol{l}\,|^{-3} = |(\mathbf{r} - \mathbf{r}' - \boldsymbol{l})^2|^{-3/2}$$

$$= [|\mathbf{r} - \mathbf{r}'|^2 - 2\,(\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{l} + l^2]^{-3/2}$$

$$= |\mathbf{r} - \mathbf{r}'|^3 \left[1 - \frac{2\,(\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{l^2}{|\mathbf{r} - \mathbf{r}'|^2} \right]^{-3/2}$$

As $\boldsymbol{l} \ll |\mathbf{r} - \mathbf{r}'|$, we may expand above expression by binomial expansion, keeping only the terms linear in $\boldsymbol{l}$. The result is

$$|\mathbf{r} - \mathbf{r}' - \boldsymbol{l}|^{-3} = |\mathbf{r} - \mathbf{r}|^{-3} \left[1 + \frac{3\,(\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^2} + \ldots \right] \qquad ...(2)$$

Using equation (2) in (1) and again keeping only those terms which are linear in $\boldsymbol{l}$, we get

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \left[(\mathbf{r} - \mathbf{r}' - \boldsymbol{l})\, |(\mathbf{r} - \mathbf{r}')|^{-3} \left\{ 1 + \frac{3\,(\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^2} \right\} - \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[3\frac{(\mathbf{r}-\mathbf{r}')\cdot \boldsymbol{l}}{|\mathbf{r}-\mathbf{r}'|^5}(\mathbf{r}-\mathbf{r}') - \frac{\boldsymbol{l}}{|\mathbf{r}-\mathbf{r}'|^3} + \dots\right] \quad \text{...(3)}$$

This equation gives that part of the electric field, due to finite electric dipole which is proportional to the separation of the charges. However, there are other contributions proportional to the square, the cube and the higher powers of separation. If the separation is amall, the contributions of higher powers to the electric field are neglibibly small. As the separation l tends to zero, all the terms vanish unless the charge becomes infinite. If $l \to 0$ and $q \to \infty$ such that $q\,l$ remains finite, then all the terms except the term linear in l vansih. In this limit a *point dipole* is formed. A point dipole has no net charge, no extent in space and is completely characterised by its *dipole moment* which is specified by **p** and dipole moment is generally defined as $\mathbf{p} = \lim\limits_{l \to 0,\, q \to \infty} q\,\boldsymbol{l}$.

The electric dipole moment is generally defined as

$$\mathbf{p} = q\,\boldsymbol{l}. \quad \text{...(4)}$$

In terms of electric dipole moement **p**, equation (3) may be written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^5}(\mathbf{r}-\mathbf{r}') - \frac{\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^3}\right] \quad \text{...(5)}$$

Similarly, the potential due to this dipole, referring fig. 2.10 may be written as

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{r}-\mathbf{r}'-\boldsymbol{l}|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] \quad \text{...(6)}$$

Expanding the first term exactly as before and retaining only terms linear in **l**, we obtain

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{r}-\mathbf{r}')\cdot\boldsymbol{l}}{|\mathbf{r}-\mathbf{r}'|^3} \quad \text{...(7)}$$

In this equation too the terms proportional l^2, l^3 and to higher powers of l are neglected. For a dipole of electric dipole moment **p**, equation (7) is better written as

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^3} \quad \text{...(8)}$$

From this potential function the electric field **E** [equ. (3)] may be obtained using the relation $\mathbf{E} = -\operatorname{grad}\phi$.

If we transform **r′** *to the origin,* then equation (8) takes the familiar form of potential due to a dipole

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon 0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \quad \text{...(9)}$$

where θ is the angle between r and **p**.

Hence the radial and transverse components of electric field at point $P(r, \theta)$ may be written, using $\mathbf{E} = -\operatorname{grad}\phi$ as

$$E_r = -\frac{\partial\phi}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^3} \quad \text{...(10)}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3} \quad \text{...(11)}$$

Tthe magnitude of resultant field is

$$E = -\sqrt{(E_r^2 + E_\theta^2)} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{(1 + 3\cos^2\theta)}. \quad \text{...(12)}$$

If α is the angle which the resultant field makes with radius vector **r**, then

$$\tan \alpha = \frac{E_\theta}{E_r} = \frac{1}{2}\tan\theta,\ i.e.\ \alpha = \tan^{-1}\left(\tfrac{1}{2}\tan\theta\right) \qquad \text{...(13)}$$

Ex. 9. *Calculate the potential and field due to a short dipole of dipole moment* $\frac{5}{3} \times 10^{-27}$ *coul-m at a point distant* 1 *cm from it (i) on its axis (ii) on its perpendicular bisector.* **(Rohilkhand 1998, 83)**

Sol. The potential due to a short dipole at distance r is

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

The field at a distance r is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{(1+\cos^2\theta)}.$$

Here $\qquad p = \frac{5}{3} \times 10^{-27}$ coul-m

and $\qquad r = 1\text{ cm} = 10^{-2}$ m.

Case (i) On the axis of dipole $\theta = 0$.

$\therefore$ Potential
$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} = 9 \times 10^9 \times \frac{(\frac{5}{3} \times 10^{-27})}{(1 \times 10^{-2})^2}$$
$$= 1{\cdot}5 \times 10^{-13} \text{ volt.}$$

$\therefore$ Electric field
$$E = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3} = 9 \times 10^9 \times 2 \times \frac{\frac{5}{3} \times 10^{-27}}{(1 \times 10^{-2})^3}$$
$$= 3{\cdot}0 \times 10^{-11} \text{ volt/m.}$$

Case (ii) On the bisector $\theta = 90°$.

$\therefore$ Potential
$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{p\cos 90°}{r^2} = 0.$$

Electric field
$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3} = 9 \times 10^{-9} \times \frac{\frac{5}{3} \times 10^{-27}}{(1 \times 10^{-2})^3}$$
$$= 1{\cdot}5 \times 10^{-11} \text{ volt/m.}$$

2·7. Potential Energy of an Electric Dipole in an External Electric Field

Let $E_{ext}(\mathbf{r})$ be the external electric field which may be described by the potential function $\phi_{ext}(\mathbf{r})$. If $-q$ and $+q$ charges are placed at $\mathbf{r}$ and $\mathbf{r}+\boldsymbol{l}$ respectively in the external field, then potential energy U is just

$$U = -q\phi_{ext}(\mathbf{r}) + q\phi_{ext}(\mathbf{r}+\boldsymbol{l}). \qquad \text{...(1)}$$

If $\boldsymbol{l}$ is small compared with $\mathbf{r}$, then $\phi_{ext}(\mathbf{r}+\boldsymbol{l})$ may be expanded in a power series in $\boldsymbol{l}$. Rataining only the terms linear in $\boldsymbol{l}$, we obtain

$$\phi_{ext}(\mathbf{r}+\boldsymbol{l}) = \phi_{ext}(\mathbf{r}) + \boldsymbol{l} \cdot \text{grad}\,\phi_{ext}(\mathbf{r}). \qquad \text{...(2)}$$

Using (2), equation (1) gives

$$U(\mathbf{r}) = -q\phi_{ext}(\mathbf{r}) + q\{\phi_{ert}(\mathbf{r}) + \boldsymbol{l} \cdot \text{grad}\phi_{ext}(\mathbf{r})\}$$
$$= q\,\boldsymbol{l} \cdot \text{grad}\,\phi_{ext}(\mathbf{r}), \qquad \text{...[3(a)]}$$
$$= \mathbf{p} \cdot \text{grad}\,\phi_{ext}(\mathbf{r}) \qquad (\text{since } \mathbf{p} = q\,\boldsymbol{l}). \qquad \text{...[3(b)]}$$

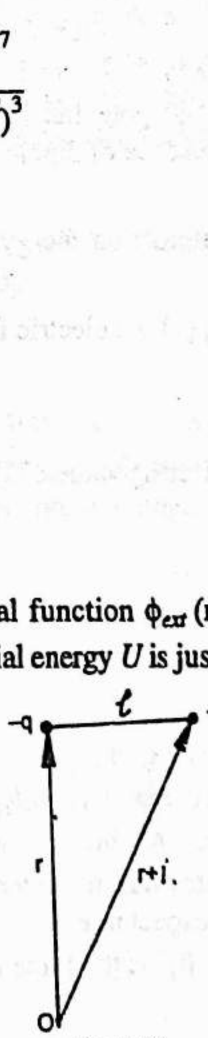


Fig. 2.11

Since $E_{ext} = -\text{grad}\ \phi_{ext}$ in equation [3(b)] Alternatively equations 3(b) may by written as

$$U(\mathbf{r}) = -\mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{r}). \qquad ...(4)$$

This equation gives the potential energy of dipole of electric dipole moment **p** in an external electric field $\mathbf{E}_{ext}$, where $\mathbf{E}_{ext}(\mathbf{r})$ is evaluated at the location of the point dipole.

2·8. Interaction of Dipoles

If a dipole of moment **p** is placed in an external electric field **E**, then potential energy of dipole is

$$U = -\mathbf{p} \cdot \mathbf{E} \text{ (refer section 2.7)} \qquad ...(1)$$

From equation (5) of section 2.6, the electric field **E** at $\mathbf{r} \gg \mathbf{r}'$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[3\frac{(\mathbf{r}\cdot\mathbf{p})}{r^5}\mathbf{r} - \frac{\mathbf{p}}{r^3}\right]. \qquad ...(2)$$

Now let us consider two electric dipoles 1 and 2 of moments $\mathbf{p}_1$, and $\mathbf{p}_2$ oriented in space at an arbitrary angle with each other (fig. 2.12). Let $\mathbf{E}_2$ and ϕ_2 be the electric field and electric potential respectively in the vicinity of dipole 2 due to presence of dipole 1, then the electric potential (interaction) energy of dipole 2 due to presence of dipole 1 is given by

$$U_{12} = \mathbf{p}_2 \cdot \mathbf{E}_2. \qquad ...(3)$$

Similarly if $\mathbf{E}_1$ and ϕ_1 are respectively the electric field and potential in the vicinity of dipole 1 due to the presence of dipole 2, then the interaction energy is

$$U_{21} = -\mathbf{p}_1 \cdot \mathbf{E}_1. \qquad ...(4)$$

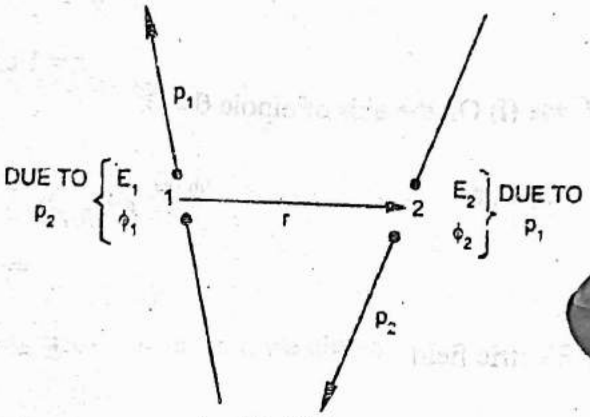


Fig. 2.12

But interaction energy of two dipoles must be same as there is no work done in the movement *i.e.*

$$U_{12} = U_{21} = U \text{ (say)}. \qquad ...(5)$$

Using (1) the electric field $\mathbf{E}_2$ is given by

$$\mathbf{E}_2(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{r}\cdot\mathbf{p}_1)}{r^6}\mathbf{r} - \frac{\mathbf{p}_1}{r^3}\right]. \qquad ...(6)$$

Substituting value of $\mathbf{E}_2$, in (3), the interaction energy of two dipoles is given by

$$U = U_{12} = -\mathbf{p}_2 \cdot \left[\frac{1}{4\pi\varepsilon_0}\left\{\frac{3(\mathbf{r}\cdot\mathbf{p}_1)}{r^5}\mathbf{r} - \frac{\mathbf{p}_1}{r^3}\right\}\right]$$

Rearranging and keeping in mind that the dot product of two vectors is commutative, we obtain

$$U = U_{12} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3}{r^5}(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})\right\} \qquad ...(7)$$

If we write down the value of $\mathbf{E}_1$ and insert in (5), we again get this expression. Thus analytically it is verified that $U_{12} = U_{21}$, equation (7) represents *the general expression for the interaction energy of two dipoles.* As interaction energy U of two dipoles contains the terms $(\mathbf{p}_1 \cdot \mathbf{p}_2)$ and $(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})$, this indicates that the interaction energy depends upon the relative orientation of dipoles and their orientation with respect to **r**.

If $\mathbf{F}_{12}$ is the force acting on dipole 2 in the field $\mathbf{E}_2$ of dipole 1, then

$$\mathbf{F}_{12} = -\,grad\, U_{12}$$

$$= -\nabla\left[\frac{1}{4\pi\varepsilon_0}\left\{\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3}{r^5}(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})\right\}\right] \qquad ...(8)$$

Similarly we may obtain force acting on dipole 1 in the field $\mathbf{E}_1$ of dipole 2 which is

$$\mathbf{F}_{21} = -\frac{1}{4\pi\varepsilon_0}\nabla\left[\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r'^3} - \frac{3}{r'^5}(\mathbf{p}_1\cdot\mathbf{r}')(\mathbf{p}_2\cdot\mathbf{r}')\right]. \qquad ...(9)$$

If the dipoles are *similar*, then $p_1 = p_2$ and $\mathbf{r} = -\mathbf{r}'$; so that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}, \qquad ...(10)$$

i.e. the interacting forces on the dipoles are equal and opposite.

The torque τ acting on the dipole of moment $\mathbf{p}$ can be obtained by differentiation of U (interaction energy) with respect to angle between $\mathbf{p}$ and $\mathbf{E}$ or by direct consideration of forces. One obtains.

$$\vec{\tau} = \frac{\partial U}{\partial \alpha} = \mathbf{p}\times\mathbf{E} \qquad ...(11)$$

where α is the angle between $\mathbf{p}$ and $\mathbf{E}$. Thus if $\vec{\tau}_{21}$ is the torque acting on dipole $\mathbf{p}_2$ due to field $\mathbf{E}_2$ of $\mathbf{p}_1$, then

$$\vec{\tau}_{21} = \mathbf{p}_2\times\mathbf{E}_2. \qquad ...(12)$$

Similarly the torque $\vec{\tau}_{12}$ acting on dipole $\mathbf{p}_1$ due to the field $\mathbf{E}_1$ of $\mathbf{p}_2$, is

$$\vec{\tau}_{12} = \mathbf{p}_1\times\mathbf{E}_1. \qquad ...(13)$$

Equations (12) and (13) indicate that tendency of the torque acting on dipoles is to align them along the line of interaction $\mathbf{r}$.

Ex. 10. *Write down the interaction potential energy of two short electric dipoles separated by a distance. If one of the dipoles in inclined at an angle θ_1 to the radius vector joining them, show that in the state of equilibrium, the other dipole would make an angle θ_2 with it given by*

$$\tan\theta_2 = -\tfrac{1}{2}\tan\theta_1.$$

(Rohilkhand 1981)

Solution. The interaction potential energy of two short electric dipoles is given by

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3}{r^5}(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})\right] \qquad ...(1)$$

If one of the dipoles of dipole moment $\mathbf{p}_1$ is inclined at an angle θ_1 to the radius vector $\mathbf{r}$ and other dipole of moment $\mathbf{p}_2$ is inclined at angle θ_2 with the same radius vector $\mathbf{r}$. then the angle between the two dipoles would be $\theta_1 - \theta_2$ as shown in fig. 2.13. So the expression (1) for interaction potential energy takes the form

$$U^* = \frac{1}{4\pi\varepsilon_0}\left[\frac{p_1 p_2\cos(\theta_1-\theta_2)}{r^3} - \frac{3}{r^5}(p_1 r\cos\theta_1)(p_2 r\cos\theta_2)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_1 p_2}{r^3}\left[\cos(\theta_1-\theta_2) - 3\cos\theta_1\cos\theta_2\right] \qquad ...(2)$$

In the state of equilibrium of dipoles the force must be zero and potential energy must be extremum. To apply the condition of extremum potential energy, let us differentiate equation (2) with respect to θ_2 and equate to zero. Then we get

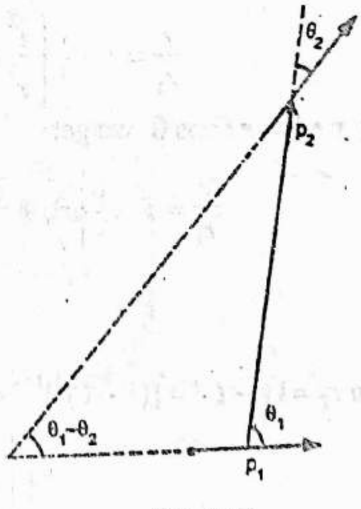


Fig. 2.13

$$\frac{\partial U}{\partial \theta_2} = \frac{1}{4\pi\varepsilon_0}\frac{p_1 p_2}{r^3}[\sin(\theta_1-\theta_2) + 3\cos\theta_1 \sin\theta_2] = 0.$$

As $1/r^3$ is not equal to zero; therefore

or $$\sin(\theta_1-\theta_2) + 3\cos\theta_1\sin\theta_2 = 0$$

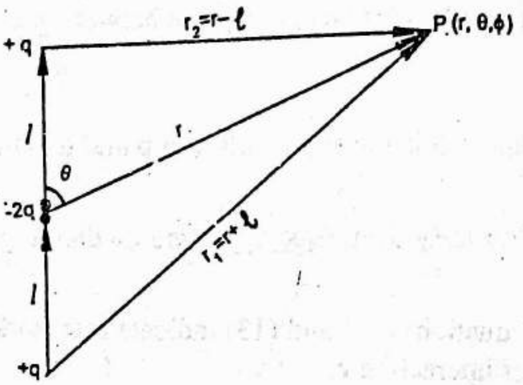
or $$\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 + 3\cos\theta_1\sin\theta_2 = 0.$$

$$\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2 = 0.$$

This gives $$\tan\theta_2 = -\tfrac{1}{2}\tan\theta_1. \qquad ...(3)$$

2.9. Electric Quadrupole and Multipoles :

An electric quadrupole is a system of four charges, each of magnitude $|q|$ and two of which bear positive sign and the other two have negative sign, arranged in the manner as shown in Fig. 2.14, the separation between charges l is assumed to be very small as compared to the distance r of a point P specified by co-ordinate (r, θ, ϕ) relative to the position of two negative charges.



(Fig. 2.14)

The electric potential at point P due to this quadrupole is given by $\phi(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r_1} + \frac{q}{r_2} - \frac{2q}{r}\right]$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}\left[\frac{r}{r_1} + \frac{r}{r_2} - 2\right]. \qquad ...(1)$$

But $$r_1 = |\,r + l\,| = [(\mathbf{r}+l)\cdot(\mathbf{r}+l)]^{1/2}$$

$$= [r^2 + l^2 + 2\mathbf{r}\cdot l\,]^{1/2} = r\left[1 + \left(\frac{l}{r}\right)^2 + \frac{2\mathbf{r}\cdot l}{r^2}\right]^{1/2}$$

i.e. $$\frac{r}{r_1} = \left[1 + \left(\frac{l}{r}\right)^2 + \frac{2\,\mathbf{r}\cdot l}{r^2}\right]^{-1/2} \qquad ...(2)$$

Assuming $r \gg l$ and expanding above expression by binomial theorem, we get

$$\frac{r}{r_1} = 1 - \frac{1}{2}\left\{\frac{l^2}{r^2} + \frac{2\mathbf{r}\cdot l}{r^2}\right\} + \frac{3}{8}\left\{\frac{l^2}{r^2} + \frac{2\,l\cdot\mathbf{r}}{r^2}\right\}^2 - \frac{15}{16}\left\{\frac{l^2}{r^2} + \frac{2\,l\cdot\mathbf{r}}{r^2}\right\}^3 + ...$$

using $\mathbf{r}\cdot l = r\,l\cos\theta$ we get

$$\frac{r}{r_1} = 1 - \frac{1}{2}\left\{\frac{l^2}{r^2} + \frac{2l\cos\theta}{r}\right\} + \frac{3}{8}\left\{\frac{l^2}{r^2} + \frac{2l\cos\theta}{r}\right\}^2 - \frac{15}{16}\left\{\frac{l^2}{r^2} + \frac{2l\cos\theta}{r}\right\}^3 - ...$$

$$= 1 - \frac{l}{r}\cos\theta + \frac{l^2}{r^2}\cdot\frac{1}{2}(3\cos^2\theta - 1). \qquad ...(3)$$

Again $r_2 = |\,\mathbf{r} - l\,| = [(\mathbf{r} - l)^2]^{1/2} = [r^2 + l^2 - 2\mathbf{r}\cdot l\,]^{1/2}$

$$= r\left\{1 + \frac{l^2}{r^2} - \frac{2\mathbf{r}\cdot l}{r^2}\right\}^{1/2} = r\left\{1 + \frac{l^2}{r^2} - \frac{2l}{r}\cos\theta\right\}^{1/2}$$

Proceeding exactly in the similar manner, we get

$$\frac{r}{r_2}=\left[1+\frac{l^2}{r^2}-\frac{2l}{r}\cos\theta\right]^{1/2}$$

$$=1-\tfrac{1}{2}\left\{\frac{l^2}{r^2}-\frac{2l}{r}\cos\theta\right\}+\frac{3}{8}\left\{\frac{l^2}{r^2}-\frac{2l}{r}\cos\theta\right\}^2-\frac{15}{16}\left\{\frac{l^2}{r^2}-\frac{2l}{r}\cos\theta\right\}^3+\ldots$$

Neglecting terms of order higher than l^2/r^2, we get

$$\frac{r}{r_2}=1-\frac{l}{r}\cos\theta+\frac{l^2}{r^2}\cdot\tfrac{1}{2}(3\cos^2\theta-1). \qquad \ldots(4)$$

Using equations (3) and (4), equation (1) gives

$$\phi(r,\theta,\phi)=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\left[\left\{1-\frac{l}{r}\cos\theta+\frac{l^2}{r^2}\cdot\tfrac{1}{2}(3\cos^2\theta-1)\right\}\right.$$
$$\left.+\left\{1+\frac{l}{r}\cos\theta+\frac{l^2}{r^2}\cdot\tfrac{1}{2}(3\cos^2\theta-1)\right\}-2\right]$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q\,l^2}{r^3}[3\cos^2\theta-1.]$$

Thus the electric potential due to a *linear quadrupole* varies inversely as the cube of distance. If the electric field strength **E** is calculated from the relation **E** = − *grad* ϕ ; then it is found that the field strength varies inversely as the fourth power of distance.

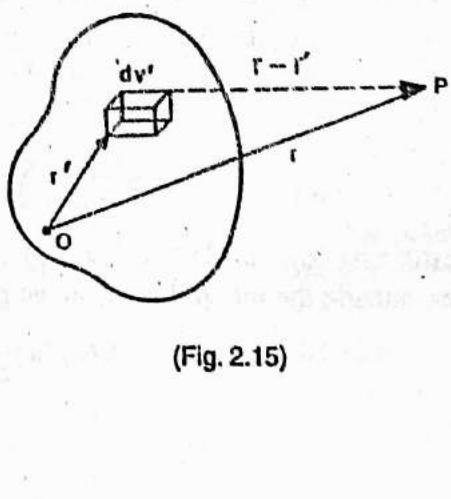
The concept of electric dipole and quadrupole can be extended to a system of larger number of positive and negative charges located at very small distances from each other. Such charge arrangements are called *multipoles.* A single point charge may be termed as a *monopole.* If a monopole is displaced through a very small distance and the original monopole is replaced by an equal and opposite charge; a *dipole* is produced. Similarly, if a dipole is displaced through a small distance and thereafter the original dipole is replaced by one of same magnitude but of opposite sign, a *quadrupole* is formed. If a quadrupole is replaced through a small distance and the original quadrupole is replaced by one of same magnitude and opposite sign ; we get what is known as an *electric octopole* and so on. This concept can be continued further to obtain multipoles of still higher order. For $(2x)^{th}$ multipole the potential at large distances from the system it is found to vary *inversely* as r^{x+1}.

2.10. Multipole Expansion of Electric Field :

In order to define the electric dipole moment of an arbitrary charge distribution, we shall consider a certain expansion of the electrostatic potential due to an arbitrary charge distribution.

Let us consider volume V' occupied by the charge distribution. Let dV' be an element of volume at position $\mathbf{r}'$ from an arbitrary point O within the charge distribution and $\rho(\mathbf{r}')$ the charge density at point $\mathbf{r}'$. Then the potential at point **r** due to the volume element dV' is

$$d\phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\frac{\rho(\mathbf{r}')\,dV'}{|\mathbf{r}-\mathbf{r}'|}.$$



(Fig. 2.15)

Hence the potential at **r** due to entire charge distribution is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV'. \qquad \text{...(1)}$$

If the point of observation P is remote from the origin, then $r' \ll r$ and we can expand $|\mathbf{r}-\mathbf{r}'|^{-1}$ in a series of ascending powers of $\mathbf{r}'/r$ *i.e.*,

$$|\mathbf{r}-\mathbf{r}'|^{-1} = |(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')|^{1/2} = (r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2)^{-1/2}$$

$$= \frac{1}{r}\left[1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right]^{-1/2} = \frac{1}{r}\left[1 + \left\{-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} - \frac{\mathbf{r}'^2}{r^2}\right\}\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left\{-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right\} + \frac{1}{2}\cdot\frac{3}{2}\left\{-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right\}^2 - \dots\right]$$

As $r' \ll r$, therefore $\left(\frac{r'}{r}\right)$ may be neglected as compared to $\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2}$; hence we may omit the terms involving the cube and higher powers of $\left(\frac{r'}{r}\right)$ so that we get

$$|\mathbf{r}-\mathbf{r}'|^{-1} = \frac{1}{r}\left[1 - \frac{1}{2}\left\{-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right\} + \frac{3}{2}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^2 + \dots\right]$$

$$= \left[\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{1}{2}\left\{\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right\} + \dots\right] \qquad \text{...(2)}$$

Substituting this in eqn. (1), we get

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\left[\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{1}{2}\left\{\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right\} + \dots\right]\rho(\mathbf{r}')\, dV' \qquad \text{...(3)}$$

The third term on the right hand side of above equation may be expressed as

$$\frac{1}{2}\left\{\frac{3(\mathbf{r}\cdot\mathbf{r}')}{r^5} - \frac{(\mathbf{r}')^2}{r^3}\right\} = \frac{1}{2r^5}[3(\mathbf{r}\cdot\mathbf{r}')^2 - \mathbf{r}'^2\cdot\mathbf{r}^2]$$

$$= \frac{1}{2r^5}\left[3\left(\sum_{i=1}^{3} x_i x_j'\right)^2 - (\mathbf{r}')^2\left(\sum_{j=1}^{3} x_j^2\right)\right]$$

$$= \frac{1}{2r^5}\left[3\left(\sum_{i=1}^{3} x_i x_i' \sum_{i=1}^{3} x_j x_j'\right) - (\mathbf{r}')^2\left(\sum_{i=1}^{3} x_j^2\right)\right]$$

$$= \frac{1}{2r^5}\left[3\left(\sum_{i=1}^{3} x_i x_i' \sum_{j=1}^{3} x_j x_j'\right) - (\mathbf{r}')^2 \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \delta_{ij}\right]$$

$$= \frac{1}{2r^5}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j\,[3x_j' x_j' - \delta_{ij}(\mathbf{r}'^2)].$$

Substituting this in (3) and keeping in mind that the variable of interaction is r' only, therefore r can be taken outside the integral sign, so we get

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\left[\frac{1}{r}\int_{V'}\rho(\mathbf{r}')\, dV' + \frac{\mathbf{r}}{r^3}\int_{V'}\mathbf{r}'\rho(\mathbf{r}')\, dV'\right.$$

$$+ \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} \frac{x_i x_j}{r^5} \int_{V'} (3x_1' x_j' - \delta_{ij} r'^2) \rho(\mathbf{r}') \, dV' + \dots \Big] \qquad \dots(4)$$

where x_i, x_j are the cartesian components of $\mathbf{r}$, x_i', x_j' are those of $\mathbf{r}'$ and δ_{ij} is Kronecker delta symbol defined as

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \qquad \dots(5)$$

It is obvious from equ. (4) that the potential due to a charge distribution may be expressed as a summation of an infinite number of terms viz.

$$\phi(\mathbf{r}) = \phi_1 + \phi_2(\mathbf{r}) + \phi_3(\mathbf{r}) + \dots \qquad \dots(6)$$

where

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r} \int_{V'} \rho(\mathbf{r}) \, dV' \qquad \dots(7)$$

$$\phi_2(\mathbf{r}) = \frac{\mathbf{r}}{4\pi\varepsilon_0 r^3} \int_{V'} \mathbf{r}' \rho(\mathbf{r}') \, dV' \qquad \dots(8)$$

$$\phi_3(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} \frac{x_i x_j}{r^5} \int_{V'} (3x_i' x_j' - \delta_{ij} r'^2) \rho(\mathbf{r}') \, dV' \qquad \dots[9(a)]$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{3} \sum_{i=1}^{3} \frac{1}{2} \frac{x_i x_j}{r^5} Q_{ij},$$

where

$$Q_{ij} = \int_{V'} (3x_i' x_j' - \delta_{ij} \bullet r_j'^2) \rho(\mathbf{r}') \, dV' \qquad \dots[9(b)]$$

and so on.

Equations (4) or (6) may be interpreted as follows :

The first term, $\phi_1(\mathbf{r})$ represents the potential potential at $\mathbf{r}$ if the total charge were concentrated at the origin and is called the potential due to *monopole moment of the charge distribution.*

The second term $\phi_2(\mathbf{r})$ represents the potential which would result if a point dipole having dipole moment equal to that of the charge distribution were located at origin of co-ordinates and is called the potential due to *dipole moment of the charge distribution.*

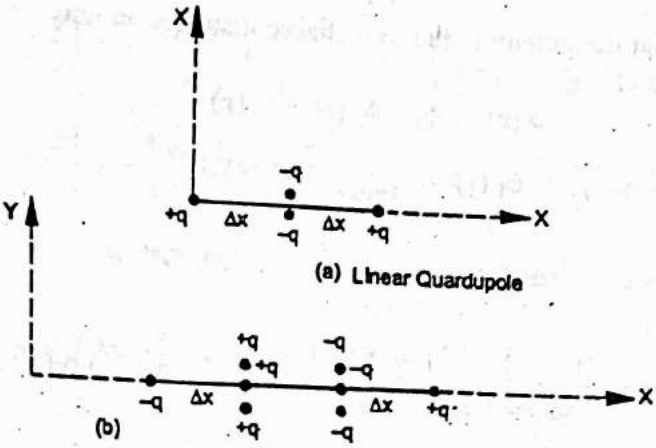
It is obvious that quantity Q_{ij} in $\phi_3(\mathbf{r})$ contains nine components corresponding to $i, j = 1, 2, 3$; out of which six are equal in pairs; thus leaving six distinct components. This set of quantities form the *quadrupole moment tensor* and represents an extension of the dipole moment concept. The term $\phi_3(\mathbf{r})$ is called the potential due to *quadrupole moment of charge distribution.* Of course there are higher order moments which are obtained if we keep higher order terms in the expansion (4). The higher order terms $\phi_4(\mathbf{r})$, $\phi_5(\mathbf{r})$ etc., are called the *multipole potentials.* These multipole potentials are important in nuclear physics.

Thus the potential due to a charge distribution may be considered as the sum of potentials due to monopole moment, dipole moment, quadrupole moment and higher order multipole moments of the charge distribution. In the case where the net charge is zero (e.g., atom), the potential due to monopole moment vanishes. For *non-polar atom* the potential due to dipole moment will also vanish : but still the potential is contributed by higher order multipole moments of the charge distribution. It may be noted that the potential contributed by higher order moments decreases rapidly with increase of order. For *polar atom* the potential is mainly due to dipole moment of the charge distribution.

If θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$ then the potential $\phi(\mathbf{r})$ given by equation (3) may be wirtten as

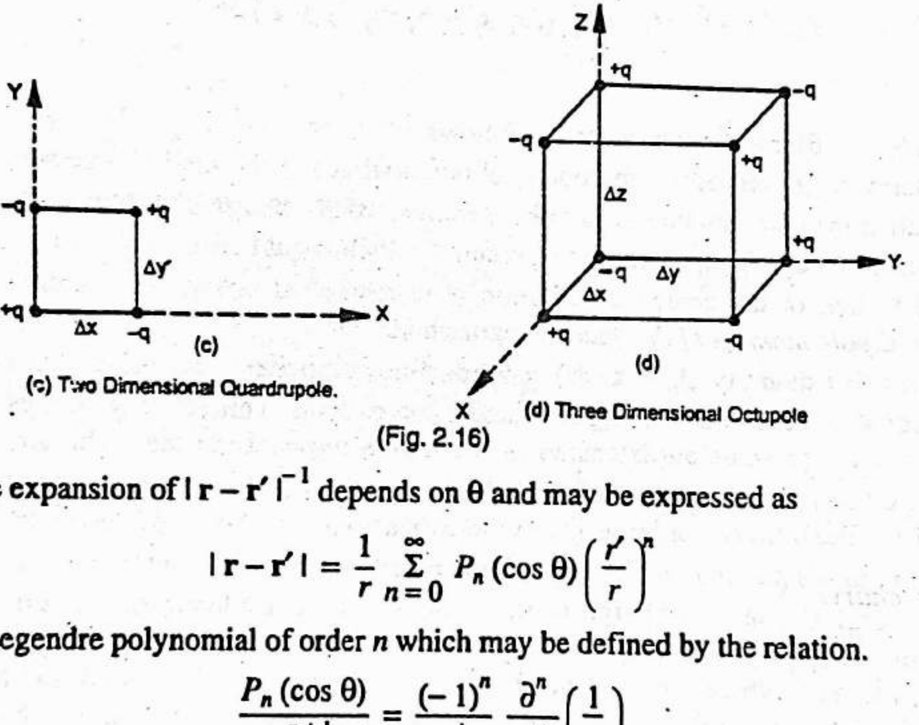
$$\phi(r, \theta) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{1}{r} \left[1 + \frac{r'}{r} \cos\theta + \frac{1}{2} \left\{ \frac{3r'^2}{r^2} \cos^2\theta - \frac{r'^2}{r^2} \right\} \right.$$

$$\left. + \frac{1}{2} \left\{ \frac{5r'^3}{r^3} \cos^3\theta - \frac{3}{r^3} \cos\theta \right\} + \dots \right] \rho\, \mathbf{r}'\, dV'. \qquad \text{...(10)}$$



(a) Linear Quardupole

(b) Linear Octupole.



(c) Two Dimensional Quardrupole.

(d) Three Dimensional Octupole

(Fig. 2.16)

It is obvious that the expansion of $|\mathbf{r} - \mathbf{r}'|^{-1}$ depends on θ and may be expressed as

$$|\mathbf{r} - \mathbf{r}'| = \frac{1}{r} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r'}{r} \right)^n \qquad \text{...(11)}$$

when $P_n(\cos\theta)$ is Legendre polynomial of order n which may be defined by the relation.

$$\frac{P_n(\cos\theta)}{r^{n+1}} = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left(\frac{1}{r} \right) \qquad \text{...(12)}$$

in which θ is the angle between x and r.

Legendre polynomials of few orders are given below

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

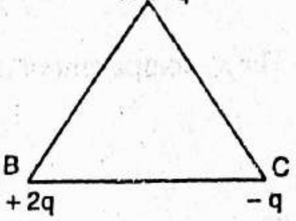
$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$

$$P_3(\cos\theta) = \tfrac{1}{2}(5\cos^2\theta - 3\cos\theta)$$

From equation (10)

it is clear that the potential function $\phi(r, \theta)$ is the function of r and θ. The first term is independent of θ while the multipole potentials are dependent on **r** and θ both. As maximum value of $\cos\theta = 1$ for $\theta = 0$, therefore $\phi(r, \theta)$ falls rapidly as the order of multipole potential increases Fig. 2.16 (a), (b), (c), (d) represents a few examples of simple multipoles.

In all these cases the net charge is zero, therefore the potential due to monopole moment vanishes. Moreover dipole moments cancel out and the potential due to quadrupole and octupole moment may be calculated by given relations



(Fig. 2.17)

Ex. 11. *Calculate the dipole moment of the system of three charges $-q$ coulomb placed at the vertices of an equilateral triangle of side l metre.*

Solution. The system of charges is equivalent to two electric dipoles $\overrightarrow{\mathbf{AB}}$ and $\overrightarrow{\mathbf{CB}}$ of moments $\overrightarrow{\mathbf{p_1}}$ and $\overrightarrow{\mathbf{p_2}}$ inclined at an angle 60° with each other *i.e.*

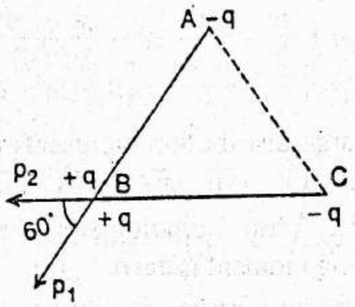
$$|\overrightarrow{\mathbf{p_1}}| = |\overrightarrow{\mathbf{p_2}}| = ql$$

∴ Net dipole moment has magnitude

$$p = \sqrt{p_1^2 + p_2^2 + 2p_1p_2\cos 60^\circ}$$

$$= \sqrt{(ql)^2 + (ql)^2 + 2(ql)(ql) \times \frac{1}{2}}$$

$$= \sqrt{3}\, ql.$$



(Fig. 2.18)

Ex. 12. *A charge of 'q'. coulomb is placed at every corner of a regular hexagon of side 'a' metre. A charge $(-6q)$ is placed at the centre. Calculate the monopole, dipole and quadrupole moments. Also calculate the potential and field at a large axial distance from the charge distribution.* **(Rohilkhand 1995)**

Solution. ***The monopole moment of charge distribution***

$$= 6q - 6q = 0$$

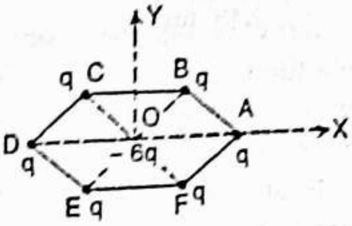
Dipole moment

The x-component of dipole moment due to charge q at $A(a, 0)$ is

$$\iiint \rho\, x\, dV = qa$$

The x component of dipole momnet due to charge q at B is $\left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$ is.

$$\iiint \rho\, x\, dV = q\frac{a}{2} = +\frac{qa}{2}.$$



(Fig. 2.19)

The x componant of dipole moment due to charge q at $C\left(-\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$ is

$$= \iiint \rho\, x\, dV = q\left(-\frac{a}{2}\right) = -\frac{qa}{2}.$$

The x -component of dipole moment due to charge q at $D\,(-a, 0)$ is

$$= \iiint \rho\, x\, dV = q\,(-a) = -q\,a$$

The x -component of dipole moment due to charge q at $E\left(-\frac{a}{2}, -\frac{\sqrt{3}}{2}a\right)$ is

$$\iiint \rho\, x\, dV = q\left(-\frac{a}{2}\right) = -\frac{qa}{2}.$$

The x component of dipole moment at $F\left(\frac{a}{2}, -\frac{\sqrt{3}}{2}a\right)$ is

$$= \iiint \rho\, x\, dV = q\frac{a}{2}$$

The x -component of dipole moment at $O\,(0, 0)$ is

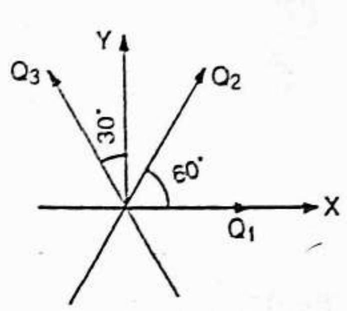
$$= \iiint \rho\, x\, dV = -6q \times 0 = 0$$

$\therefore$ Net x -component of dipole moment due to given charge distribution is

$$= qa + \frac{qa}{2} - \frac{qa}{2} - qa - \frac{qa}{2} + \frac{qa}{2} + 0 = 0.$$

Aliter :

Dipole moment : The given charge distribution represents 6 dipoles of charge pairs $(-q, q)$; named as $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$; each having dipole moment (qa). They form 3 dipole pairs of equal and opposite moments. Therefore net dipole moment is **zero.**



(Fig. 2.20)

Quadrupole moment : The given charge distribution forms 3 quadrupoles each having charges $(q, -2q, q)$ at separation (a, a). The quadropole moment of each quadrupole is $2qa^2$.

$$Q_1 = Q_2 = Q_3 = 2qa^2 = Q' \text{ (say)}$$

These are shown in fig. 2.19

The x-component of quadrupolė moment is

$$Q_x = Q_1 + Q_2\cos 60^\circ - Q_3\cos 60^\circ = Q_1 = 2q\,a^2$$

$$Q_y = Q_2\cos 30^\circ + Q_3\cos 30^\circ = Q_2\frac{\sqrt{3}}{2} + Q_3\frac{\sqrt{3}}{2}$$

$$= Q'\sqrt{3} = 2\sqrt{3}\,qa^2$$

$\therefore$ Net quadrupole moment

$$Q_d = \sqrt{Q_x^2 + Q_y^2} = \sqrt{(2q\,a^2)^2 + (2\sqrt{3}\,qa^2)^2}$$

$$= 2qa^2\sqrt{(1+3)} = 4qa^2$$

The electric potential at point $V(r, \theta)$ from the charge distribution is

$$V(r, \theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q_d}{2r^3}(3\cos^2\theta - 1)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{4qa^2}{2r^3}(3\cos^2\theta - 1)$$

The electric field components at far point $E(r, \theta)$, from the charge distribution is

$$E_r = -\frac{\partial V}{\partial r} \text{ and } E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_0}\frac{Qd}{2r^3}(3\cos^2\theta - 1)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3}{2}\frac{Q_d}{r^4}(3\cos^2\theta - 1)$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{4\pi\varepsilon_0}\frac{Q_d}{2r^3}(3\cos^2\theta - 1)\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{Q_d}{2r^4}\{3 \times 2\cos\theta \cdot (-\sin\theta)\}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{6Q_d}{2r^4}\cos\theta\sin\theta$$

∴ Net electric field

$$E = \sqrt{E_r^2 + E_\theta^2} = \frac{1}{4\pi\varepsilon_0}\frac{3Q_d}{2r^4}\sqrt{5\cos^4\theta - 2\cos^2\theta + 1}$$

As $Q_d = 4qa^2$, we have

$$E = \frac{1}{4\pi\varepsilon_0}\frac{6qa^2}{r^4}\sqrt{\{5\cos^4\theta - 2\cos^2\theta + 1\}}$$

Ex. 13. *Four charges are arranged as shown in Fig. 2.20. Calculate the monopole moment, dipole moment and quadrupole moment of the system. Hence calculate the field at a point P which. is located at a distance r from the centre of configuration.*

Solution :

Monople moment $= \int_{V'} \rho(r')\,dV'$

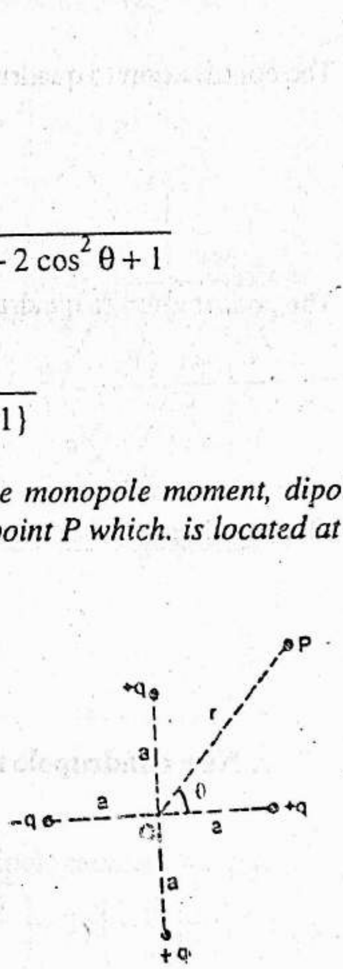
$$= -q - q + q + q = 0$$

Dipole moment : The x and y components of dipole moment may be computed by adding up to contributions due to individual charges.

The contribution to p_x due to charge $-q$ at $(a, 0)$

$$= \int_{V'} \rho(x')\,x'\,dV'$$

$$= -qa$$



(Fig. 2.21)

The contribution to p_x due to charge $-q$ at $(-a, 0)$

$$= \int_V \rho(x')\, x'\, dV'$$

$$= -q \cdot (-a) = qa$$

The contributions to p_x due to charges $+q$ as $(0, a)$ and $+q$ at $(0, -a)$

$$= q \times 0 + q \times 0 = 0$$

Net x-component of dipole moment.

$$p_x = -qa + qa + 0 = 0$$

Similarly y-component of dipole moment,

$$p_y = 0 + qa - qa + 0 = 0$$

Thus the ***net dipole moment of charge configuration*** $= 0$

Quadrupole moment :

$$\int \frac{r'^2}{2} (3\cos^2\theta - 1)\, \rho(r')\, dV'$$

where r' is the distance of the charge $\rho(\mathbf{r}')\, dV'$ from the origin and θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$

The *contribution to quadrupole moment* of charge $-q$ at $(a, 0)$ is

$$= \frac{a^2}{2} (-q) (3\cos^2\theta - 1)$$

$$= -\frac{qa^2}{2} (3\cos^2\theta - 1)$$

The contribution to quadrupole moment of charge $-q$ at $(-a, 0)$ is

$$= \frac{a^2}{2} (-q) \left\{3\cos^2(180^\circ - \theta) - 1\right\}$$

$$= \frac{-qa^2}{2} (3\cos^2\theta - 1)$$

The contribution to quadrupole moment of charge $+q$ at $(0, a)$

$$= \frac{qa^2 [3\cos^2(90^\circ - \theta) - 1]}{2}$$

$$= \frac{qa^2}{2} (3\sin^2\theta - 1)$$

The contribution to quadrupole moment of charge $+q$ at $(0, -a)$ is

$$= \frac{qa^2}{2} [3\cos^2(90 + \theta) - 1]$$

$$= \frac{qa^2}{2} (3\sin^2\theta - 1)$$

∴ New quadrupole moment of charge configuration is

$$\frac{-qa^2}{2} (3\cos^2\theta - 1) - \frac{qa^2}{2} (3\cos^2\theta - 1)$$

$$+ \frac{qa^2}{2} (3\sin^2\theta - 1) + \frac{qa^2}{2} (3\sin^2\theta - 1)$$

$$= qa^2 [3\sin^2\theta - 1 - 3\cos^2\theta + 1]$$

$$= 3qa^2(\sin^2\theta - \cos^2\theta) = 3\,qa^2(2\sin^2\theta - 1)$$

The potential at P due to this charge distribution is

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{3\,qa^2(2\sin^2\theta - 1)}{r^3}$$

∴ The electric field strength at P is given by

$$\vec{E}_r = -\frac{\partial\phi}{\partial r}\hat{r} = -\frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_0}\frac{3qa^2(2\sin^2\theta - 1)}{r^3}\right]\hat{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{9qa^2(2\sin^2\theta - 1)}{r^4}$$

and

$$\vec{E}_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r}\frac{3qa^2\cdot 4\sin\theta\cos\theta}{r^3} = -\frac{1}{4\pi\varepsilon_0}\frac{6qa^2\sin 2\theta}{r^4}\hat{\theta}$$

∴ The resultant field strength

$$\vec{E} = \vec{E}_r + \vec{E}_\theta = \frac{1}{4\pi\varepsilon_0}\cdot\frac{3qa^2}{r^4}\left[3(2\sin^2\theta - 1)\hat{r} - 2\sin 2\theta\,\hat{\theta}\right].$$

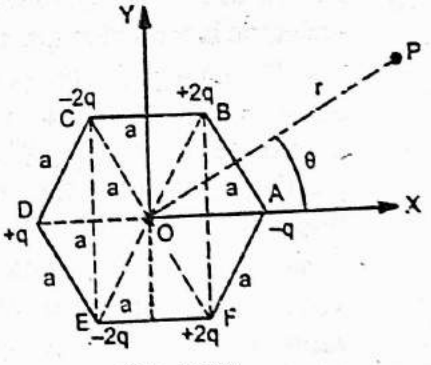
Ex. 14. *Calculate the resultant dipole moment for the system of charges shown in fig* 2.21 *and hence determine to electric potential and field at a point quite far away from the charges.* **(Kanpur 1985)**

Solution. *The electric monopole* of the charge configuration is zero.

Dipole moment. The x-component of dipole moment due to charge $-q$ at point $A(a, 0)$ is

$$= \iiint \rho x\,dV = -q\bullet a = -qa.$$

The x-componet of dipole moment due to charge $+2q$ at $B\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right) = 2q\bullet\frac{a}{2} = +qa$



(Fig. 2.22)

The x-component of dipole moment due to charge $-2q$ at $C\left(-\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$

$$= -2q\left(-\frac{a}{2}\right) = +qa$$

The x-component of dipole moment due to charge $+q$ at $D(-a, 0)$

$$= +q(-a) = -qa$$

The x-component of dipole moment due to charge $-2q$ at $E\left(-\frac{a}{2}, \frac{-a\sqrt{3}}{2}\right)$

$$= -2q\left(-\frac{a}{2}\right) = qa$$

The x-component of dipole moment due to charge $+2q$ at $F\left(+\frac{a}{2}, \frac{-a\sqrt{3}}{2}\right)$

$$= 2q.\frac{a}{2} = +qa$$

∴ x-component of dipole moment $p_x = +2qa$

Similarly adding to contributions of all the charges to y-component of dipole moment we get,

$$p_y = \iiint \rho\,dV = 0 + 2q\left(\frac{a\sqrt{3}}{2}\right) - 2q\left(\frac{a\sqrt{3}}{2}\right) + 0 - 2q\left(-\frac{a\sqrt{3}}{2}\right) + 2q\left(-\frac{a\sqrt{3}}{2}\right)$$
$$= 0$$

∴ Total dipole moment,

$$\mathbf{p} = p_x \hat{i} + p_y \hat{j} = 2qa\, \hat{i}$$

∴ The potential at distant point due to dipole moment at P

$$\phi_p = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{2qa \cos\theta}{r^2}$$

Therefore the field components are

$$E_r = \frac{\partial\phi}{\partial r} = \frac{1}{4\pi\varepsilon_0} \frac{4qa\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{4\pi\varepsilon_0} \frac{2qa\sin\theta}{r^3}$$

∴ Net field strength is, $\vec{\mathbf{E}} = E_r \hat{r} + E_\theta \hat{\theta}$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2qa}{r^3} [2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta}]$$

2·11. Dielectric Polarisation :

An insulator or alternatively known as dielectric is a material in which the electrons are tightly bound to the nucleus of the atom; so that there are no free electrons or the number of such electrons is too low. An ideal dielectric is a material in which there is no free charge at all. Nevertheless every material medium is composed of molecules, which in turn contain positive and negative charged entities (atomic nuclei and electrons) and in fact an equal quantity of each kind, if the dielectric is electrically neutral. However the presence of the electric field may change the behaviour of dielectric because of its molecular structure. If the change in behaviour of the dielectric is independent of the direction of the applied field, the dielectric is termed as *isotropic* while if the change in behaviour of the dielectric depends on the field direction, the dielectric is termed as *anisotropic*.

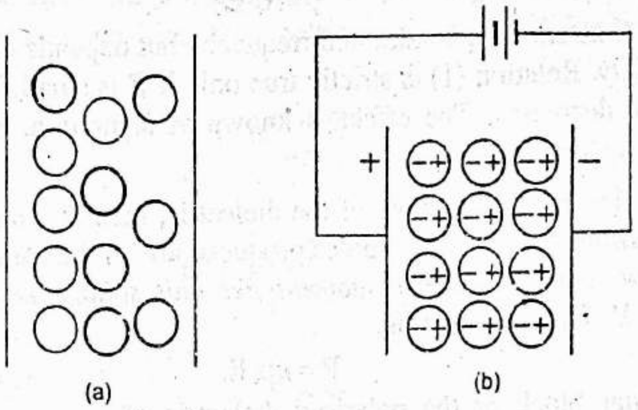
The change in behaviour of the dielectric is due to the fact that the applied electric field exerts a force on each charged particle, positive particles being pushed in the direction of the field and negative particles oppositely, so that positive and negative parts of each molecule are displaced from their equilibrium position in opposite directions. These displacements are, however, limited (in most cases to very small fractions of a molecular diameter) by strong restoring forces which are set up by changing charge configuration in the molecule. The overall effect from the macroscopic point of view is most easily visualied as a displacement of the entire positive charge in the dielectric relative to the negative charge. *The relative displacement of the charges is called* **polarisation** and *the dielectric is said* to be **polarised** and acts like an electric doublet or dipole.

The molecules of a dielectric may be classified into two classes (i) *Polar molecules* and (ii) *Non-polar molecules.*

In any molecule the positive charge of the molecule may be conidered to act, as far as the electric effects of the molecule as a whole are concerned, as the equivalent positive electric charge at some specific point in the molecule-the centre of gravity of positive charge. Similarly the negative charge due to electron in orbits may be considered to act at a point- the centre of gravity of the negative charge. When these two centres of gravity coincide, then the molecule has no resultant charge and zero (permanent) dipole moment such a molecule is said to non-polar e.g. air, hydrogen, benzene, carbon disulphide and carbon tetrachloride. On the other hand when the two centres of gravity are separated from each other by a short distance (order of 0·1 Å in practice), then the molecule as a whole possesses polarity and has a permanent dipole moment;

such a molecule is said to be polar molecule e.g. water, glass, $CHCl_3$. The polar character appears due to asymmetry of the atoms in the molecule.

When either of these types of molecule is placed in an external electric field, the small displacements of the orbitals electrons will cause the distance between the centres of gravity to alter. Thus non-polar molecules become induced dipoles whereas polar molecules, which are already dipoles, will be oriented by the field and may have resultant dipole moment modified since they possess (i) permanent dipole moment and (ii) induced dipole moment. The orientation of the induced dipoles or of the permanent dipoles in an external electric field will be such as to set or tend to set the axis of dipole along the lines of forces. As already indicated this phenomenon is known as electric polarisation. The behaviou of a dielectric in the field between two parallel plates is show in fig. 2.23.

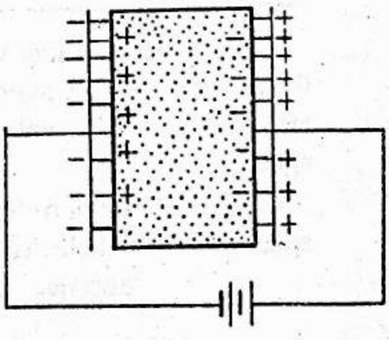


(a) In the absence of the field. (b) In the presence of the field

(Fig. 2.23) Polarisation of a dielectric, (i) non-polar molecules are represented as spheres while poiar molecules would be represented as ellipsoids. (ii) Induced dipoles are shown to be aligned along lines of force whereas permanent dipoles would be similarly aligned.

When no electric field is applied the molecules do not show any polarity since either they are unelectrified non polar molecules or they will be polar molecules which will be oriented at random and will be in such large numbers that the resultant electric effect due to all of them is zero. When a potential difference is applied across the parallel plates, the electric field is produced. This electric field tends to align the induced or permanent dipoles along the electric lines of force, the degree of alignment depending on the field strength. The negative charges on the dipoles will face the positively charged plate, while the positively charges will face the negative plate. *The induced field in the dielectric is therefore opposite in direction to the external applied field.*

Let us now consider a slab of dielectric placed between parallel plates such that there is a gap between the plates and the walls of the dielectric (fig. 2.23). Such a gap is always present, though it may be only of molecular dimensions in width, even though the dielectric is said to fill up the space between the plates. If the potential difference is applied between the plates, an electric field is produced This electric field produces an induced field within dielectric in direction opposite to the external field, hence the field in the dielectric is less than the original field in free space (or in air for practical purposes). The induced charges within the polarised dielectric are called fictitious or bound charges a term introduced by Faraday, whereas the charges on the plates are called real charges. If the dielectric is homogeneous and isotropic, then all the molecules are polarised to the same extent. In this case the fictitious charges within the main body of the medium will neutralise one other, because the negative side of its polarised molecule is adjacent to the positive side of its neighbour. However at the surface fo the dielectric in contact with the plates, the



(Fig. 2.24)

fictitious or bound charges are not neutralised; thus electric field induces charges only on the surface of the dielectric. These induced charges are opposite to the charges on the plates; consequently the field within the dielectric is reduced as compared with that in free space.

Definition of dielectric polarisation. A polarised dielectric, even though it is electrically neutral on the average produces an electric field, both at the exterior points and inside the dielectric as well. The polarisation of the dielectric thus depends on the total electric field in the medium, but a part of the electric field is produced by the dielectric itself. If the average electric field acting on a molecule is **E** (which is in generally different from the applied field) the average dipole moment **p** induced in each molecule of the dielectric is proportional to **E** provided **E** is not too great *i.e.*

$$\mathbf{p} \propto \mathbf{E}$$

or

$$\mathbf{p} = \alpha\,\mathbf{E}$$

where α is a constant at constant temperature and frequency but depends on the dielectric and is known as the *molecular polarisability*. Relation (1) is strictly true only if **E** is small. When **E** becomes large, the rate of increase of **p** with **E** decreases. The effect is known as saturation. Saturation effects only become effective in fields of the order 10^7 volt/m.

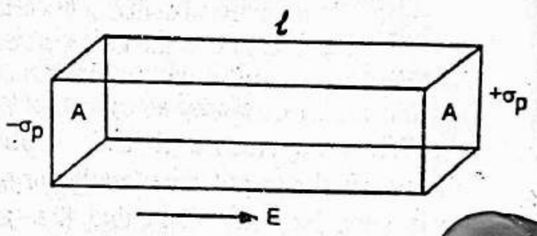
If there are n molecules per unit volume of the dielectric, then the *total induced electric moment per unit volume* $= n\,\alpha\,\mathbf{E}$, assuming the induced dipoles produced are all the same and that they are independent of one another. *The total induced electric moment per unit solume is called the polarisation* and is represented by the vector **P**. Thus polarisation,

$$\mathbf{P} = n\alpha\,\mathbf{E}. \qquad \text{...(2)}$$

Consider a rectangular block of the polarised dielectric of length l and uniform crossection A. Let the surface charge densities of fictitious charge appearing at the end faces be $+\sigma_p$ and $-\sigma p$. Therefore charge induced on each face $= \sigma_p A$.

∴ Total induced electric dipole moment

$$(= ql) = \sigma_p Al. \qquad \text{...(3)}$$



(Fig. 2.25)

As P is the polarisation *i.e.* electric dipole moment per unit volume and (Al) is the volume of the block, therefore the *total dipole moment* $= PAl$.

Equating (3) and (4), we get

$$PAl = \sigma_p\, Al$$

∴

$$P = \sigma_p. \qquad \text{...(5)}$$

Thus polarisation may also be defined as the surface density of charge appearing at faces perpendicular to the direction of the applied field.

2·12. Dielectric Constant (Relative Permittivity) and Displacement Vector

We have seen that when a dielectric is placed, between parallel plates having real charges $+q$ and $-q$, then bound charges appear on the surfaces of the dielectric perpendicular to the field. These bound charges are opposite to the real charges present on the plates. Thus the effect of a dielectric is less than that in free space.

The *relative permittivity* (or dielectric constant) may be defined as the ratio of the electric field in free space to that in dielectric and is represented by the symbol K_e (or ϵ_r)

$$\varepsilon_r = \frac{\text{Field in free space}}{\text{Field in dielectric}} = \frac{\mathbf{E}_0}{\mathbf{E}} \qquad \text{...(1)}$$

If ε_0 and ε are the absolute permitivities of free space and dielectric respectively, then

$$\mathbf{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q\,\mathbf{r}}{r^3} \text{ and } \mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{q\,\mathbf{r}}{r^3} \qquad ...(2)$$

Substituting these values in (1), we get

$$K_e = \frac{\varepsilon}{\varepsilon_0} \qquad ...(3)$$

i.e. relative permitivity (or dielectric constant) may be defined as the ratio of the absolute permitivity of the dielectric to that of free space.

From (2) it is clear that the electric field strength depends upon the medium. Therefore let us introduce a new electric field vector **D**, called the **electric displacement or electric induction** and defined *in such a way that it depends only on the distribution and magnitude of charges which produce electric field but is independent of the nature of the medium.* Thus the electric displacement **D** due to a single point charge q at a point **r** is given by

$$\mathbf{D} = \frac{1}{4\pi}\frac{q\,\mathbf{r}}{r^3} \qquad ...(4)$$

Equation (4) may be written as

$$\mathbf{D} = \varepsilon\left(\frac{1}{4\pi\varepsilon}\frac{q\,\mathbf{r}}{r^3}\right) \qquad ...(5)$$

Using (2), we may write

$$\mathbf{D} = \varepsilon\,\mathbf{E}. \qquad ...(6)$$

Thus it is obvious that

$$\left.\begin{aligned} \mathbf{D} &= \varepsilon_0\mathbf{E} \text{ (in free space)} \\ &= \varepsilon\,\mathbf{E} \text{ in a dielectric of absolute permittivity } \varepsilon \end{aligned}\right\} \qquad ...(7)$$

and the vectors **D** and **E** have the same direction.

2·13. External Field of a Dielectric Medium.

Let us consider a finite piece of polarised dielectric which is characterised at each point **r′** by a polarisation **P** (**r′**). The polarisation gives rise to an electric field and we want to find this field at point **r** outside the dielectric body. It is convenient to calculate first the electric potential ϕ (**r**) and then obtain electric field **E** by the relation $\mathbf{E} = -\text{ grad }\phi$.

Let the volume of the dielectric be divided into small elementary volumes. Let $\Delta V'$ be the volume element of the dielectric medium characterised by the dipole moment $\Delta\,\mathbf{p}$. They be definition of *polarisation*

$$\mathbf{P}(\mathbf{r}') = \frac{\Delta\,\mathbf{p}}{\Delta V'}, \text{ we get}$$

$$\Delta\,\mathbf{p} = \mathbf{P}(\mathbf{r}')\,\Delta V' \qquad ...(1)$$

∴ The potential at point **r** due this small volume element is

$$\Delta\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\Delta\,\mathbf{p}\bullet(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{P}(\mathbf{r}')\bullet(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\Delta V' \text{ using (1)} \qquad ...(2)$$

where $(\mathbf{r}-\mathbf{r}')$ is a vector from volume element $\Delta V'$ to the point A (x, y, z) under consideration.

The magnitude of $(\mathbf{r}-\mathbf{r}')$ is given by

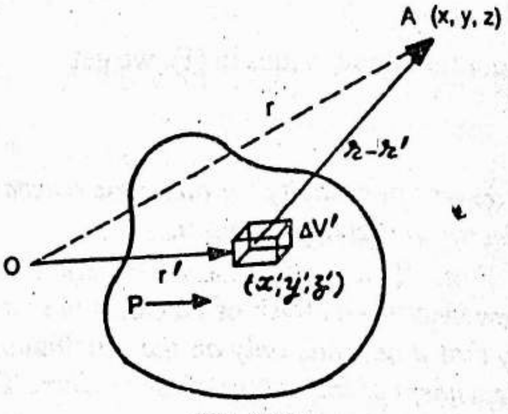
$$|\mathbf{r}-\mathbf{r}'| = \sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]} \qquad ...(3)$$

The net potential at point **r** is obtained by summing up the contribution from all parts of the dielectric *i.e.*

$$\phi(\mathbf{r}) = \Sigma\, \frac{1}{4\pi\varepsilon_0}\, \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\Delta V' \qquad ...(4)$$

If the volume element $\Delta V'$ is infinitesimal, we write dV' for $\Delta V'$ and replace summation by integration *i.e.*

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dV' \qquad ...(5)$$



(Fig. 2.26)

From this result $\phi(\mathbf{r})$ may be evaluated directly if the function from of **P** is known. It is sometimes useful to express equation (5) in a rather different way by means of simple mathematical transformations.

If ∇' operator involves with respect to the primed coordinates, then

$$\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \qquad ...(6)$$

It may be seen by direct application of the gradient operator in cartesian coordinates. Using (6) we may write

$$\frac{\mathbf{P}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \qquad ...(7)$$

This equation may be further transformed by means of following vector identity

$$\text{div}'(S\,\mathbf{A}) = S\,\text{div}'\mathbf{A} + \mathbf{A}\cdot\nabla' S \qquad ...(8)$$

where S is any scalar point function and **A** is an arbitrary vector point function. Here again prime indicates the differentiation with respect to the primed coordinates. Using $S = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$ and $\mathbf{A} = \mathbf{P}$, eqn. (8) yields

$$\text{div}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\text{div}'\,\mathbf{P} + \mathbf{P}\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$\therefore \quad \mathbf{P}\cdot\nabla'\left(\frac{1}{|\mathbf{r}-r'|}\right) = \text{div}'\,\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\text{div}'\,\mathbf{P}. \qquad ...(9)$$

Comparing (7) and (9) we note that integrand of (5) takes from

$$\frac{\mathbf{P}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \text{div}'\,\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\text{div}'\,\mathbf{P} \qquad ...(10)$$

Hence the potential $\phi(\mathbf{r})$., from (5), may be written as

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \left[\text{div}'\,\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\text{div}'\,\mathbf{P}\right]dV'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_V \text{div}'\,\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|}\,dV' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\text{div}'\,\mathbf{P}\,dV' \qquad ...(11)$$

From Gauss divergence theorem changing volume integral of div' $\frac{\mathbf{P}'}{|\mathbf{r}-\mathbf{r}'|}$ into surface integral, we get

$$\int_V \text{div}'\,\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|}\,dV' = \int_S \frac{\mathbf{P}\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}\,da' \qquad ...(12)$$

using (12) the potential $\phi(\mathbf{r})$ finally takes the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\mathbf{P}\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}da' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{-\text{div}'\,\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|}dV' \quad ...(13)$$

This quantities $\mathbf{P}\cdot\mathbf{n}$ and $-$ div $\mathbf{P}$, appearing in the integrals of (13), are two scalar functions obtained from the polarisation $\mathbf{P}$ and have the dimensions of charge per unit area and charge per unit volume respectively. Therefore they are known as *polarisation* (or bound) *volume charge density* respectively and are denoted by

$$\left.\begin{aligned}\sigma_p &= \mathbf{P}\cdot\mathbf{n} = P_n \\ \rho_p &= -\,\text{div}\,\mathbf{P}\end{aligned}\right\} \quad ...(14)$$

The surface density of bound charge is given by the normal component of polarisation vector normal to the surface, while the volume density of bound charge is a masure of the nonuniformity of the polarisation inside the material.

Using (14), eqn. (1) for potential $\phi(\mathbf{r})$ takes the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\left[\int_S \frac{\sigma_p\,da'}{|\mathbf{r}-\mathbf{r}'|} + \int_V \frac{\rho_p\,dV'}{|\mathbf{r}-\mathbf{r}'|}\right] \quad ...(15)$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{dq_p'}{|\mathbf{r}-\mathbf{r}'|} \quad ...(16)$$

It is clear that the potential $\phi(\mathbf{r})$ arises due to a charge distribution. In other words, the dielectric material has been replaced by an appropriate polarisation (or bound) charge distribution.

From (16), (15) and (13) it is obvious that the net *polarisation charge of a dielectric* is

$$\begin{aligned} q_p &= \int_S \sigma_p\,da' + \int_V \rho_p\,dV' \\ &= \int_S \mathbf{P}\cdot\mathbf{n}\,da' + \int_V (-\text{div}'\,\mathbf{P})\,dV' \end{aligned} \quad ...(17)$$

and that it must be zero since we had assumed that the dielectric as a whole is electrically neutral. This result immediately follows since q_p vanishes as a consequence of the divergence theorem.

Now we have two distinct expressions for electrostatic potential $\phi(\mathbf{r})$ due to a polarised dielectric given by (15) and (16). Both of them are correct : but it is found that the latter expression is more convenient in most cases. The electric field $\mathbf{E}$ is obtained by the relation $\mathbf{E} = -\,\text{grad}\,\phi$. As ϕ is a function of (x, y, z) it is desirable to use the gradient operation with respect to unprimed coordinates. Evidently, ∇' (gradient operator with respect to primed coordinates) operating on a function of $|\mathbf{r}-\mathbf{r}'|$ is equal to $-\nabla$ (gradient operator with respect to unprimed coordinates) appear only in the function $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ and noting that

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

and using eqn. (6), we get

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_p(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}da' + \int_V \frac{\rho_p(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV'\right] \quad ...(18)$$

This equation gives general expression for the electric field at an external point $\mathbf{r}$ due to a dielectric medium.

2·14. The Electric Field in a Material Medium :

Let us now find an expression for the electric field in a small polarised material medium. We are actually interested in the macroscopic electric field *i.e.* the average electric field in a small region of the dielectric which, nevertheless, contains a large number of molecules. the actul electric field inside a dielectric is different from the marcoscopic field, since the polarised medium of positive nuclei and electrons in which fields of millions volts/cm. exist and which may point in any direction depending on the location of point under consideration. *The macroscopic electric field is the force per unit charge on a test charge embedded in the dielectric in the limit where the test charge is so small that it does not itself affect the charge distribution.* This test charge must be dimensionally small from the macroscopic point of view (which we call *point charge*), but it will be large compared with the size of a molecule.

It is very difficult to use this fundamental definition of macroscopic electric field **E** directly to obtain an expression for the field, since we would have to calculate the force on a charged body of extended size, and then go to the limit as the size of the object is decreased. However for this purpose we can use another property of the leectric field. The electrostatic field in a dielectric must have the same basic properties which are applicable to electric field **E** in vacuum; in particular, **E** is a conservative field and hence derivable from a scalar potential ϕ *i.e.*

$$\mathbf{E} = -\operatorname{grad}\phi \qquad ...(1)$$

since curl grad $\phi = 0$

This implies that

$$\operatorname{curl}\mathbf{E} = 0$$

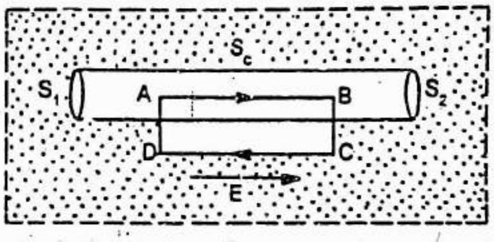
or

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0. \qquad ...(2)$$

Let us apply equation (2) to the path *ABCD* a shown in fig. 2.27 where the segment *AB* lies in a *needleshaped cavity* cut out of the dielectric and the segment *CD* lies in the dielectric proper since the segment *AB* and *BC* may be made arbitrarity small, the line intergral reduces to

$$\oint \mathbf{E}\cdot d\mathbf{l} = \int_A^B \mathbf{E}_v\cdot d\mathbf{l} + \int_C^D \mathbf{E}\cdot d\mathbf{l} = 0 \qquad ...(3)$$

where the subscripts v and d refer to vacuum and dielectric respectively. If l is the length of the path *AB* or *CD*, then (3) may be expressed as.



(Fig. 2.27)

$E_{vt}\, l - E_{dt}\, l = 0$ where subscript t stand for tangential component. This implies

$$E_{vt} = E_{dt}. \qquad ...(4)$$

This equation is valid for any orientation of the needle shaped cavity. Further if the needle is oriented along the direction of **E** then we have $E_{dt} = E_d$ and by symmetry the field in the cavity is along the direction of the needle *i.e.*, $E_{vt} = E_v$. Thus in this case equation (4) takes the form

$$E_v = E_d$$

i.e. the electric field in a dielectric is equal to the electric field inside a needle shaped cavity in the dielectric provided the cavity axis is oriented to the direction of the electric field.

This statement is strictly true only for isotropic dielectrics; because for anisoftropic dielectrics the symmetry argument fails.

The problem of calculating the electric field inside a dielectric, thus reduces to calculating the electric field inside a needle shaped cavity in the dielectric. But the electric field in the cavity is an external field and hence may be determined by means of results already described in section 2.13. Just as we discussed earliar, we assume here that the polarisation of the dielectric is a given function $\mathbf{P}(x', y', z')$ and calculate the potential and electric field arising from this polarisation. Taking the field point **r** at the centre of the cavity, the potential from eqn. (15) of section 2.13 is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\int_{S'+S} \frac{\sigma_p(x', y', z')\, dV'}{|\mathbf{r}-\mathbf{r}'|} + \int_{V-V'} \frac{\rho_p(x', y', z')\, dV'}{|\mathbf{r}-\mathbf{r}'|}\right\} \qquad ...(5)$$

where $V - V'$ is the volume of the dielectric excluding the needle, S_0 is the exterior surface of the dielectric and $S' = S_1 + S_2 + S_c$ are the needle surfaces. But from fig. 2.27 $\sigma_p = 0$ on the cylindrical surface S_c of the needle may be assumed arbitratrily thin to make the surface S_1 and S_2 to possess negligibly small area; so that only the exterior surfaces of the dielectric contribute to the surface integral. Consequently the surface integral of (5) becomes identical in form to the surface integral of (15) of section 2.13. The volume integral of (5) exculdes the cavity; but it may be seen that the contribution of the cavity to the volume integral is negligible if the cavity is arbitrarily thin. Thus it is not necessary to exclude the volume V' and hence equation (5) becomes similar in form to eqn. (15) of section 2.27. In other words equation (15) of section 2.13 gives the potential $\phi(\mathbf{r})$ irrespective of whether the point **r** is located inside or outside the dielectric.

The electric field **E** at **r** may be calculted by the relation $\mathbf{E}(\mathbf{r}) = -\operatorname{grad} \phi(\mathbf{r})$ But this differs only by a negligible amount from equation (18) of section 2.27. Thus equation (18) of section 2.13 *i.e.*,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_p(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,da' + \int_V \frac{\rho_p(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dV'\right]$$

gives the contribution of the dielectric medium to the electric field **E(r)**, ***independently of whether the point* r *lies inside or out side the medium.***

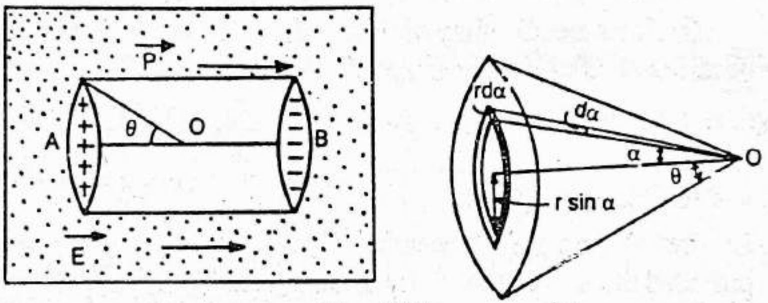
Ex. 15. *Show that the cavity definition of the electric field inside a dielectric depends on the shape of cavity.* **(Meerut 1976)**

Solution. Consider a cylindrical cavity as show is fig. 2.28. (a) with its axis along the direction of field **E**. At any point O

$$\mathbf{E}_c = \mathbf{E}_1 + \mathbf{E}_2 \qquad \text{...(1)}$$

where $\mathbf{E}_1$ is the field at O due to macroscopic field **E** in the dielectric and $\mathbf{E}_2$ is the field at O due to bound charges over the walls of the cavity surface. For a very small cavity the contribution due to macroscopic field at O is the same as that at O in the absence of the cavity *i.e.*,

$$\mathbf{E}_1 = \mathbf{E} \qquad \text{...(2)}$$



(a) (Fig. 2.28) (b)

i.e., for cylindrical cavity the equivalent charge distribution over its walls comprises layers of surface charge densities $\pm\sigma_P = \pm P$ over the two circular ends. As polarisation **P** is tangential to curved surface we have $\sigma_P = \mathbf{P} \cdot \mathbf{n} = P1 \cos 90^\circ = O$ over the curved surface and therefore there is no charge induced on curved surface.

If O is the centre of cylinder, the field strength at O due to charge on either circular ends will be

$$\mathbf{E}' = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_p \mathbf{r}}{r^3}\,da = \frac{1}{4\pi\varepsilon_0}\sigma_P\,\mathbf{n}\int d\omega \qquad \text{...(3)}$$

[where $d\omega = \dfrac{ds}{r^2}$ is the solid angle subtended by surface element (da) at O]

But for a circular area refer fig. 2.28 (b)

$$\int d\omega = \int \frac{da}{r^2} = \int_0^\theta \frac{(2\pi r \sin\alpha)\, r d\alpha}{r^2} = 2\pi(1-\cos\theta)$$

$$\therefore \quad \mathbf{E}' = \frac{1}{4\pi\varepsilon_0}\sigma_P \cdot \mathbf{n}\, 2\pi(1-\cos\theta). \qquad \text{...(4)}$$

Hence the total field strength at O due to the two circular end layers of charges

$$\mathbf{E}_2 = 2\mathbf{E}' = 2 \times \frac{1}{4\pi\varepsilon_0} \mathbf{P}\, 2\pi (1 - \cos\theta) = \frac{\mathbf{P}}{\varepsilon_0}(1 - \cos\theta) \qquad \text{...(5)}$$

(Since $\sigma_P \cdot \mathbf{n} = \mathbf{P}$)

Substituting values of $\mathbf{E}_1$ and $\mathbf{E}_2$ from (2) and (5) in (1) we get

$$\mathbf{E}_c = \mathbf{E} + \frac{\mathbf{P}}{\varepsilon_0}(1 - \cos\theta) \qquad \text{...(6)}$$

Obviously the electric field inside a cavity depends on θ *i.e.*, upon **the shape of the cavity.** Hence the result.

Ex. 16. *The following cavities are cut in a dielectric in which polarisation* **P** *is uniform and fixed*

(i) *a long needle shaped cavity*

(ii) *a narrow slit at right angles to its polarisation*

(iii) *a spherical cavity.*

Find the electric field strength at the centre of each due to charge produced on the surface of the cavity. Hence write the expression for net field strength with the cavity if the applied field is E.

Solution. Let us consider a dielectric slab of infinite extent polarised along x-axis due to applied field **E** and in which cavities of type. (i), (ii), and (iii) are cut (fig. 2.29). The field due to induced charges from ex. (15) is given by

$$\mathbf{E}_2 = \frac{\mathbf{P}}{\varepsilon_0}(1 - \cos\theta) \qquad \text{...(1)}$$

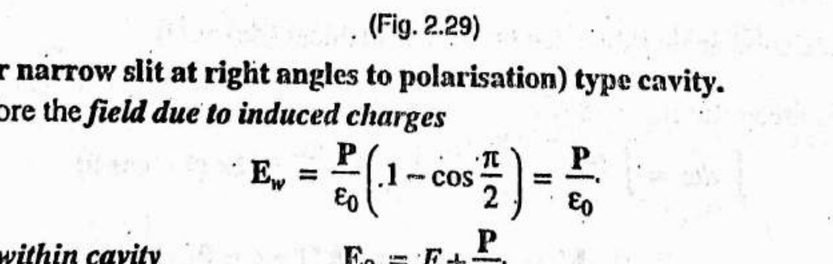
and net field within cavity
$$\mathbf{E}_0 = \mathbf{E} + \mathbf{E}_2. \qquad \text{...(2)}$$

(i) For a needle shaped cavity the side surfaces are very small and approach zero in the limit when the needle shaped cavity has length only and so $\theta = 0$; Therefore *field due to induced charges*

$$\mathbf{E}_2 = \frac{\mathbf{P}}{\varepsilon_0}(1 - \cos\theta) = \frac{\mathbf{P}}{\varepsilon_0}(1 - 1) = 0; \qquad \text{...(3)}$$

and so field inside cavity,
$$\mathbf{E}_0 = \mathbf{E} \qquad \text{...(4)}$$

i.e., the electric field strength inside a needle shaped cavity is same as the electric field in a dielectric provided the axis of the cavity is along the electric field **E**.



(Fig. 2.29)

(ii) For disc (or narrow slit at right angles to polarisation) type cavity.

$\theta = \pi/2$, therefore the ***field due to induced charges***

$$\mathbf{E}_w = \frac{\mathbf{P}}{\varepsilon_0}\left(1 - \cos\frac{\pi}{2}\right) = \frac{\mathbf{P}}{\varepsilon_0}. \qquad \text{...(5)}$$

Hence the ***net field within cavity***
$$\mathbf{E}_0 = E + \frac{\mathbf{P}}{\varepsilon_0}. \qquad \text{...(6)}$$

(iii) For a **spherical cavity** the expression (1) for field due to induced charges does not hold. In this case the electric field due to bound charges is given by

$$\mathbf{E}_2 = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_p\,\mathbf{r}\,da}{r^2} = \frac{1}{4\pi\varepsilon_0}\int \frac{(\mathbf{P}\bullet\mathbf{n})}{r^2}\,\mathbf{n}\,da$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{P}\cos\theta}{r^2}\,\mathbf{n}\,da$$

By symmetry the component of field from each surface element is parallel to **P** or **E** *i.e.*

$$\mathbf{E}_2 = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{P}\cos\theta}{r^2}\cos\theta\,da$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^{\pi} \frac{\mathbf{P}\cos\theta}{r^2}\cos\theta\,(2\pi r^2\sin\theta\,d\theta)$$

$$\left(\begin{aligned}\text{since surface element } da &= (2\pi\, r\sin\theta\,)\,r d\theta\\ &= 2\pi r^2\sin\theta\,d\theta\end{aligned}\right)$$

$$= \frac{\mathbf{P}}{2\varepsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = \frac{\mathbf{P}}{3\varepsilon_0} \qquad \text{...(7)}$$

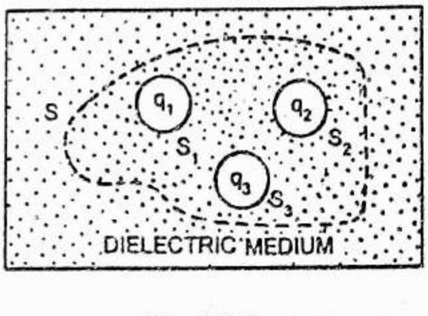
$\therefore$ Net field within cavity

$$\mathbf{E}_c = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0} \qquad \text{...(8)}$$

2·15. Gauss's Theorem in a Dielectric : The Electric Displacement :

The Gausss theorem relates the electric flux and charge. The theorem states that the net electric flux across an arbitrary closed surface drawn in an electric field is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface. Now we want to extend this theorem for a region containing free charge embedded in a dielectric.

In fig. 2.30. the dotted surface S in an imaginary closed surface drawn in a dielectric medium. There is certain amount of free charge Q in the volume bounded by surface S. Let us assume that this free charge exists on the surfaces of three conductors in amounts q_1, q_2, q_3. In a dielectric there also exists certain amount of polarisation (bound) charge Q_P. Hence by Gauss's theorem



(Fig. 2.30)

$$\oint \mathbf{E}\bullet\mathbf{n}\,da = \frac{1}{\varepsilon_0}(Q+Q_P) \qquad \text{...(1)}$$

where $Q = q_1 + q_2 + q_3$, is the total free charge and Q_P is the polarisation (bound) charge given by

$$Q_P = \int_{S_1+S_2+S_3}\mathbf{P}\bullet\mathbf{n}\,da + \int_V(-\operatorname{div}\mathbf{P})\,dV \qquad \text{...(2a)}$$

Here V is the volume of the dielectric enclosed by S. As there is no bounday of the dielectric at S, therefore the surface integral in (2*a*) *does not* contain a contribution from S. If we transform volume integral in (2a) into surface integral by means of Gauss divergence theorem, we must include contributions from all surface bounding V, namely S, S_1, S_2 and S_3 *i.e.*,

$$\int_V(-\operatorname{div}\mathbf{P})\,dV = -\left[\int_{S_1+S_2+S_3}\mathbf{P}\cdot\mathbf{n}\,da + \oint_S \mathbf{P}\cdot\mathbf{n}\,da\right]$$

using above equation, we note that

$$Q_P = -\oint_S \mathbf{P}\cdot\mathbf{n}\,da \qquad \text{...(2b)}$$

Substituting this value in (1), we get

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}Q - \frac{1}{\varepsilon_0}\oint_S \mathbf{P}\cdot\mathbf{n}\,da$$

or

$$\oint_S \left(\mathbf{E} + \frac{\mathbf{P}}{\varepsilon_0}\right)\cdot\mathbf{n}\,da = \frac{Q}{\varepsilon_0}$$

Multiplying throughout by ε_0

$$\oint_S (\varepsilon_0\,\mathbf{E} + \mathbf{P})\cdot\mathbf{n}\,da = Q \qquad \text{...(3)}$$

This equation states that the flux of the vector $(\varepsilon_0\,\mathbf{E} + \mathbf{P})$ through a closed surface is equal to the *total free charge* enclosed by the surface. This vector quantity is named as *electric displacement* **D** *i.e.*

$$\mathbf{D} = \varepsilon_0\,\mathbf{E} + \mathbf{P} \qquad \text{...(4)}$$

Evidently electric displacement **D** has the same units as **P** *i.e.* charge per unit area.

In terms of electric displacement vector **D**, equations (3) becomes

$$\oint_S \mathbf{D}\cdot\mathbf{n}\,da = Q$$

i.e. the flux of electric displacement vector across an arbitrary closed surface is equal to the total free charge enclosed by the surface. This result is usually referred to as 'Gauss's theorem in a dielectric. Equation (5) is applicable to a region of space bounded by any closed surface S. If we consider the entire volume V to be divided into a large number of infinitesimal volume elements, then Gauss theorem may also be expressed as

$$\oint_S \mathbf{D}\cdot\mathbf{n}\,da = \int_V \rho\,dV \qquad \text{...(6)}$$

where ρ is the charge density at a point within volume element dV such that $dV \to 0$.

$$\int_V \operatorname{div}\mathbf{D}\,dV = \int_V \rho\,dV$$

or

$$\int_V (\operatorname{div}\mathbf{D} - \rho)\,dV = 0$$

As volume is arbitrary, therefore we get

$$\operatorname{div}\mathbf{D} - \rho = 0$$

or

$$\operatorname{div}\mathbf{D} = \rho \qquad \text{...(1)}$$

This result is called differential form of "Gauss's theorem" in a dielectric.

The main advantage of this method is that the total electrostatic field at each point in the dielectric medium may by expressed as the sum of parts

$$\mathbf{E}(x, y, z) = \frac{1}{\varepsilon_0}\mathbf{D}(x, y, z) - \frac{1}{\varepsilon_0}\mathbf{P}(x, y, z) \qquad \text{...(8)}$$

where the first term $\frac{\mathbf{D}}{\varepsilon_0}$ is related to free charge density through the divergence and the second term $\frac{\mathbf{P}}{\varepsilon_0}$ is proportional to the polarisation of the medium. In vacuum $\mathbf{P} = 0$, therefore electric field **E** is given by

$$E = \frac{\mathbf{D}}{\varepsilon_0}. \qquad \text{...(9)}$$

$$Q_P = -\oint_S \mathbf{P}\cdot\mathbf{n}\,da \qquad \text{...(2b)}$$

Substituting this value in (1), we get

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\,Q - \frac{1}{\varepsilon_0}\oint_S \mathbf{P}\cdot\mathbf{n}\,da$$

or

$$\oint_S \left(\mathbf{E} + \frac{\mathbf{P}}{\varepsilon_0}\right)\cdot\mathbf{n}\,da = \frac{Q}{\varepsilon_0}$$

Multiplying throughout by ε_0

$$\oint_S (\varepsilon_0\,\mathbf{E} + \mathbf{P})\cdot\mathbf{n}\,da = Q \qquad \text{...(3)}$$

This equation states that the flux of the vector $(\varepsilon_0\,\mathbf{E} + \mathbf{P})$ through a closed surface is equal to the *total free charge* enclosed by the surface. This vector quantity is named as *electric displacement* **D** *i.e.*

$$\mathbf{D} = \varepsilon_0\,\mathbf{E} + \mathbf{P} \qquad \text{...(4)}$$

Evidently electric displacement **D** has the same units as **P** *i.e.* charge per unit area.

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$$\oint_S \mathbf{D}\cdot\mathbf{n}\,da = \int_V \rho\,dV \qquad \text{...(6)}$$

where ρ is the charge density at a point within volume element dV such that $dV \to 0$.

or

$$\int_V \operatorname{div}\mathbf{D}\,dV = \int_V \rho\,dV$$

$$\int_V (\operatorname{div}\mathbf{D} - \rho)\,dV = 0$$

As volume is arbitrary, therefore we get

$$\operatorname{div}\mathbf{D} - \rho = 0$$

or

$$\operatorname{div}\mathbf{D} = \rho \qquad \text{...(1)}$$

This result is called differential form of "Gauss's theorem" in a dielectric.

The main advantage of this method is that the total electrostatic field at each point in the dielectric medium may by expressed as the sum of parts

$$\mathbf{E}\,(x, y, z) = \frac{1}{\varepsilon_0}\,\mathbf{D}\,(x, y, z) - \frac{1}{\varepsilon_0}\,\mathbf{P}\,(x, y, z) \qquad \text{...(8)}$$

where the first term $\frac{\mathbf{D}}{\varepsilon_0}$ is related to free charge density through the divergence and the second term $\frac{\mathbf{P}}{\varepsilon_0}$ is proportional to the polarisation of the medium. In vacuum $\mathbf{P} = 0$, therefore electric field **E** is given by

$$E = \frac{\mathbf{D}}{\varepsilon_0}. \qquad \text{...(9)}$$

2·16. Electric Susceptibility and Dielectric Constant.

We have seen that the polarisation of a dielectric medium occurs in response to the electric field in the medium. The degree of polarisation depends on electric field and the properties of molecules which make up the dielectric material. For macroscopic point of view the behaviour of the material is completely specified by an experimentally determined relationship $\mathbf{P} = \mathbf{P}(\mathbf{E})$ where $\mathbf{E}$ is the macroscopic electric field. The relationship holds at a point. Accordingly if E varies from point to point, then $\mathbf{P}$ will vary accordingly.

Let us limit ourselve to materials having no permanent polarisation. For these materials $\mathbf{P}$ vanishes when $\mathbf{E}$ vanishes. Moreover if the material is isotropic, then the polarisation will have the same direction as the electric field responsible for polarisation *i.e.*

$$\mathbf{P} \propto \mathbf{E}$$

or

$$\mathbf{P} = \chi(\mathbf{E})\, \mathbf{E} \qquad \text{...(1)}$$

where $\chi(\mathbf{E})$ is a scalar quantity (for isotropic materials) and is called the *electric susceptibility* of the material. A large number of materials are isotropic ; this category includes fluids, polycrystalline and amorphous solids and some crystals.

We also have

$$\mathbf{D} = \varepsilon(\mathbf{E})\, \mathbf{E} \qquad \text{[From eqn. (6) of section (2·12)]} \qquad \text{...(2)}$$

and

$$\mathbf{D} = \varepsilon_0 \mathbf{E}\ \mathbf{P} \qquad \text{[from eqn., (4) of section (2·15)]} \qquad \text{...(3)}$$

Susbtituting values of $\mathbf{P}$ and $\mathbf{D}$ from (1) and (2) is equation (3), we get

$$\varepsilon(\mathbf{E})\, \mathbf{E} = \varepsilon_0\, \mathbf{E} + \chi(\mathbf{E})\, \mathbf{E} \qquad \text{...(4)}$$

or

$$\varepsilon(\mathbf{E}) = \varepsilon_0 + \chi(\mathbf{E})$$

Evidently $\varepsilon, \varepsilon_0$ and χ all have same units.

Experimental observations show that χ and ε are characteristics of material and they are independent of electric fields except perhaps for very intense fields. Such materials are called *linear dielectrics* and they obey the relations

$$\mathbf{D} = \chi\, \mathbf{E} \qquad \text{...(5)}$$

$$\mathbf{D} = \varepsilon\, \mathbf{E} \qquad \text{...(6)}$$

The electrical behaviour of the material is now completely specified by either the permittivity ε or the susceptibilty χ.

Let us now express a relation between dielectric constant K and susceptibility χ.

We know

$$\varepsilon = K\varepsilon_0 \qquad \text{...(7)}$$

i.e.

$$K = \frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_0 + \chi}{\varepsilon_0} \qquad \text{using (4)}$$

or

$$K = 1 + \frac{\chi}{\varepsilon_0} \qquad \text{...(8)}$$

As already pointed out we are considering linear dielectrics. However if the electric field in a dielectric is made very intense, it will begin to pull electrons completely out of the molecules and the material will become conducting. ***The maximum electric field which a dielectric can withstand without breakdown is called its dielectric strength.***

2·17. Point Charge in a Dielectric Fluid

Let us consider a point charge $+q$ located in a homogeneous isotopic medium of infinite extent. Let us assume that the dielectric medium is linear and characterised by a dielectric constant K. Under the influence of the point charge $+q$, the medium is polarised radially the centres of negative charge of constituent molecules of the medium are being pulled in towards $+q$ and the centres of positive charges are being pushed away. By the presense of the medium the radial nature of the field is not changed, since **E, D** and **P** are all parallel to one another at each point. By symmetry of the problem **E, D** and **P** can depend only on distance from the point charge, not on any angular coordinate. Let us draw a spherical surface of radius r concentrically about q. For convenience let us assume that q is located at the origin. Now applying Gauss's theorem to this spherical surface, we obtain :

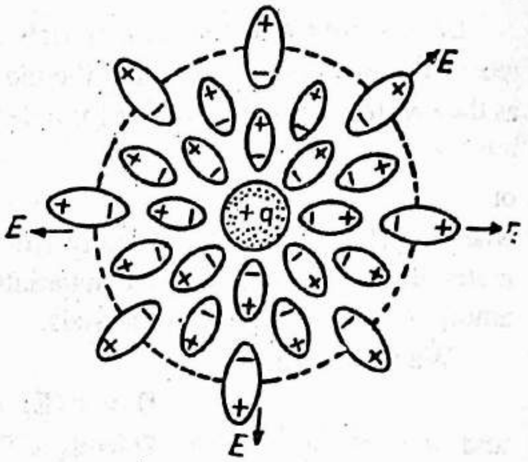
$$\oint_S \mathbf{D} \cdot \mathbf{n}\, da = q$$

or $$D\, 4\pi r^2 = q$$

or $$D = \frac{q}{4\pi r^2}$$

on in vector form

$$\mathbf{D} = \frac{q}{4\pi r^3}\,\mathbf{r}. \qquad \text{...(1)}$$



(Fig. 2.31)

The relation between electric field **E** and electric displacement **D** is $\mathbf{D} = \varepsilon \mathbf{E}$, ε being permitivity of medium.

Hence *electric field* **E** is given by

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\mathbf{D}}{K\varepsilon_0} \quad (\text{using } \varepsilon = K\varepsilon_0)$$

$$= \frac{q}{4\pi K\varepsilon_0 r^3}\,\mathbf{r} \quad [\text{using (1)}]. \qquad \text{...(2)}$$

From equation (4) of section (2·15), the relation between **E**, and **P** is

$$\mathbf{D} = \varepsilon_0\, \mathbf{E} + \mathbf{P}.$$

Hence *polarisation* **P** is given by

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\, \mathbf{E} = \varepsilon_0 K \mathbf{E} - \varepsilon_0\, \mathbf{E} = (K-1)\, \varepsilon_0\, \mathbf{E}$$

$$= (K-1)\, \varepsilon_0 \frac{q\mathbf{r}}{4\pi K\varepsilon_0 r^3} \quad [\text{using (2)}]$$

i.e. $$\mathbf{P} = \frac{(K-1)\, q}{4\pi K r^3} \qquad \text{...(3)}$$

From equation (2) it is obvious that the electric field in a dielectric is smaller by the factor K than would be the case when medium is absent. Now let us try to find why the dielectric has weakened the electric field. The electric field has its origin due to all the charges, bound and free. The bound charge is made up from two contributions, a volume density $\rho_p = -\operatorname{div} \mathbf{P}$ ans a surface density $\sigma_p = -\mathbf{P} \cdot \mathbf{n}$ on the suface of the dielectric in contract with the point charge. Using eqn. (3), we see that

$$\operatorname{div} \mathbf{P} = \operatorname{div}\left[\frac{(K-1)\, q\mathbf{r}}{4\pi K r^3}\right] = \frac{(K-1)\, q}{4\pi K} \operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) = 0\,;$$

consequently $\rho_p = -\operatorname{div} \mathbf{P} = 0$ *i.e.* there is no volume density of bound charge in this case.

It may be noted that the point charge q is a point in the macroscopic sense ; but it is not a point on a molecular scale. Let b be the radius of this point charge on the molecular scale. Then total *surface bound charge* is given by

$$Q_P = \lim_{b \to 0} 4\pi b^2 \, (-\mathbf{P} \cdot \mathbf{n})_{r=b} = - \lim_{b \to 0} 4\pi \, b^2 \left[\frac{(K-1)\, b}{4\pi K r^3} \, \mathbf{r} \cdot \mathbf{n} \right]_{r=b}$$

$$= - \lim_{b \to 0} 4\pi b^2 \left[\frac{(K-1)\, qr}{4\pi K r^3} \right]_{r=b} = - \frac{(K-1)\, q}{K}. \qquad ...(4)$$

Hence the *total charge* within the Gaussian surface

$$q + Q_P = q - \frac{(K-1)\, q}{K} \text{ using (4)}$$

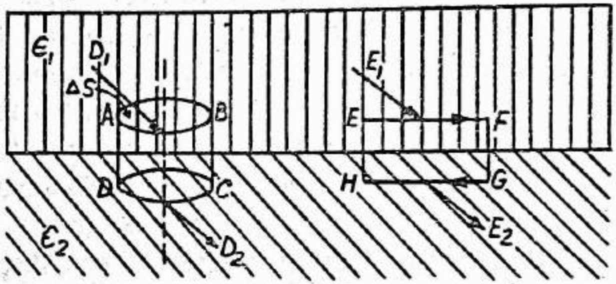
$$= q/K.$$

Thus the presence of the medium reduces the point charge $+q$ to a value q/K. This explains why the electric field is a factor K smaller than it should be if the medium were absent. The schematic diagram of the point charge $+q$ in a dielectric medium is shown in fig. 2·31.

2·18. Boundary Conditions on Field Vectors :

Let us know how the field vectors **E** and **D** change in passing an interface between the two media. The two media any be two dielectrics with different properties, or a dielectric and a conductor. Vacuum is treated as a dielectric with permitivity ε_0.

Consider two dielectric media with permittivities ε_1 and ε_2, in contact as shown in the fig. 2·32. Let us construct a small pillbox shaped surface $ABCD$ which intersects the interface and encloses an area ΔS of the interface and whose axis is normal to the interface. In order to find the boundary condition for **D**, we apply Gauss's theorem (eqn. 5 of section 2·15) to this pill box surface. If the height AD, BC of the pill box is negligibly small in comparison with the diameter of bases ; the only contribution, to $\oint \mathbf{D} \cdot d\mathbf{a}$ over its surface will come from the component of **D** normal to the interface. Since there are no free charges at the interface,



(Fig. 2.32)

$$\oint \mathbf{D} \cdot d\mathbf{a} = 0,$$

i.e.

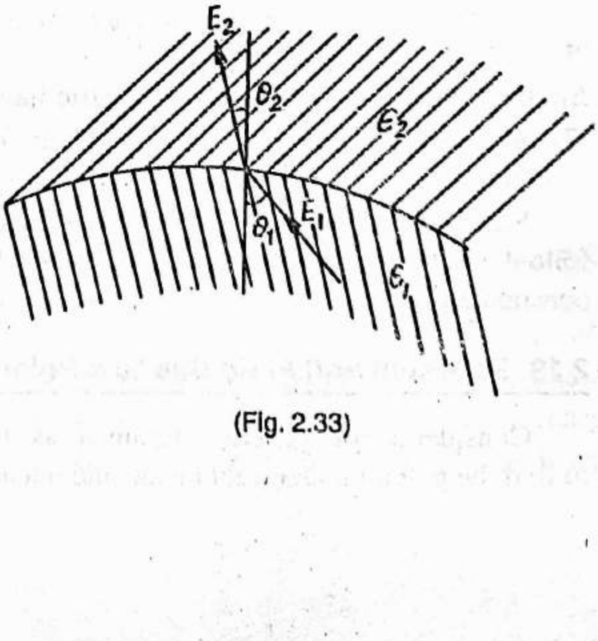
$$-D_{1n}\, \Delta S + D_{2n}\, \Delta S = 0$$

$$D_{1n} = D_{2n}$$

where D_{1n} and D_{2n} represent, respectively, the normal component of the displacements in the two media. Thus the normal component of the displacement has the same value on the two sides of the surface of separation. Since $\mathbf{D} = \varepsilon \mathbf{E}$, eq. (1) may also be rewritten as

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}. \qquad ...(2)$$

The boundary condition for **E** is found from the fact that no work is done in taking a unit test charge around a closed path. Consider a rectangular path $EFGH$ of length Δl, as shown in fig. 2·32, with the side EF lying in the medium of permitivity ε_1 and side GH lying in the medium of permitivity ε_2. If the sides EH and FG are negligibly small compared with EF, HG, then the work done in taking a unit charge around this rectangle is



(Fig. 2.33)

$$\oint E \cdot dl = 0$$

i.e.
$$(E_{1t} - E_{2t})\,\Delta l = 0$$

or
$$E_{1t} = E_{2t} \qquad \text{...(3)}$$

where E_{1t} and E_{2t} represent, respectively, the tangential components of the electric intensity in the two media. Therefore, the tangential component of the electric intensity has the same value on both sides of the surface of separation.

Equations (1) or (2) and (3) are our two fundamental boundary conditions. It follows from them that lines **D** (or E) will, in general, be refracted at the boundary between two dielectrics, as shown in fig. 2·33. If θ_1 is the angle between **E** and the normal to the surface in medium 1 and θ_2 is the similar angle, in medium 2, then equations (2) and (3) becomes

$$\left.\begin{aligned} \varepsilon_1 E_1 \cos\theta_1 &= \varepsilon_2 E_2 \cos\theta_2 \\ \text{and} \qquad E_1 \sin\theta_1 &= E_2 \sin\theta_2 \,. \end{aligned}\right\} \qquad \text{...(4)}$$

Dividing one equation by the other the relation between the angles made by the lines of force in the two media with the normal to the surface is given by

$$\varepsilon_1 \cot\theta_1 = \varepsilon_2 \cot\theta_2 \,.$$

We have considered the above description for two dielectric media with different properties. One very important special case is where one of the two media is a perfect conductor. A perfect conductor is a material in which charges move freely when an electric field is applied. If charges are placed on or in the material, they will move about as long as there is the slightest force on them. They will move to an equilibrium position such that the forces are zero, all electric fields in the conductr are zero. If there were net charge in the body of the conductor, Gauss's law would require an electric field in the vicinity of this charge, so this is an impossible condition for the static case. All free charges in perfect conductor in equilibrium must then reside on the surface and must be distributed so that the component of electric field intensity tangential to the surface and the total electric field intensity inside the surface are zero. If we apply Gauss' theorem to an elementary pillbox shaped surface intersecting the surface, as in fig. 2·33, we have $D_{1n}\,\Delta S = \sigma\,\Delta S$. (Since $D_{2n} = 0$ within the conductor, we have taken medium 2 as a perfect conductor) where σ is the surface charge density on the conducting surface. Thus,

$$D_{1n} = \sigma. \qquad \text{...(6)}$$

The second condition is obtained from the fact that no work is done in taking a unit test charge around a closed path a portion of which lies in each medium. Thus taking the line integral of **E** around the rectangular path, such as *EFGH* in fig. 2·32, we must get it equal to zero, *i.e.*

$$E_{1t}\,\Delta l = 0$$

or
$$E_{1t} = 0. \qquad \text{...(7)}$$

E_{1t} the tangential component of electric intensity at the surface of the conductor is zero. Equation (6) and (7) represent the boundary conditions for displacement and electric field in a dielectric just outside a conductor.

Since the tangential component of electric field is zero along the perfectly conducting surface, it follows that the surface must be an equipotential surface, Moreover, since the entire body will be at a constant electrostatic potential, the field is zero inside a perfect conductor.

2·19. Potential and Field due to a Polarized Sphere :

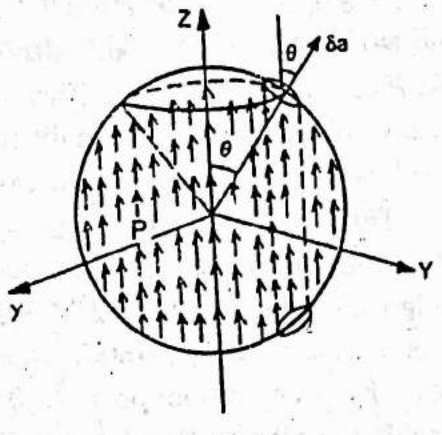
Consider a solid sphere of radius R as shown in fig. 2·34, which is polarized uniformly. We are in need to find the potential and field inside and outside this solid sphere. Let the density of polarization be **P**. Let us cosider a thin cylindrical column of this polarized sphere,with its length parallel to **P**. The areas δa of the upper and lower faces of this column make an angle θ with **P**. At the upper and lower surfaces of this column, there will be charges.

$$+\ P\delta a\cos\theta \text{ and } -\ P\delta a\cos\theta.$$

respectively, due to polarization. The factor cos θ arises, because the component of δa along **P** in $\delta a\cos\theta$. Thus the surcace densities of charges at the ends of this column is,

$$\pm\ P\delta a\cos\theta/\delta a = \pm\ P\cos\theta.$$

Since we have considered the sphere as polarized uniformly, the density of charge will be the same at each circular point of the sphere forming a cone of semi-vertex θ at the centre of the sphere, with the axis of the cone lying along **P**, as shown in fig. 2·34.. The total surface charge due to polarization on the upper half of this sphere is,



(Fig. 2.34)

$$= \int_0^{+\pi/2} 2\pi R\sin\theta.\, Rd\theta.\, P\cos\theta$$

$$= \pi R^2 P\int_0^{\pi/2}\sin 2\theta\, d\theta$$

$$= \pi R^2 P\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \pi R^2 P \qquad \text{...(1)}$$

Similarly the total surface charge on the lower half of this sphere is $-\pi R^2 P$, *i.e.*, the total surface charge on this polarized sphere is zero.

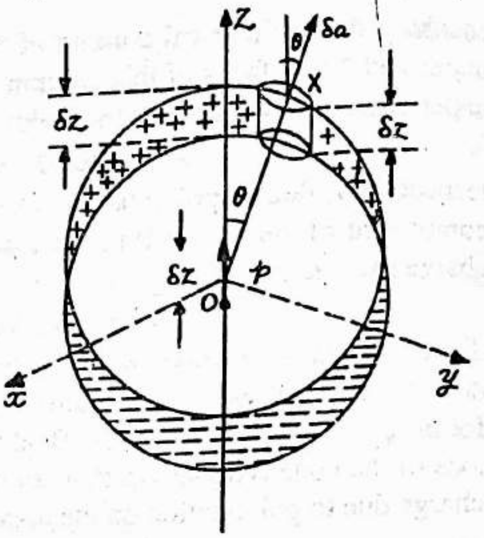
This type of polarized sphere in regard of its electrical properties may be thought of as composed of two identical spheres, one having a uniform positive charge of volume density $+P/\delta z$ and the other a uniform negative charge of density $-P/\delta z$, and with the centres of the two spheres spaced by an infinitesimal distance δz apart, as shown in fig. 2·35. All over the region of overlap the net charge density is zero, but there is very thin layer of positive charge over the upper half of the surface and an identical layer of negative charge over the lower surface. The total charge inside the upper sphere is $\frac{4}{3}\pi R^3 . \frac{P}{\delta z}$ and that inside the lower sphere is $-\frac{4}{3}\pi R^3 . \frac{P}{\delta z}$. For the calculation of field at external point, these charges may be considered as concentrated at the centre of their respective sphere, which are at a distance δz apart. The two point charges $\left(\pm\frac{4}{3}\pi R^3\frac{R}{\delta z}\right)$ at a separation of δz, thus form a dipole with dipole moment,

$$\mathbf{p} = \frac{4}{3}\pi R^3 \cdot \frac{P}{\delta z}\,\delta z$$

$$= \frac{4}{3}\pi R^3 \mathbf{P} \qquad \text{...(2)}$$

Since **P** is along **δz**, the polarized sphere of fig. 2·34 also has the same total dipole moment as given by eq. (2), since the dipole moment per unit volume is *P*.

Now let us consider an infinitesimal small cylindrical volume of surface area δa and height δz at a point *X* in the superposed spherical charges in a radial direction making an angle θ with the line of displacement of their centres, as shown in fig. 2·35. Inside this twisted cylinder there exists a positive charge equal to that in a rectangular cylinder of volume $\delta a\cos\theta.\,\delta z$. Since the volume density of charge in this



(Fig. 2.35)

region is $P/\delta z$, the positive charge inside the small cylinder is $\delta a \cos\theta \, \delta z \, P/\delta z = \delta a \, P \cos\theta$. Since the surface area of the twisted cylinder is δa, the surface density of charge is $\delta a \, P \cos\theta / \delta a = P \cos\theta$. Therefore the surface density of charge at any point on the surface in the radial direction θ is $P \cos\theta$. This result is also same as that in the case of polarized sphere of fig. 2·34.

The two sphere system of fig. 2·35, which is equivalent to a short dipole of dipole moment given eq. (2) is thus equivalent to the polarized sphere of fig. 2·34 for the calculation of electrostatic problems at external point. In order to find the potential and electric intensity at any distant point (r, θ) outside the polarized sphere, we can thus use the results of section 2·6.

Thus

$$\phi_0 = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\left(\frac{4}{3}\pi R^3 P\right)\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{PR^3\cos\theta}{3\varepsilon_0 r^2} \qquad \text{...(3)}$$

and

$$(E_r)_0 = \frac{1}{4\pi\varepsilon_0}\,\frac{2p\cos\theta}{r^3} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2\cdot\left(\frac{4}{3}\pi R^3 P\right)\cos\theta}{r^3} = \frac{2PR^3\cos\theta}{3\varepsilon_0 r^3} \qquad \text{...(4)}$$

$$(E_\theta)_0 = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} = \frac{\left(\frac{4}{3}\pi R^3 P\right)\sin\theta}{4\pi\varepsilon_0 r^3.} = \frac{PR^3\sin\theta}{3\pi\varepsilon_0 r^3}. \qquad \text{...(5)}$$

Now let us find the potential and field at points inside the polarised sphere. For this, we first find the potential on the surface of the polarized sphere. This is obtained by putting $r = R$ in eqn. (3). Thus,

$$\phi_{(R,\theta)} = \frac{PR\cos\theta}{3\varepsilon_0}. \qquad \text{...(6)}$$

Since $R\cos\theta = Z$, we see that the potential a point on the sphere depends only on its Z-coordinate :

$$\phi_{(R,\theta)} = \frac{PZ}{3\varepsilon_0}. \qquad \text{...(7)}$$

Equation (7) gives the potential at every point on the boundary of the region, inside which ϕ must satisfy Laplace's equation. Thus ensures that there can exist one and only one solution for ϕ_{in} and that it must be that given by equation (7) at the surface. Therefore ϕ_{in} may be taken as equal to that given by equation (7) satisfy the required boundary condition.

Thus,

$$\phi_{in} = \frac{PZ}{3\varepsilon_0}. \qquad ...(8)$$

The field inside the polarized sphere is now obtained as,

$$(E_x)_{in} = (E_y)_{in} = 0$$

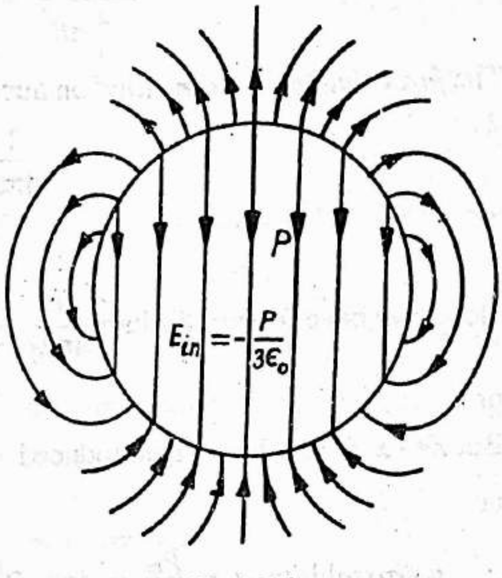
and

$$(E_z)_{in} = -\frac{\partial}{\partial z}\phi_{in} = -\frac{P}{3\varepsilon_0} \qquad ...(9)$$

As the direction of $\mathbf{P}$ is the only thing that distinguishes the Z-axis, we can write our result in more general form

$$\mathbf{E}_{in} = -\frac{\mathbf{P}}{3\varepsilon_0}. \qquad ...(10)$$

This is the macroscopic field $\mathbf{E}$ inside the polarized sphere, it remains constant throughout and points opposite to the direction of polarization. Fig. 2·36 shows the nature of lines of electric intensity inside and outside the uniformly polarized sphere.



(Fig. 2.36)

SOLVED EXAMPLES

Ex. 17. *Consider an electron of charge $-e$ moving in a circular orbit of radius a_0 about charge $+e$ in a field directed at right angles, to the plane of the orbit. Show that the polarizability α is approximately $4\pi\varepsilon_0 a_0^3$.*

Sol. When an atom, consisting of a charge $+e$ at the nucleus and an electron of charge $-e$ revolving around it in a circular orbit of radius a_0, is placed in an electric field, then the atom gets polarized and the electric field inside the atom becomes,

$$\mathbf{E}_{in} = -\frac{\mathbf{P}}{3\varepsilon_0},$$

where $\mathbf{P}$ is the density of polarization. The dipole moment of the polarized atom will be given by,

$$\mathbf{p} = \frac{4}{3}\pi a_0^3 . \mathbf{P}$$

since the polarizability α is given by relation,

$$\mathbf{P} = \alpha \mathbf{E}$$

where $\mathbf{E}$ is the average electric field acting on the atom, and this is equal to $\mathbf{E}_{in}$. Thus,

$$\frac{4}{3}\pi a_0^3 \mathbf{P} = \alpha \frac{\mathbf{P}}{3\varepsilon_0}$$

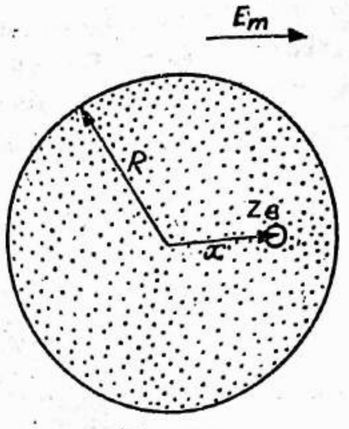
$$\alpha = 4\pi\varepsilon_0 a_0^3.$$

Ex. 18. *Assuming the negative charge in an atom to be distributed uniformly in a spherical volume of radius R around the nucleus, find an expression for its polarisability.*

Sol. Let Z be the atomic number.

The charge on nucleus = Ze. If $\mathbf{E}_m$ is the polarising field, then positive charge will be displaced through a distance x (say) from the centre of electron cloud along the direction of field due to the force $q\mathbf{E}_m = Ze\,\mathbf{E}_m$; while the electron cloud tends to restore the initial configuration. By Gauss theorem the negative charge cloud attracting the nucleus is that part of cloud which lies within the sphere of radius x. Assuming uniform density of electron cloud, the attracting electron cloud has charge

$$\frac{(Ze)\left(\frac{4}{3}\pi x^3\right)}{\frac{4}{3}\pi R^3} = \frac{Ze\,x^3}{R^3}.$$



(Fig. 2.37)

The *force due to electron cloud* on nucleus

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{(Ze)\,(Ze\,x^3/R^3)}{x^2} = \frac{1}{4\pi\varepsilon_0}\,\frac{(Ze)^2\,x}{R^3}.$$

Hence we have for equilibrium $\frac{1}{4\pi\varepsilon_0}\,\frac{(Ze)^2\,x}{R^3} = Ze\,E_m$

or

$$Zex = 4\pi\varepsilon_0 R^3 E_m.$$

But $Ze \cdot x$ = dipole moment induced = p_m (say) :

or

$$p_m = 4\pi\varepsilon_0 R^3 E_m$$

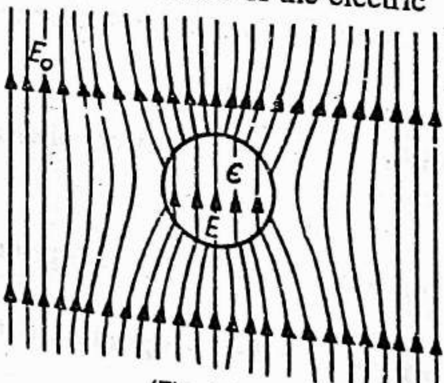
$\therefore$ *polarisability* $\alpha = \frac{p_m}{E_m} = 4\pi\varepsilon_0 R^3$.

2·20. Dielectric Sphere in a Uniform Field :

Let us consider a dielectric sphere of radius 'a' placed in a region of space containing an initially uniform electric field $\mathbf{E}_0$, (fig. 2·38). Let us assume dielectric to be linear isotropic, and homogeneous and to be characterized by the dielectric constant K. We take the origin of coordinate system at the centre of the sphere and the direction of $\mathbf{E}_0$ as the polar direction (Z-direction). Under the influence of the electric field, the dielectric sphere will be polarized uniformly, say the density of polarization is $\mathbf{P}$. Now the field inside and outside the dielectric sphere will not be equal to E_0. However, at points far away from the sphere the field will remain practically $\mathbf{E}_0$.

Now the field inside the dielectric sphere is the sum of the uniform applied field $\mathbf{E}_0$ and a field $\mathbf{E}'_{in}$ generated by the polarized matter itself

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'_{in}.$$



(Fig. 2.38)

The field $\mathbf{E}'_{in}$ inside the dielectric generated by the polarization is given by

$$\mathbf{E}'_{in} = -\frac{\mathbf{P}}{3\varepsilon_0} \qquad \text{...(2)}$$

[See eq. (10) of section 2·19]

so that effective field inside the polarized sphere is given by,

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{3\varepsilon_0}. \qquad \text{...(3)}$$

Now the popular relation between $\mathbf{P}$, the density of polarization and $\mathbf{E}$, the field acting on the dielectric, is

$$\mathbf{P} = \chi \mathbf{E} \qquad \left[\begin{array}{l} \chi = \text{electrical susceptibility} \\ K = 1 + \dfrac{\chi}{\varepsilon_0} \end{array}\right] \quad \ldots(4)$$
$$= \varepsilon_0 (K-1)\mathbf{E}$$

Substituting this value of $\mathbf{P}$ into eq. (3) we get

$$\mathbf{E} = \mathbf{E}_0 - \frac{\varepsilon_0 (K-1)}{3\varepsilon_0}\mathbf{E}$$

$$\mathbf{E} = \frac{3}{2+K}\mathbf{E}_0 . \qquad \ldots(5)$$

Because K is greater than one, the factor $\frac{3}{2+K}$ will be less than one, so the resultant field inside the dielectric is less than $\mathbf{E}_0$ and also that it is along the direciton of $\mathbf{E}_0$. The polarization is,

$$\mathbf{P} = \varepsilon_0 (K-1)\ \mathbf{E} = 3\varepsilon_0 \frac{(K-1)}{(K-2)}\mathbf{E}_0 . \qquad \ldots(6)$$

Similarly the field outside the dielectric sphere will be the sum of the uniform applied field of $\mathbf{E}_0$ and E'_{out} generated by the polarized matter itself :

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'_{out} . \qquad \ldots(7)$$

The field $\mathbf{E}'_{out}$ of a outside point due to the polarization of the dielectric is given by,

$$\mathbf{E}'_{out} = \frac{2\,Pa^3 \cos\theta}{3\varepsilon_0 r^3}\hat{r} + \frac{Pa\sin^2\theta}{3\varepsilon_0 r^3}\hat{\theta}$$

[Using eqs. (4) and (5) of section 2·19].

The resultant field at a outside point (r, θ) is thus,

$$\mathbf{E} = \mathbf{E}_0 + \frac{2\,Pa^3 \cos\theta}{3\varepsilon_0 r^3}\hat{r} + \frac{Pa^3 \sin\theta}{3\varepsilon_0 r^3}\hat{\theta} .$$

$$= \mathbf{E}_0 + \frac{2\,(K-1)\,a^3}{(K+2)\,r^3} E_0 \cos\theta .\, \hat{r} + \frac{(K-1)\,a^3}{(K+2)\,r^3} E_0 \sin\theta .\, \hat{\theta} . \qquad \ldots(8)$$

In fig. 2·38 the electric field inside and outside the dielectric sphere is also sketched.

2·21. Molecular Field in a Dielectric. The Clausius-Mossotti-Relation :

The electric field which is responsible for polarizing a molecule of the dielectric is called the molecular field E_m. This is the electric field at a molecular position in the dielectric ; it is produced by all external sources and by all polarized molecules in the dielectric with the exception of the one molecule at the point under consideration. It is clear that $\mathbf{E}_m$ need not be the same as the macroscopic electric field because this field is related to the force on a test charge which is large in comparison with molecular dimensions.

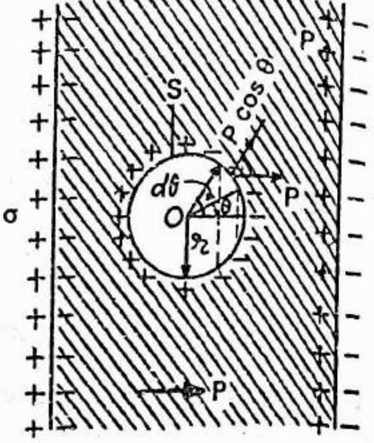
In order to compute the molecular field, we cut out a small piece of the dielectric, leaving a spherical cavity surrounding the point at which the molecular field is to be computed. The dielectric which is not left will be treated as a continuum from the macroscopic point of view. Now we put the dielectric back into the cavity, molecule by molecule, except for the molecule at the centre of the cavity where the molecular field is to be computed. The molecules which have just been replaced are to be treated, not as a continuum, but as individual dipoles. The above statement may be summarized, that for the calculation of molecular field, we regard the dielectric as consisting of two parts : (a) a sphere of radius r (centred at O, where the molecule at which the molecular field is to be computed, is located) within which the continuous nature of the dielectric must be taken into account, and (b) the rest of the dielectric slab, which is sufficiently distant from O, may be regarded as a continuous medium.

Let us suppose that the dielectric sample has been polarised by placing it in the uniform electric field between the plates of a parallel plate condenser (fig. 2·39). Suppose the density of polarization is uniform and that it is along the field which produces it.

The molecular field at O consists of four components :

$$\mathbf{E}_m = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3. \qquad ...(1)$$

E_0 is the field due to the charged condenser plates that would exist at O if the dielectric were absent. If the dimensions of the condenser plates are large compared with their separations, then E_0 is given by



(Fig. 2.39)

$$E_0 = \frac{\sigma}{\varepsilon_0} \qquad ...(2)$$

where σ is the surface charge density.

$\mathbf{E}_1$ is the depolarizing field that arises from the polarization charges on the surfaces of the dielectric that are in contact with the capacitor plates. Because the density of these bound charges on the surface next to the positive capacitor plate is $-P$ and $+P$ on the surface next to negative plate, $\mathbf{E}_1$ has the value,

$$\mathbf{E}_1 = -\frac{\mathbf{P}}{\varepsilon_0}. \qquad ...(3)$$

$\mathbf{E}_2$ is the field at O from the polarization bound charges on the surface S of the cavity which is formed when the spherical section of the dielectric is removed. This value of $\mathbf{E}_2$ will be the same as the field inside a uniformly polarized dielectric sphere having density of polarization as $-\mathbf{P}$. Thus,

$$\mathbf{E}_2 = \frac{\mathbf{P}}{3\,\varepsilon_0}. \qquad ...(4)$$

[See eq. (10) of section 2·19]

Alternatively, $\mathbf{E}_2$ may be computed independently as follows. The value of $\mathbf{E}_2$ is found by summing the contributions to the field at O of all elements of polarization charge, on the surface S, of the cavity. The bound charge density on the spherical surface S is given by the component of the polarization normal to S, that is

$$= -P\cos\theta.$$

Thus the charge on the surface element dS that lies between θ and $\theta + d\theta$ is,

$$= -P\cos\theta.\ 2\pi r\sin\theta.\ r\,d\theta.$$

Because of the symmetry of the arrangement, all components of the field that are not normal to the capacitor plates cancel one another. Therefore, the polarization charges on dS give rise to a field at O, in the direction of $\mathbf{P}$, of value,

$$dE_2 = \frac{(P\cos\theta)\,(\cos\theta)\,2\pi r^2\sin\theta\,d\theta}{4\pi\varepsilon_0 r^2}.$$

The value of E_2 is then given by,

$$E_2 = \int_0^{\pi} dE_2$$

$$= \frac{2\pi P}{4\pi\varepsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$$

$$= \frac{P}{3\varepsilon_0} \qquad ...(5)$$

or, vectorically,
$$E_2 = \frac{\mathbf{P}}{3\,\varepsilon_0}.$$

The last term in equation (1), *i.e.* $\mathbf{E}_3$, is the field at 0 due to the electric dipoles inside the spherical section. There are a number of important cases for which this term vanishes. If there are a great many dipoles in the cavity, and if these are oriented parallel but randomly distributed in position, and also it there are no correlation between the positions of the dipoles, then $\mathbf{E}_3 = 0$. This is the situation which might prevail in a gas or a liquid. Similarly, if the dipoles in the cavity are located at the regular atomic positions of a cubic crystal, then again $\mathbf{E}_3 = 0$. In the general case, $\mathbf{E}_3$ is not zero, and if the material contains several species of molecules, $\mathbf{E}_3$ may differ at the various molecular positions. However, we restrict our discussion on the rather large class of materials in which $\mathbf{E}_3 = 0$.

Combining equations (2), (3) and (4), we find the molecular field,

$$E_m = \frac{\sigma}{\varepsilon_0} - \frac{P}{\varepsilon_0} + \frac{P}{3\,\varepsilon_0} \qquad ...(6)$$

But, $D = \sigma$, the free charge density on the condenser plates

$$\therefore \quad E_m = \frac{D-P}{\varepsilon_0} + \frac{P}{3\,\varepsilon_0} \qquad ...(7)$$

or
$$\mathbf{E}_m = \mathbf{E} + \frac{\mathbf{P}}{3\,\varepsilon_0} \qquad ...(8)$$

where, $\mathbf{E} = (\mathbf{D}-\mathbf{P})/\varepsilon_0$, is the macroscopic electric field in the dielectric, since $\mathbf{D} = \varepsilon_0\,\mathbf{E} + \mathbf{P}$.

Eq. (8) gives the electric field E_m that acts upon a single molecule of the dielectric. Under the action of this field, the molecule displays an electric moment $\mathbf{p}$ in the direction of the field which is dependent upon the field strength, *i.e.*

$$\mathbf{p} = \alpha\,\mathbf{E}_m \qquad ...(9)$$

where α is the *molecular polarizability*. It there are n molecules per unit volume of the dielectric, then the polarization (electric dipolement/volume) is obtained as,

$$\mathbf{P} = n\,\mathbf{p} = n\alpha\,\mathbf{E}_m$$
$$= n\,\alpha\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right) \qquad ...(10)$$

This equation may be rewritten in terms of the dielectric constant K, since $P = (K-1)\,\varepsilon_0\,\mathbf{E}$. In this way equation (10) becomes,

$$\frac{n\alpha}{3\,\varepsilon_0} = \frac{(K-1)}{(K+2)} \qquad ...(11)$$

If the molecular weight of the substance is M and its density ρ, and N the Avogadro number, then $n = \frac{N_\rho}{M}$ and equation (11) becomes

$$\frac{N\alpha}{3\,\varepsilon_0} = \frac{M}{\rho}\cdot\frac{(K-1)}{(K+2)} \qquad ...(12)$$

The quantity $N\alpha/3\varepsilon_0$ is sometimes called the *molar polarizability*. This equation indicates that molecular polarizability of a dielectric of known density can be determined by measuring its dielectric constant. Again since α is approximately proportional to the cube of the radius of the molecule, so if K is found and n is known for a gas at a given temperature and pressure, the approximate diameter of the atom may be found using eq. (11) for monatomic gases. For the monatomic gases the result are in agreement the kinetic theory methods of determining atomic dimensions.

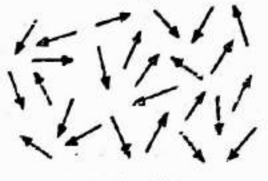
Eq. (11) or (12) is known as the *Clausius-Mossotti relation.* Remember again, this relation is applicable to those dielectrics for which $E_3 = 0$ such as gases and non-polar liquids.

2·22. Polar Molecules. The Langevin-Debye Formula :

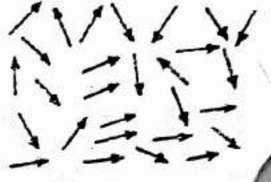
Consider a gaseous or liquid dielectric polar molecules each molecule having a permanent dipole moment **p.** In the absence of an electric field a macroscopic piece of the polar dielectric is not polarized. since the individual dipoles point in random directions as shown in fig. 2·40. (a), so that net moment per unit volume is zero. But when an electric field is applied, two things happen; first, there is an induced dipole moment which gives just the same kind of electronic polarizability we found in the previous section for a non-polar molecule. For the moment, we shall ignore this induced dipole contribution, but its effect will be added later on. Second, the electric field produces torques on individual dipoles which tend to align them with the field so that a net moment per unit volume results, [fig. 2.40 (b)]. If the field is strong enough, the dipoles may be completely aligned, and the polarization achieves the saturation value,

$$\mathbf{P}_S = n\mathbf{p} \qquad ...(1)$$

where n is the number of molecules per unit volume.



(a) No Field



(b) Field E_{in}

(Fig. 2.40)

At field strengths normally encountered, the polarization of a polar dielectric is usually far from its saturation value and if the temperature of the dielectric is raised the polarization becomes even smaller. This lack of complete alignment of dipoles is due to the thermal energy of the molecules, which tends to produce random dipole orientations. The average effective dipole moment per molecule may be calculated by means of a principle from statistical mechanics which states that in a state of thermal equilibrium the probability of any one molecule having potential energy W due to its position or orientation in the field is proportional to $e^{-W/kT}$, where k is Boltzmann's constant and T is the absolute temperature. The potential energy of a polar molecule of moment **p** in an electric field $\mathbf{E}_{in}$ is

$$W = -\,\mathbf{p}\cdot\mathbf{E}_{in} = -\,pE_{in}\cos\theta \qquad ...(2)$$

where θ is the angle between **p** and the electric field $\mathbf{E}_{in}$. Therefore the number of dipoles per unit volume per unit solid angle making an angle θ with the direction of the electric field may be expressed as

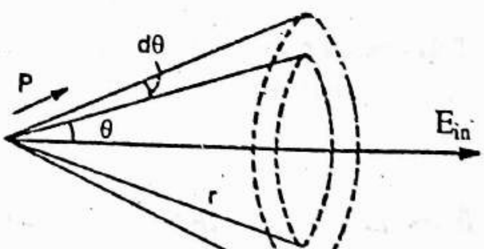
$$n(\theta) = Ce^{-(-p\,E_{in}\cos\theta)/kT} = Ce^{p\,E_{in}\cos\theta/kT} \qquad ...(4)$$

where C is a constant of proportionality and its value can be determined by the fact that the total number of dipoles per unit volume having all possible values of θ is just equal to n.

The conical solid angle (fig. 2.41) included between θ and $(\theta + d\theta)$ is equal to $2\pi r\sin\theta\; r\,d\theta/r^2 = 2\pi\sin\theta\,d\theta$, so that the number of dipoles per unit volume lying in this conical solid angle is,

$$n(\theta)\,d\theta = dn = Ce^{pE_{in}\cos\theta/kT}\,2\pi\sin\theta.d\theta \qquad ...(4)$$

Hence the total number of molecules per unit volume over all possible value of θ is n, we have



(Fig. 2.40)

$$\int dn = n = \int_0^{\pi} Ce^{pE_{in}\cos\theta/kT}\,2\pi\sin\theta\,d\theta$$

or

$$C = \frac{n}{2\pi\int_0^{\pi} e^{pE_{in}\cos\theta/kT}\sin\theta\,d\theta} \qquad ...(5)$$

The component of the dipole moment **p** in the direction of the field of a molecule, whose moment lies in the conical solid angle $2\pi \sin\theta \, d\theta$ of fig. 2.41, is $p\cos\theta$. The net contribution to the polarization perpendicular to the field will vanish from symmetry. Since there are $n(\theta)\, d\theta$ molecules per unit volume in the solid angle, we get the total dipole moment per unit volume as,

$$P = \int_0^{\pi} n(\theta)\, d\theta\, p \cos\theta$$

$$= \int_0^{\pi} C e^{pE_{in}\cos\theta/kT}\, 2\pi \sin\theta\, p\cos\theta\, d\theta$$

$$= np \frac{\int_0^{\pi} e^{pE_{in}\cos\theta/kT} \sin\theta . \cos\theta . d\theta}{\int_0^{\pi} e^{pE_{in}\cos\theta/kT} \sin\theta . d\theta} \qquad \text{[using equation (5)]} \qquad ...(6)$$

Let us write $\frac{pE_{in}}{kT} = x$, and $\cos\theta = y$, then equation (6) becomes,

$$p = np \frac{\int_{-1}^{+1} e^{xy}\, y dy}{\int_{-1}^{+1} e^{xy}\, dy}$$

$$= np \frac{\left[\frac{ye^{xy}}{x} - \frac{e^{xy}}{x^2} \right]_{-1}^{+1}}{\left[\frac{e^{xy}}{x} \right]_{1}^{+1}}$$

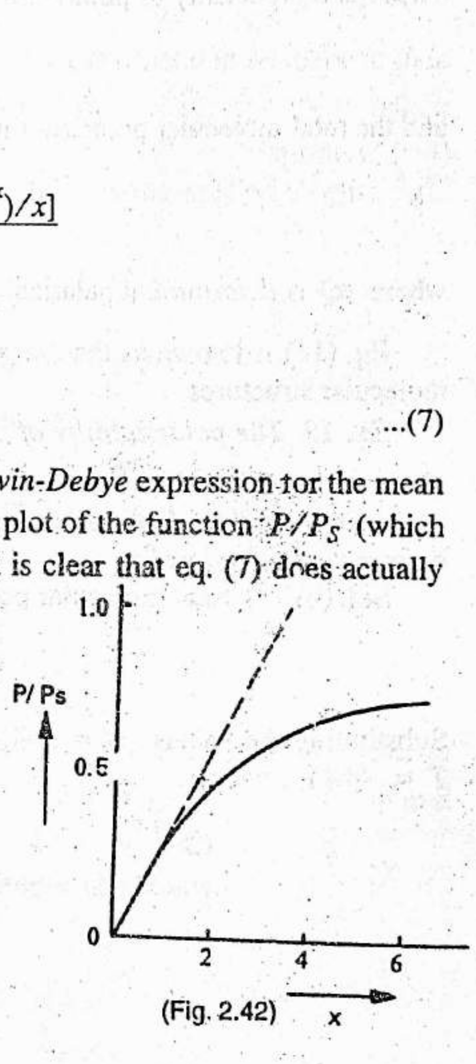
$$= np \frac{[(e^{x} + e^{-x}) - (e^{x} - e^{-x})/x]}{(e^{x} - e^{-x})}$$

$$= np \left[\frac{(e^{x} - e^{-x})}{(e^{x} - e^{-x})} - \frac{1}{x} \right]$$

$$= P_S \left[\coth x - \frac{1}{x} \right] \qquad ...(7)$$

where $P_S = np$, is the saturation value of polarization. This is the *Langevin-Debye* expression for the mean effective polarization of a polar gas in a field E_{in} at a temperature T. A plot of the function P/P_S (which is known as Langevin-function) is given in fig. 2.42. From this figure it is clear that eq. (7) does actually give a saturation effect at large field strengths. For value of the field and temperature ordinarily encounterd. x is small and the curve is linear. The molecular dipole moment of most polar materials is such that $x \ll 1$ for a full range of field strengths, even for those approaching the dielectric strength of the material, so long as the temperature is above about 250 K. Thus a dielectric material containing polar molecules is, in general, linear.

Since it is the linear region of eq. (7) which is important, it is appropriate to expand coth x in a power series of x and keep only the important terms. Thus eq. (7) becomes



(Fig. 2.42)

$$P = P_S\left[\left(\frac{1}{x}+\frac{x}{3}\right)-\frac{1}{x}\right] = P_S\,\frac{x}{3}$$

$$= \frac{np^2}{3kT}\,E_{in}\quad\left(\begin{array}{l}\text{since } x = \dfrac{pE_{in}}{kT}\\ \text{and } P_S = np\end{array}\right)$$

or

$$\mathbf{P} = \frac{np^2}{3kT}\,\mathbf{E}_{in} \qquad ...(8)$$

From this equation it is clear that the polarization is proportional to the field E_{in}. Also, as we expect, the polarization depends inversely on the temperature, because at higher temperatures there is more disalignment by collisions. This $1/T$ dependence is called Curie's law.

Since $\mathbf{P} = n\alpha\,\mathbf{E}_{in}$, a comparison of this with eq. (8) yields for the polarizability α (*i.e.* molecular dipole moment per unit polarizing field) as,

$$\alpha = \frac{p^2}{3kT} \qquad ...(9)$$

The result has been derived by neglecting induced dipole moments, and represents what we call as ***orientational polarizability***. Induced dipole effects produce further dipole moment due to deformation of the polar molecules. Since the induced electric moment is always parallel to the field and is proportional to its magnitude, the polarization due to this effect is given by

$$\mathbf{P} = n\alpha_d\,\mathbf{E}_{in} \qquad ...(10)$$

Thus the total density of polarization of a polar dielectric is given by the sum of eqs. (8) and (10),

$$\mathbf{P} = n\left(\alpha_d + \frac{p^2}{3kT}\right)\mathbf{E}_{in} \qquad ...(11)$$

and the total molecular polarizability is,

$$\alpha = \alpha_d + \frac{p^2}{3kT} \qquad ...(12)$$

where α_d is *deformation* palarizability and, $\frac{p^2}{3kT} = \alpha_0$ is the *orientational* polarizability.

Eq. (12) is known as the *Langevin-Debye equation*. and it has been of great importance in interpreting molecular structures.

Ex. 19. *The polarizability of* NH_3 *molecule is found approximately by the measurement of dielectric constant as* $2{\cdot}42 \times 10^{-39}$ *coulomb*^{2}*—m/Newton and* $1{\cdot}74 \times 10^{-39}$ *coulomb*^{2}*-m/Newton at 309 K and 448 K respectively. Calculate for each temperature the polarizability due to permanent dipole moment and due to deformation of molecules.* **(Rohilkhand 1990, Agra 1970)**

Sol. (b) The total molecular polarizability of a polar gas is given by,

$$\alpha = \alpha_d + \frac{p^2}{3kT}.$$

Substituting the values : $\alpha = 2{\cdot}42 \times 10^{-39}$ coul^2-m/N.; at T = 309 K and $\alpha = 1{\cdot}74 \times 10^{-39}$ coul^2-m/N., at $T = 448$ K., we get,

$$2{\cdot}42 \times 10^{-39} = \alpha_d + \frac{p^2}{3k \times 309}$$

and
$$1{\cdot}74 \times 10^{-39} = \alpha_d + \frac{p^2}{3k \times 448}$$

From these equations, we get,

$$\frac{2{\cdot}42 \times 10^{-39} - \alpha_d}{1{\cdot}74 \times 10^{-39} - \alpha_d} = \frac{448}{309}$$

or
$$(448 - 309)\,\alpha_d = (448 \times 1{\cdot}74 - 309 \times 2{\cdot}42) \times 10^{-39}$$

or
$$139\,\alpha_d = 31{\cdot}74 \times 10^{-39}$$

or
$$\alpha_d = \frac{31{\cdot}74 \times 10^{-39}}{139} = 0{\cdot}228 \times 10^{-39}\ \text{coul}^2\text{–m/N}$$

This is the value of deformation polarizability.

The value of polarizability due to permanent dipole moment (*i.e.*, the orientational polarizability) at 309K is given by

$$\begin{aligned}\alpha_0 &= (2{\cdot}42 - 0{\cdot}228) \times 10^{-39}\\ &= 2{\cdot}192 \times 10^{-39}\ \text{coul}^2\text{–m/N.}\end{aligned}$$

and that at 448K is given by,

$$\begin{aligned}\alpha_0 &= (1{\cdot}74 - 0{\cdot}228) \times 10^{-29}\\ &= 1{\cdot}512 \times 10^{-39}\ \text{coul}^2\text{–m/N.}\end{aligned}$$

Ex. 20. *Calculate the induced dipole moment per unit volume of helium gas placed in an electric field of* 6×10^5 V/m. *The molecular polarisability of helium is* $2{\cdot}33 \times 10^{-41}$ *farad–*m^2 *and the density of helium is* $20{\cdot}6 \times 10^{25}$ *molecules/*m^3. **(Kumaun 1998)**

Solution. If p is average dipole moment induced in each molecule and E is the applied electric field strength, then

$$p = \alpha E$$

Given $\alpha = 2{\cdot}33 \times 10^{-41}$ farad m^2

Electric field
$$E = 6 \times 10^5\ \text{V/m}$$

∴
$$\begin{aligned}p &= \alpha E = 2{\cdot}33 \times 10^{-41} \times 6 \times 10^5\\ &= 1{\cdot}4 \times 10^{-35}\ \text{coul} - \text{m}\end{aligned}$$

∴ Number of helium molecules per unit volume

$$n = 20{\cdot}6 \times 10^{25}\ \text{m}^{-3}$$

∴ Induced dipole moment per unit volume

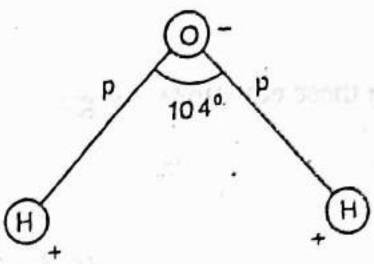
$$\begin{aligned}P &= np = 20{\cdot}6 \times 10^{25} \times 1{\cdot}4 \times 10^{-35}\\ &= \mathbf{2{\cdot}88 \times 10^{-9}\ coul/m^2}\end{aligned}$$

Ex. 21. *Calculate the total dipole moment of a water-molecule assuming that all the ten electrons in the molecule circulate symmetrically about the oxygen atom. Given that* OH *distance is* $0{\cdot}96 \times 10^{-10}$ m *and that the angle between two* O—H *bonds is* 104°. (*charge on electron* $e = -1{\cdot}6 \times 10^{-19}$ C)

Solution. A water molecule consists of one oxygen (O) and two Hydrogen (H) atoms.

The dipole moment of each (O—H) dipole $p = q.d.$

$$= 1{\cdot}6 \times 10^{-19} \times 0{\cdot}96 \times 10^{-10} = 1{\cdot}536 \times 10^{-28} \text{ C-m}$$



(Fig. 2.43)

The net dipole moment of H_2O molecule is

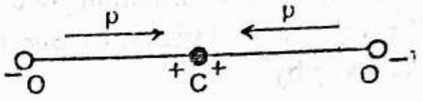
$$P = \sqrt{p_1^2 + p_2^2 + 2p_1p_2 \cos\theta}$$

As $\quad p_1 = p_2 = p$

$$\therefore\ P = \sqrt{p^2 + p^2 + 2pp\cos 104^\circ} = p\sqrt{2 + 2\cos 104^\circ}$$
$$= p\sqrt{2\,(1 + \cos 104^\circ)}$$
$$= p \cdot \sqrt{2\,(2\sin^2 52^\circ)} = p \cdot (2\sin 52^\circ)$$
$$= 2p\sin 52^\circ = 2 \times 1{\cdot}536 \times 10^{-28} \times 0{\cdot}6156$$
$$= \mathbf{1{\cdot}89 \times 10^{-38}\ \textbf{coul} - \textbf{metre}}$$

Ex. 22. *What is the dipole moment of* CO_2 *molecule* (CO_2 *molecule is symmetric linear molecule).*

Solution. CO_2 molecule is a symmetric linear molecule. It may be supposed to be a combination of two (C—O) dipoles having equal and opposite dipole moments. Therefore net dipole moment of CO_2 molecule is zero.



(Fig. 2.44)

QUESTIONS AND PROBLEMS

(A) QUESTIONS

1. (a) State and prove Gauss theorem.
   (b) Use the theorem of find expressions for the field strength due to (i) a uniformly charged, (ii) an infinite uniform flat sheet of charge. **(Rohilkhand 1991, 77; Agra 76, 74, 72; Kanpur 72; Meerut 89)**
2. Extablish Gauss's theorem for an electrostatic field. Deduce Laplace's and Poisson's equations. **(Meerut 1985)**
3. (a) A circular disc of radius R has uniform surface charge density σ. Find the the electric field at a point on the axis of the disc at a distance z from the plane of the disc.
   (b) A right circular cylinder of radius R and height L is oriented along Z-axis. It has a non-uniform volume density of charge given by $\rho(z) = \rho_0 + \beta z$ with reference to an origin at the centre of the cylinder. Find the electric field at the centre of cylinder.

$$\left[\ \textbf{Ans.}\ \text{(a)}\ E = \left(\frac{\sigma}{2\varepsilon_0}\right)\left\{1 - \frac{z}{\sqrt{(z^2 + R^2)}}\right\}\right.$$

$$\left.\text{(b)}\ E = \left(\frac{\beta}{2\varepsilon_0}\right)\left[\frac{L}{2}\left\{\frac{L}{2} - \sqrt{\left(\frac{L^2}{4} + R^2\right)}\right\} + R^2 \log\left\{\frac{L}{2R} + \sqrt{\left(1 + \frac{L^2}{4R^2}\right)}\right\}\right]\ \right]$$

4. (a) Derive an expression for the electric field due to an electric dipole and show that the magnitude of the electric field of an electric dipole decreases monotonically as the angle θ between the dipole moment and radius to the observer increases from 0 to $\pi/2$ radians, when the distance of the observer is constant. **(Agra 1989, 76, 72)**
5. Explain what is meant by a dipole, a quadrupole and an octupole. Prove that the potential due to an arbitrary charge distribution of finite extent at a large distance from the distribution can be always expressed as a sum of multipole potentials with coefficients which represent various order moments of the charge distribution. Explain how you can generate multipole potentials from that of a single charge. **(Rohilkhand 1990, 86; Meerut 76)**
6. Define electric monopole, dipole, quadrupole and octopole. Show that the potential of charge distribution over a finite volume can always be expressed as a series of multipole potentials, referred of a suitable origin of coordinates. Explain how can you generate multipole potentials from that of a single charge. **(Rohilkhand 1988)**
7. Show that the potential produced by an arbitrary charge distribution is the same as the sum of the potentials due to (i) a point charge equal to the net charge of the distribution, (ii) a point dipole, with a dipole moment equal to the dipole moment of the distribution, (iii) a point quadrupole with a quadrupole moment of the distribution and so on, all located at the origin. Describe the relative importance of the contributions due to (i), (ii) and (iii). **(Rohilkhand 97, 96, 94)**

8. Show that the potential at any external point due a charge distribution can be expressed as the contribution of the moments of monopole, dipole, quadrupole etc. **(Rohilkhand 1999, Agra 1998)**

9. What do you understand by electric dipole and quadrupole moment ? Obtain expressions for electric potential and field at a point in space due to (i) a dipole and (ii) a quadrupole. **(Meerut 1988, 83)**

10. (a) What do you mean by electric monopole, a dipole, quadrupole and octopole ? Explain with necessary diagrams.

    Or Explain the meaning of point multipoles and multipole potentials.

    (b) Show that the potential of a charge distribution over a finite volume can always be expressed as a series of multipole potentials, referred to a suitable origin of coordinates. **(Rohilkhand 1986; Agra 92)**

11. An electric quadrupole is formed by a charge $-q$ at the origin and charge $+q$ at the points $(\pm\ a, 0, 0)$. Show that the potential ϕ at a distance r large compared with a is approximately given by

$$\phi = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qa^2(3\cos^2\theta - 1)}{r^3}$$

where θ is the angle between r and the line through the charge and that the value of electric field E on the axis of quadrupole for point distant r from its centre (assume $r >> a$) is given by

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{6\,qa^2}{r^4}.$$

12. Calculate the torque exerted by one dipole on another dipole. Also derive the interaction energy between the two dipoles. **(Rohilkhand 1995)**

13. (a) Give multipole expansion of potential of charge distribution. **(Rohilkhand 1984)**

    (b) Show that the interaction of two dipoles of moments $\mathbf{p_1}$ and $\mathbf{p_2}$ is given by

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p_1}\cdot\mathbf{p_2}}{r^3} - \frac{3}{r^5}(\mathbf{p_1}\cdot\mathbf{r})(\mathbf{p_2}\cdot\mathbf{r})\right]$$

    where r is the radius vector joining the centres of the two dipoles. **(Rohilkhand 1997, 83)**

14. (a) Establish a relation between electric field strength **E**, polarisation **P** and electric induction **D**. **(Agra 1993)**

    (b) Deduce an expression for the polarisability α of a hydrogen atom assuming its negative charge to be uniformly distributed in spherical volume of radius 'a' around the proton. **(Rohilkhand 1993; Agra 90, 75)**

15. (a) A uniform electric field E_0 is set up in a medium of specific inductive capacity K. Show that the field inside a spherical cavity in the medium is given by

$$\mathbf{E} = \frac{3K}{2K+1}\mathbf{E_0}$$

    (b) Show that the field inside a cylindrical cavity in a dielectric of constant K is $\frac{2K}{K+1}$, when the axis of cylinder is at right angles to the uniform field $\mathbf{E_0}$.

16. A sphere of a material of dielectric constant K is placed in air where the field is originally of strength $\mathbf{E_0}$. Show that the field at external points can be determined by regarding it as a superposition of the original field and that due to a doublet at the centre of sphere of moment

$$4\pi\varepsilon_0 \frac{(K-1)\,a^2}{(K+2)}\mathbf{E_0}.$$

a being radius of the sphere.

17. What are polar and non-polar molecules of a dielectric ? Obtain Langevin's expression for polarizability when a gaseous dielectric is subjected to an electrostatic field.

    What is the effect of induced polarisation ? **(Rohilkhand 1996; Agra 85, 72)**

18. Discuss Langevin's theory of dielectric polarisation. **(Rohilkhand 1992)**

19. (a) Discuss the possible ways in which one can define the field inside a material medium placed in an electrostatic field of given strength. Show that the cavity definition of this field depends on the sphere of the cavity.

    (b) Define the polarizability of a material and derive the Clausius-Mossotti relation connecting this quantity with the dielectric constant of the material. **(Meerut 1996, 88)**

20. (a) Find the general expression for the interaction energy of two dipoles.

    (b) What is dielectric polarisation ? Deduce Clausius-Mossotti relation. Point out its limitations.

(Rohilkhand 1982, 77)

21. Derive an expression for the total field acting on a molecule inside an isotropic dielectric medium. Hence show that $\frac{K-1}{K+2}=\frac{N\alpha}{3\varepsilon_0}$

where $K=$ dielectric constant, $\alpha=$ polarisability and $N=$ number of molecules per unit volume. **(Rohilkhand 1992)**

22. Derive and discuss Clausius Mossotti relation. How is it subjected to experimental test ? Briefly give its modification. **(Kanpur 1997; Agra 81)**

Describe the nature of electrostatic field in a material medium. Derive the Clausius Mossotti relation and interpret it. **(Agra 93)**

23. (a) Derive an expression for the effective field (E_{eff}) inside a dielectric medium placed in an external electric field E.

(b) Define polarizability. Show how the specific inductive capacity is related with the polarizability of molecules and its dipole moment. **(Agra 1994; Rohilkhand 85, 79)**

24. Discuss electrical polarization. Show that $\mathbf{E}_{eff}=\mathbf{E}+\frac{\mathbf{P}}{3\varepsilon_0}$ where the symbols have their usual meanings. **(Rohilkhand 1997, 84)**

25. A dielectric sphere is placed in an uniform electrostatic field. Draw a neat diagram showing a distortion produced in the field. Calculate the electric field outside and inside the dielectric sphere. **(Rohilkhand 1985, 79)**

26. Obtain the expression for potential inside and outside of a dielectric sphere placed in a uniform electric field. Discuss the nature of the field. **(Agra 1983)**

27. What are polar and non-polar molecules ? Give some examples. Derive Clausius-Mossotti relation for the permittivity of such molecules. How was this relation utilised by Debye to study molecular structure. **(Meerut 1991)**

28. Write short notes on **(Rohilkhand**
    (i) Dielectric polarisation
    (ii) Clausius Mossotti relation.

(B) PROBLEMS

1. Charges $-2q$, $+q$ and $+q$ are placed at the vertices of an equilateral triangle. Calculate the total dipole moment vector (p) of this system.

[**Ans.** If equilateral triangle is an $x-y$ plane then taking $+q$, $+q$ line as X-axis, the dipole moment vector p is along y-axis of magnitude $p=1{\cdot}722\ qa$, a being the side of triangle.

2. A parallel plate condenser having plate separation 1·4 cm, conductor surface of area 100 cm^2, is charged to a potential difference of 100 volts with no dielectric present, between the conductor plates. Now a dielectric slab of thickness 0·5 cm and dielectric constant $K=7{\cdot}0$ is introduced between the conductor plates calculate **E**, **D** and **P** in (a) the dielectric (b) the air gap.

[**Ans.** (a) $\mathbf{E}=1{\cdot}43\times10^{3}$ volt/metre, $\mathbf{D}=6{\cdot}9\times10^{-2}\ \text{coul/m}^2$ and $\mathbf{P}=7{\cdot}5\times10^{-3}\ \text{coul/m}^2$.

(b) $\mathbf{E}=1{\cdot}000\times10^{4}$ volt/m, $\mathbf{D}=8{\cdot}9\times10^{-6}\ \text{coul/m}^2$, $\mathbf{P}=0$].

3. The density of polarisation in a uniformly polarised carbon is $4{\cdot}8\times10^{-4}\ \text{coul/m}^2$. Calculate the distance of separation of the centre of cloud of electrons constituting a carbon atom from the centre of its positive charge. The density of carbon is $3{\cdot}5\times10^{3}\ \text{kg/m}^2$ and Avogadro number $N=6\times10^{26}$ per kg. mole.

[**Ans.** $2{\cdot}38\times10^{-15}$ m.]

4. For a drop of water of radius 0·1 cm, the dipole moment for each molecule is $6{\cdot}0\times10^{-36}$ coul-m. Calculate the electric polarisation of the drop and the electric field strength inside and 10 cm away from the drop in the direction of polarisation. Avagadro number $N=6\times10^{26}$ per kg. mole.

[**Ans.** $P=2\times10^{4}\ \text{coul/m}^2$, $E_{in}=7{\cdot}68\times10^{9}$ volt/m, $E=4{\cdot}61\times10^{4}$ volt/m.]

3

Force and Energy Relations in Electrostatic Field

3.1. Introduction

Many problems of mechanics are greatly simplified by means of energy cosiderations. Hence when the mechanical behaviour of an electrical system is to be studied, it is beneficial to use energy methods. In electrostatics there are *two* different but essentially equivalent ways of considering the total energy of the field. (i) *method of action at a distance* considers only the charge and the forces operating between them (ii) *method of direct action* locates the energy in the intervening space too, and implies that wherever an electric field is present there is also a certain continuous distribution of energy. The method of direct action becomes apparent in the case of rapidly-varying fields, for it leads simply and naturally to the existence of electromagnetic waves.

In general the energy of a system of charges, just like that of any other mechanical system, may be divided into its potential and kinetic contributions. *Under static conditions,* however, the entire energy of the charge system exists as potential energy. Hence we are particularly interested in the potential energy arising from electrical interaction fo the charges named as *electrostatic energy.*

In section 2·3 we have shown that the electrostatic energy of a point charge is closely related to the electrostatic potential at the position of the point charge. If q is positive test charge, then the work done on the charge in moving it from position '1' to position '2' is'

$$W = \int_{r_1}^{r_2} \mathbf{F}_m \cdot d\mathbf{r} = -q \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= q \int_{r_1}^{r_2} \text{grad}\, \phi \cdot d\mathbf{r} = q\,(\phi_2 - \phi_1). \qquad \text{...(1)}$$

Here mechanical force $\mathbf{F}_m$ has been chosen so as to balance exactly the electrical force $q\mathbf{E}$ along the path. Under these conditions equation (1) represents the change in electrostatic energy of the charge over the path interval from position '1' to position '2'.

Similar arguments may be applied to more complicated systems of charges. For example the electrostatic energy of an arbitrary charge distribution may be calculated as the work done in assumbling this distribution of charge without imparting to other forms of energy. In this chapter we shall evaluate energy and force due to charges in terms of fields.

3.2. Electrostatic Field Energy in Free Space and in Dielectric

Let us first calculate the potential energy of a group of n point charges. The state of zero potential energy is when all the point charges are infinitely separated from one another. The total potential energy of a system of n point charges may be obtained by calculating the work done to assemble the charges, bringing in one at a time.

Consider a point charge q, at infinite distance from the point charge q_1. If the charge q_1 is kept stationary, then work required to bring the charge q_2 from infinity to a distance $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ from charge q_1 is

$$\Delta W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}. \qquad ...(1)$$

Now let us suppose that a third charge q_3 is brought from infinity at a point whose distance from charge q_1 and q_2 are r_{13} and r_{23} respectively. Then work required to bring charge q_2 is

$$\Delta W_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}}. \qquad ...(2)$$

Therefore the *total work done to assemble this arrangement of three charges initially at infinite distance from one another is* the sum of ΔW_2 and ΔW_3, namely

$$W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}}. \qquad ...(3)$$

Similarly we may write expressions for work required to bring in the fourth charge q_4, fifth charge q_5 etc. Thus the work required to assemble a system of n charges initially at infinite distance from one another *i.e.*, *total electrostatic energy of the assembled n-charge system* may be written as

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \dots + \frac{q_i q_j}{r_{ij}} + \dots\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}{}' \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}} \qquad ...(4)$$

where the prime on the second summation means that the terms $i = j$ are to be omitted and the factor $\frac{1}{2}$ occurs before the double sum on account of the fact that every combination of i and j occurs twice in taking the double sum.

Now
$$\sum_{j=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_{ij}} = \phi_i \qquad ...(5)$$

signifies the electrostatic potential at position occupied by the ith point charge, caused by all other charges.

Thus the total electrostatic energy of the system of n point charges may be expressed as

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i \phi_i. \qquad ...(6)$$

If the point charges had been assembled in a *linear dielectric medium* instead of vacuum, then the absolute permittivity $\varepsilon = \varepsilon_0 K$ would replace ε_0 in equation (1) to (5) but equation (6) would remain unchanged.

Now let us calculate the *electrostatic energy of an arbitrary charge distribution* with volume charge density ρ and surface charge density σ. Some of the charge may reside on the surface of the conductors if we explicitly assume that there are conductors in the system. In addition, let us assume that the *dielectrics* in the system, if present, are *linear* : this restriction is necessary in order that the work expended in bringing

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Therefore the *total work done to assemble this arrangement of three charges initially at infinite distance from one another is* the sum of ΔW_2 and ΔW_3, namely

$$W = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{r_{23}}. \qquad ...(3)$$

Similarly we may write expressions for work required to bring in the fourth charge q_4, fifth charge q_5 etc. Thus the work required to assemble a system of n charges initially at infinite distance from one another *i.e.*, *total electrostatic energy of the assembled n-charge system* may be written as

$$U = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \ldots + \frac{q_i q_j}{r_{ij}} + \ldots \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n}{}' \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{r_{ij}} \qquad ...(4)$$

where the prime on the second summation means that the terms $i = j$ are to be omitted and the factor $\frac{1}{2}$ occurs before the double sum on account of the fact that every combination of i and j occurs twice in taking the double sum.

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$$\sum_{j=1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_{ij}} = \phi_i \qquad ...(5)$$

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If the point charges had been assembled in a *linear dielectric medium* instead of vacuum, then the absolute permittivity $\varepsilon = \varepsilon_0 K$ would replace ε_0 in equation (1) to (5) but equation (6) would remain unchanged.

Now let us calculate the *electrostatic energy of an arbitrary charge distribution* with volume charge density ρ and surface charge density σ. Some of the charge may reside on the surface of the conductors if we explicitly assume that there are conductors in the system. In addition, let us assume that the *dielectrics* in the system, if present, are *linear* : this restriction is necessary in order that the work expended in bringing the system to its final charged state shall be independent of the way in which this final state is reached. Let us further assume that we assemble the charge distribution by bringing in charge increments δq from a reference potential $\phi_A = 0$. If the charge distribution is partly assembled and the potential at a particular point in the system is $\phi'(x, y, z)$, then the work required to place δq at this point is

$$\delta W = \phi'(x, y, z)\, \delta q\,. \qquad ..(7)$$

The charge increment δq may be added to a volume element Δv located at point (x, y, z), such that $\delta q = \delta\rho\, \Delta v$ or δq may be added to a surface element Δa located at this point (x, y, z) such that $\delta q = \delta\sigma\, \Delta a$. Then the *total work done in assembling the entire charge distribution or the total electrostatic energy of the assembled charge distribution* will be obtained by summing contributions of the form (7). For convenience of calculation, let us consider particular assembly procedure in which all parts of the system are brought to their final charge in concert *i.e.*, at any instant of time during the charging process all charge densities will be at the same fraction of their final values. Let this fraction be α. If the final values of charge densities are given by the function $\rho(x, y, z)$ and $\sigma(x, y, z)$, then the charge densities at any arbitrary time are $\alpha\rho(x, y, z)$ and $\alpha\sigma(x, y, z)$ and the increments in these charge densities are $\delta\rho = \rho(x, y, z)\, \delta\alpha$ and $\delta\sigma = \sigma(x, y, z)\, \delta\alpha$. Then the *total electrostatic energy*, which is obtained by summing contribution (7), is given by

$$U = \Sigma\, \delta W = \int_0^1 \delta\alpha \int_V \rho(x, y, z)\, \phi'(\alpha ; x, y, z)\, dv$$

$$+ \int_0^1 \delta\alpha \int_S \sigma(x, y, z)\, \phi'(\alpha ; x, y, z)\, da. \qquad ...(8)$$

Since all the charges are at same fraction α of their final values, the potential $\phi'(\alpha ; x, y, z) = \alpha\phi(x, y, z)$ where ϕ is the final values of potential at point (x, y, z). Making this substituion, we obtain

$$U = \int_0^1 \alpha\, \delta\alpha \int_V \rho(x, y, z)\, \phi(x, y, z)\, dv + \int_0^1 \alpha\, \delta\alpha$$

$$\int_S \sigma(x, y, z)\, \phi(x, y, z)\, da$$

$$= \tfrac{1}{2}\int_V \rho\, \phi\, dv + \tfrac{1}{2}\int \sigma\, \phi\, da\,. \qquad ...(9)$$

This is the required expression for the electrostatic energy of a charge distribution.

Remembering that the conductors are present in the system, it is convenient to separate out the contribution from the conductors explicitly. The last integral in (9) involves, in part, integrations over the surface of these conductors. Since a conductor is an equipotential region. each of above integrations may be easily done : Considering i^{th} conductor, we may write

$$\tfrac{1}{2}\int_{\text{conductor } i} \sigma\, \phi\, da = \tfrac{1}{2} q_i\, \phi_i \qquad ...(10)$$

where q_i is the charge on i^{th} conductor. Hence equation (9) takes the form

$$U = \tfrac{1}{2}\int_V \rho\, \phi\, dv + \tfrac{1}{2}\int_{S'} \sigma\, \phi\, da + \tfrac{1}{2}\sum_i q_i\, \phi_i \qquad ...(11)$$

where S' denotes the surface integral over non-conducting surfaces only the last summation is over all conductors.

In a particular case if all of the free charge resides on the surface of conductors, equation (11) reduces to

$$U = \tfrac{1}{2}\, \Sigma\, q_i\, \phi_i \qquad ...(12)$$

3.3. Energy Density of an Electrostatic Field

The electrostatic energy of an arbitrary charge distribution is given by [eqn. (9) of preceding section]

$$U = \frac{1}{2}\int_V \rho\,\phi\,dv + \frac{1}{2}\int \sigma\,\phi\,da \qquad ...(1)$$

In some cases it is rather convenient to convert above equation to an integral involving the field vectors **E** and **D** of the *system* by means of mathematical transformation.

Let us now assume that *the charge system is bounded i.e.* it is possible to construct a closed surface of finite dimensions which encloses all of the free charge. In addition let us assume that *all surface densities of free charge σ reside on conductor surfaces.* This assumption is really no restriction at all, since a surface charge density on a dielectric–dielectric interface may be spread out slightly and then treated as a volume charge density ρ. The relations of ρ and σ with electric displacement vectors **D** are

$$\rho = \text{div}\,\mathbf{D} \qquad ...(2)$$

throughout the dielectric regions

and

$$\sigma = \mathbf{D}\cdot\mathbf{n} \text{ on conductor surface} \qquad ...(3)$$

where **n** is a unit vector normal to surface under consideration.

Using (2) and (3); equations (1) becomes

$$U = \frac{1}{2}\int_V \phi\,\text{div}\,\mathbf{D}\,dv + \frac{1}{2}\int_S \phi\,\mathbf{D}\cdot\mathbf{n}\,da. \qquad ...(4)$$

Hence the volume integral refers to the region where div **D** is non-zero and this is the region external to conductors. The surface integral is over the conductor surfaces.

The integrand in the first integral of (4) may be transformed by means of vector identity

$$\text{div}\,(\phi\,\mathbf{D}) = \phi\,\text{div}\,\mathbf{D} + \mathbf{D}\cdot\text{grad}\,\phi$$

or

$$\phi\,\text{div}\,\mathbf{D} = \text{div}\,(\phi\,\mathbf{D}) - \mathbf{D}\cdot\text{grad}\,\phi\,. \qquad ...(5)$$

Making this substitution (3) takes the form

$$U = \frac{1}{2}\int_V \text{div}\,(\phi\,\mathbf{D})\,dv - \frac{1}{2}\int_V \mathbf{D}\cdot\text{grad}\,\phi\,dv + \frac{1}{2}\int_S \phi\,\mathbf{D}\cdot\mathbf{n}\,da. \qquad ...(6)$$

Converting volume integral into a surface integral through the use of Gauss divergence theorem and substituting $\mathbf{E} = -\,\text{grad}\,\phi$ in second integral, we may write equation (6) as

$$U = \frac{1}{2}\int_{S+S'} \phi\,\mathbf{D}\cdot\mathbf{n}'\,da + \frac{1}{2}\int_V \mathbf{D}\cdot\mathbf{E}\,dv + \frac{1}{2}\int_S \phi\,\mathbf{D}\cdot\mathbf{n}\,da. \qquad ...(7)$$

Here $(S + S')$ denotes to surface integral over the entire surface bounding the volume V. It consists S (the surfaces of all conductors in the system) and also S' (a surface which bounds our system from outside and which may be chosen to locate at infinity). In both cases the unit vector **n′** is directed *out of* volume V. In the last integral the unit vector **n** is directed out of conductor, hence into V. Thus the two surface integrals over S cancel each other. Hence equation (7) takes the form

$$U = \frac{1}{2}\int_{S'} \phi\,\mathbf{D}\cdot\mathbf{n}'\,da + \frac{1}{2}\int_V \mathbf{D}\cdot\mathbf{E}\,dv\,. \qquad ...(8)$$

Now we shall show that the integral over S' vanishes. If the charge distribution, which is arbitrary but bounded, bears a net charge, then at large distances from the charge system, the potential fall off inversely as the distance *i.e.*, as r^{-1} and **D** falls over as r^{-2}. The area of a closed surface which passes through a point at a distance r is proportional to r^2. Hence value of the integral over S' which bounds our charge system at a distance r is proportional to r^{-2} and hence when S' is removed out to infinity, its contribution bears zero net charge, then the potential at large distance acts like some multipole and falls off *more rapidly* than r^{-2}. Hence again the contribution from **S′** vanishes. Thus

$$\int_{S'} \phi \mathbf{D} \cdot \mathbf{n}' \, da = 0 \text{ (always)}.$$

Using the fact equation (8) gives for *electrostatic energy*

$$U = \frac{1}{2}\int_V \mathbf{D} \cdot \mathbf{E} \, dv. \qquad \text{...(9)}$$

where the volume integral is taken over the volume of the system external to conductors *i.e.* over various dielectrics in the system. However the integration may be extended to include all space since the electrostatic field **E** is equal to zero within the conductor. This electrostatic energy may be imagined to be stored in the electric field. Equation (9) also prescribes that the energy is distributed with a density $\frac{1}{2}\mathbf{D} \cdot \mathbf{E}$ per unit volume. Hence energy density (*i.e.* energy per unit volume) in an electrostatic field is

$$u = \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \qquad \text{...(10a)}$$

Since equation (9) holds for *linear dielectrics*, each dielectric is characterised by a constant (absolute permittivity ε ; we have $\mathbf{D} = \varepsilon \mathbf{E}$; hence equation (10a) may also be written as

$$u = \frac{1}{2}\varepsilon \mathbf{E}^2 \qquad \text{...(10b)}$$

In free space or vacuum $\varepsilon = \varepsilon_0$ (since $K = 1$), hence *energy density of electrostatic field in free space* is given by

$$U = \frac{1}{2}\varepsilon_0 \mathbf{E}^2. \qquad \text{...(11)}$$

The relations (10) and (11) hold good for *isotropic homogeneous dielectrics* in which **D** and **E** are proportional to each other. If the medium is not isotropic but anisotropic, the permittivity ε has different values in different directions and hence the coefficient which relates **D** to **E** is a tensor $\varepsilon_{\alpha\beta}$. For such media the relation between **D** and **E** may be written as

$$D_\alpha = \varepsilon_{\alpha\beta} E_\beta = \varepsilon_0 K_{\alpha\beta} E_\beta \qquad \text{...(12)}$$

where $\varepsilon_{\alpha\beta} = \varepsilon_0 K_{\alpha\beta}$, K being relative permittivity or dielectric constant of medium. Substituitng these values in (9), we get expression for electrostatic energy as

$$U = \frac{1}{2}\int \varepsilon_0 K_{\alpha\beta} E_\beta E_\alpha \, dv \qquad \text{...(13)}$$

The comparison of equations (6) and (13) show that equation (1) is also valid for linear non-isotropic media where all field components increase proportionally.

The symmetry property of tensor $K_{\alpha\beta}$ may be judged from integral of (13) as

$$K_{\alpha\beta} E_\beta E_\alpha = K_{\alpha\beta} E_\alpha E_\beta = K_{\beta\alpha} E_\beta E_\alpha \qquad \text{...(14)}$$

where we have simply interchanged indices in the last step. Hence

$$\int (K_{\alpha\beta} - K_{\beta\alpha}) E_\alpha E_\beta \, dv = 0 \qquad \text{...(15)}$$

which show that for arbitrary value of E_α and E_β, the tensor

$$K_{\alpha\beta} - K_{\beta\alpha} = 0, \text{ i.e. } K_{\alpha\beta} = K_{\beta\alpha}$$

Thereby indicating that the dielectric constant tensor $K_{\alpha\beta}$ is a symmetric tensor except when the field energy vanishes identically.

Ex. 1. *An electron can be assumed to be uniformly charged sphere having a total charge e and radius* R_0.

(a) Calculate the electrostatic energy of the electron.

(b) If this energy is equal to rest energy m_0c^2 *of the electron. What must be its radius ?* **[Meerut 1975]**

Sol. (a) We known that field strength, inside and outside a uniformly charged sphere are given by

$$E_{in}\left(=\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\right)=\frac{1}{4\pi\varepsilon_0}\frac{er}{R_0^3} \qquad ...(1)$$

$$E_{out}\left(=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)=\frac{1}{4\pi\varepsilon_0}\frac{e}{r^2} \qquad ...(2)$$

The energy density of electrostatic field in free space is given by

$$U=\tfrac{1}{2}\,\varepsilon_0 E^2 \qquad ...(3)$$

Hence the net electrostatic energy for free space is

$$U=\int_{\text{all space}}\tfrac{1}{2}\,\varepsilon_0\,E^2\,dV \qquad ...(4)$$

Now assuming the entire space to be divided into two parts (i) a spherical volume of radius R_0 and (ii) the remaining free space (outside this small sphere) of radius from R_0 to infinity.

Assuming the spherical volumes to be formed of a large number of thin shells, we may write equation (4) as

$$U=\int_0^{R_0}\tfrac{1}{2}\,\varepsilon\,E_{in}^2\bullet 4\pi r^2\,dr+\int_{R_0}^{\infty}\tfrac{1}{2}\,\varepsilon_0\,E_{out}^2\bullet 4\pi r^2\,dr$$

$$=\int_0^{R_0}\tfrac{1}{2}\,\varepsilon_0\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{er}{R_0^3}\right)^2 4\pi\,r^2\,dr+\int_R^{\infty}\tfrac{1}{2}\,\varepsilon_0\,E^2\left(\frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}\right)^2 4\pi r^2\,dr$$

$$=\frac{e^2}{8\pi\varepsilon_0}\left[\int_0^{R_0}\frac{r^4}{R_0^6}\,dr+\int_{R_0}^{\infty}r^{-2}\,dr\right]$$

$$=\frac{e^2}{8\pi\varepsilon_0}\left[\frac{1}{5R_0}+\frac{1}{R_0}\right]=\frac{1}{4\pi\varepsilon_0}\left(\frac{3}{5}\frac{e^2}{R_0}\right) \qquad ...(5)$$

(b) According to problem $U=m_0c^2$

or

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{3}{5}\frac{e^2}{R_0}\right)=m_0c^2$$

This gives

$$R_0=\left(\frac{1}{4\pi\varepsilon_0}\,\frac{3e^2}{5m_0c^2}\right)$$

$$=9\times10^9\times\frac{3\times(1{\cdot}6\times10^{-19})^2}{5\times9{\cdot}1\times10^{-31}\times(3\times10^8)^2}$$

$$=1{\cdot}4\times10^{-15}\text{ m.}$$

3.4. Thermodynamical Interpretation of Electrostatic Energy U

The electrostaitc energy in a dielectric medium of absolute permittivity ε is

$$U=\tfrac{1}{2}\int_V\varepsilon\,\mathbf{E}^2\,dv=\tfrac{1}{2}\int_V\mathbf{D}\bullet\mathbf{E}\,dv \qquad ...(1)$$

where the integration extends over all dielectrics (including vacuum). The question is whether electrostatic energy U can be interpreted thermodynamically ? We have seen that actually electrostatic energy U is the work done in bringing the system to its charged condition. Hence U is really a work term and thus the immediate problem is to find the conditions under which a work increment may be identified with a thermodynamic property of the system.

From the first law of thermodynamics (which expresses conservation of energy) for a reversible process.

$$dQ = dU + dW \qquad \text{...(2a)}$$

where $dQ = T\,dS$ (T denotes absolute temperature and S denotes entropy of system) represents heat added to the system, dU represents the change in internal energy and dW is the work done by the system. If dW_m is the mechanical work *done on the system*, then $dW = -\,dW_m$.

Keeping these facts in mind, the first law of thermodynamics (2a) takes the form

$$T\,dS = dU - dW_m$$

or

$$dU = T\,dS + dW_m \qquad \text{...(2b)}$$

If the process is *adiabatic* $dQ = T\,dS = 0$: hence

$$dU = dW_m$$

i.e. for adiabatic process the work increment dW_m may be identified with the change in internal energy (dU) of the system. But the temperature of the system will change, in general, during an adiabatic process and hence the dielectric coefficients, which are temperture dependent, will also change. But in deriving the expression for electrostatic energy, we have assumed that various dielectric coefficients remain constant during the charging process. Hence we *must consider that the process is isothermal* and consequently it is *not possible* to identify work increment dW_m with change in internal energy dU.

According to *thermodynamics* the *Helmholtz free energy* F of a system is defined by

$$F = U - TS \qquad \text{...(3)}$$

Differentiating above euqation, we get

$$\begin{aligned} dF &= dU - d(TS) \\ &= dU - T\,dS - S\,dT \end{aligned} \qquad \text{...(4)}$$

Using (2*b*) above equation may be expressed as

$$dF = -\,S\,dT + dW_m \qquad \text{...(5)}$$

For an isothermal process $dT = 0$, hence equation (5) takes the form

$$dF = dW_m\,.$$

Thus for an isothermal process dF is equal to dW_m *i.e.* the work increment dW_m (or *the increment in electrostatic energy*) is identical to increment in Helmholtz free energy of the system. Thus we may say that the *electrostatic field energy forms part of the free energy system.* This energy represents the maximum work which can be extracted at a later time from the electrostatic field.

Consequently for a system kept at a constant temperature (iso-thermal process), the free energy plays the same role as does the potential energy for a temperture independent mechanical system.

3.5. Thomson's Theorem of Extremum Free Energy

We have seen that thermodynamically the maximum electrostatic energy convertible into work is known as (Helmholtz) free energy, therefore in dielectric media the expression $U = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,dv$ represents the *free energy.*

Thomson's theorem states that the *free energy of the system in a dielectric medium is extremum in equilibrium state.* The theorem may be proved as follows :

Let us consider a virtual process in which charges in equilibrium on a number of conductors are displaced by an infinitesimal amount along the constant potential conductor surface in such a way that the total charge remains unchanged. Then the variation δU in free energy,

$$U = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,dv = \frac{1}{2}\varepsilon\int \mathbf{E}^2\,dv \quad (\text{since } \mathbf{D} = \varepsilon\,\mathbf{E}) \qquad \text{...(1)}$$

is given by

$$\delta U = \tfrac{1}{2}\varepsilon \int \delta(\mathbf{E}^2)\, dv = \varepsilon \int \mathbf{E} \cdot \delta \mathbf{E}\, dv = \int \mathbf{E} \cdot \delta \mathbf{D}\, dv. \qquad ...(2)$$

But $\mathbf{E} = -\operatorname{grad} \phi = -\nabla \phi$, therefore

$$\delta U = \int (-\nabla \phi \cdot \delta \mathbf{D})\, dv. \qquad ...(3)$$

The vector identify $\operatorname{div}(\phi \mathbf{A}) = \phi \operatorname{div} \mathbf{A} + \mathbf{A} \cdot \nabla \phi$ in this case, replacing $\mathbf{A}$ by $\delta \mathbf{D}$, becomes

$$\operatorname{div}(\phi\, \delta \mathbf{D}) = \phi \operatorname{div} \delta \mathbf{D} + \delta \mathbf{D} \cdot \nabla \phi$$

i.e.

$$\delta \mathbf{D} \cdot \nabla \phi = \operatorname{div}(\phi\, \delta \mathbf{D}) - \phi \operatorname{div} \delta \mathbf{D}$$

or

$$\nabla \phi \cdot \delta \mathbf{D} = \nabla \cdot (\phi\, \delta \mathbf{D}) - \phi\, \nabla \cdot \delta \mathbf{D}$$

using this equation (3), becomes

$$\delta U = \int [\phi\, \nabla \cdot \delta \mathbf{D} - \nabla \cdot (\phi\, \delta \mathbf{D})]\, dv. \qquad ...(4)$$

But $\operatorname{div} \mathbf{D} = \rho$ implies $\nabla \cdot \delta \mathbf{D} = \delta\rho$, where ρ is the charge density. Thus equation (4), may be written in the form

$$\delta U = \int_V \phi\, \delta\rho\, dv - \int_V \phi\, \nabla \cdot (\delta \mathbf{D})\, dv. \qquad ...(5)$$

Changing volume integral into surface integral by Gauss's divergence theorem in last term of (5), we get

$$\delta U = \int_V \phi\, \delta\rho\, dv - \int_V \phi\, \delta \mathbf{D} \cdot d\mathbf{a} \qquad ...(6)$$

But the surface integral vanishes as in section 3·3 and the volume integral can be broken into to sum of individual integrals over conductors of the system. Thus

$$\delta U = \sum_i \int_{V_i} \phi_i\, \delta\rho_i\, dv_i.$$

If ϕ_i is constant over each conductor, then

$$\delta U = \sum_i \phi_i \int_{V_i} \delta\rho_i\, dv_i \qquad ...(7)$$

But $\int_{V_i} \delta\rho_i\, dv_i = 0$, since charge q_i on each coductor is constant.

Therefore $\delta U = 0$.

This implies that U = Extremum

Thus, *the elementary equilibrium condition that each conductor be an equipotential is equivalent is to make the free energy an extremum.* This requirement is usually known as *Thomson*'s theorem.

3.6. Capacity and Capacitors

It is of practical importance to know how much, charge can be placed on an insulated conductor before the electric intensity just outside its surface reaches such a magnitude that the medium surrounding the conductors no longer behaves like an insulator. Then as more charge is given to the conductor, the conductor loses charge to its surroundings. To determine when this point will be reached we must known (i) the size and shape of the conductor, (ii) the size, shape and location of other conductors in the vicinity of the conductor, (iii) the dielectric strength of the intervening medium, and (iv) the potential difference between the given conductor and each conductor in its vicinity or the charge on each conductor.

In practical there are *two important* special cases :

1. *When the given conductor is isolated* (*i.e.* very far from other conductors). The potential of such a conductor may be defined as the potential difference between it and a point at infinite distance. Let q be the charge on the conductor when its potential is V. Then *capacity* (or *capacitance*) C of the conductor is defined as the ratio

$$C = \frac{q}{V}. \qquad ...(1)$$

2. *When there are two conductors in the vicinity of each other which can store equal and opposite charge* (± *q*), *independently of whether other conductors in the system are charged.* Such a combination is known as a *capacitor.* The independence of other charge implies that one of the pair of conductors is sheilded by the other; in other words, the potential contributed to each of the pair by external charges must be the same. If V_{ab} is the potential difference between the conductors of the pair, then *capacity of the capacitor* is defined as

$$C = \frac{q}{V_{ab}}. \qquad ...(2)$$

It is common practice to put the charge q on one of the two conductors of the capacitor, while the other is connected to ground. The charge $-q$ will then be attracted from the earth to the grounded conductor. The value of introducing the concept of *capacitanee* (or capacity) is that if q and V_{ab} are varied. C is constant upto the breakdown point of the surrounding medium, for the following reason.

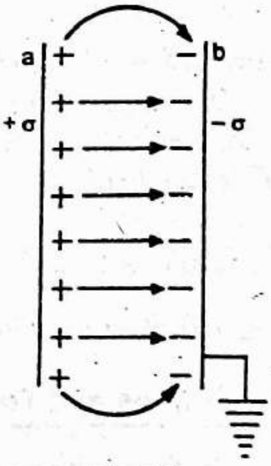
The electric field strength at any point is proportional to charge q. If this field results in the production and polarisation charges, these charges and the electric field strength due to them will also be proportional to q, provided that the medium is a class A dielectric. Since the potential difference between two points in such a field is the line integral of electric field strength, we note that

$$V_{ab} \propto q$$

or C is independent of q and V_{ab}. The value of capacitance C does depend on the size, shape and location of conductors present and the value of dielectric constant of intervening medium. From the definition of C, it is obvious that the unit of capacitance is *cculomb/volt* or *farad* defined as 1 farad = coul/volt.

If the two conductors making up the capacitor have simple geometrical shapes, the capacitance may be obtained analytically. Thus for example, it is easy to calculate the capacitance of two parallel plates, two co-axial cylinders, two concentric spheres or that of a cylinder and a plane. As an illustration let us derive an expression for the capacitance a *parallel plate capacitor.* A parallel plate capacitor consists of two parallel metal plates of large area and relatively small separation with one of the plates earthed. The electric field between the region of parallel plates is uniform, except for the fringing field at the edges. If the plate separation d is very small, the fringing field may be neglected.

Let a and b be two parallel plates containing free surface charge densities $+\sigma$ and $-\sigma$. If A is the area of each plate and the region between the plates is filled with a dielectric of (absolute) permittivitiy ε, then the electric field between the plates is



(Fig.3.1)

$$E = \frac{\sigma}{\varepsilon} \qquad ...(1)$$

and the potential difference between plates a and b is

$$V_{ab} = Ed. \qquad ...(2)$$

If q is the magnitude of charge on each plate, then the *capacitance of parallel plate capacitor is*

$$C = \frac{q}{V_{ab}} = \frac{\sigma A}{Ed} \text{ (since } q = \sigma A\text{)}$$

$$= \frac{\sigma A}{\sigma/\varepsilon\, d} \qquad \text{using (1)}$$

i.e.

$$C = \frac{\varepsilon A}{d}. \qquad ...(3)$$

Obviously capacitance of a parallel plate capacitor is directly proportional to permittivity of intervening medium and area of each plate, but inversely proportional to the plate separation.

3.7. Work Required to Charge a Capacitor and the Energy Stored in an Electric Field

Suppose that we gradually build up to charge on a capacitor, bringing up a little charge at a time. When the charge has reached the value αq, $(0 < \alpha < 1)$, the potential will be αV, where q and V are the final values of charge and potential respectively. Let us now bring the additional charge $dq = (q d\alpha)$, then the required work dW will be given by

$$dW = dq\,(\alpha V) = (q d\alpha)\,(\alpha V) = q\,V\alpha\,d\alpha\,.$$

Integrating above expression between the limits $\alpha = 0$ to $\alpha = 1$, we get an expression for the total work to charge the capacitor

$$w = \int_0^1 qV\,\alpha\,d\alpha = \tfrac{1}{2}\,qV. \qquad \text{...(1)}$$

But

$$q = CV.$$

$\therefore$

$$W = \tfrac{1}{2}\,CV^2. \qquad \text{...(2)}$$

In many respects a capacitor behaves like a spring with charge corresponding to extension, potential corresponding to stretching force and $1/C$ to the stiffness factor k of the spring. The greater the charge on a capacitor, the greater is the stored enereqy which will be released during discharge.

From equation (3) of the section 3 5, the capacitance C of a parallel plate capacitor is $C = (\varepsilon A/d)$ and $V = Ed$, therefore,

$$W = \tfrac{1}{2}\left(\frac{\varepsilon A}{d}\right)(Ed)^2 = \tfrac{1}{2}\,\varepsilon E^2\,Ad.$$

where Ad = Volume of dielectric region between plates of the parallel plate capacitor.

If we again take the view that the work done in charging the capacitor goes into energy stored in the field, then noting that the field is uniform throughout the volume (Ad), we may say that the *energy density of electric field.*

$$u = \frac{\tfrac{1}{2}\,\varepsilon E^2\,Ad}{Ad} = \tfrac{1}{2}\,\varepsilon E^2, \qquad \text{...(3)}$$

which is identical with expression [10 (b)] of section 3·3.

3.8. Forces and Torques

So far we have derived the electrostatic energy of a charge system by alternative means. Now let us show how the force on one of the objects in a charge system may be calculated from the knowledge of the electrostatic energy. Let us consider an isolated system composed of a number of parts (conductor, point charges, dielectrics) and let us allow one of these parts to make a small displacement $d\mathbf{r}$ under the influence of the electrical force acting upon it. The mechanical work performed by the system in these circumstances is

$$dW_m = \mathbf{F}\cdot d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz, \qquad \text{...(1)}$$

because the system is isolated, this work is done at the expense of the electrostatic energy U, *i.e.*

$$dU + dW_m = 0. \qquad \text{...(2)}$$

Comparing (1) and (2), we get,

$$-dU = F_x\,dx + F_y\,dy + F_z\,dz\,.$$

Thus implies,

$$F_x = -\frac{\partial U}{\partial x}, F_y = -\frac{\partial U}{\partial y}, F_z = -\frac{\partial U}{\partial z}. \qquad ...(4)$$

On the other hand, if the object under consideration in constrained to move in such a way that it rotates about an axis, then equation (1) may be repalced by

$$dW_m = \vec{\tau} \cdot \vec{d\theta} \qquad ...(6)$$

where $\vec{\tau}$ is the electrical torque and $\vec{d\theta}$ is the angular displacement. Writing $\vec{\tau}$ and $\vec{d\theta}$ in terms of their components (τ_1, τ_2, τ_3) ; $(d\theta_1, d\theta_2, d\theta_3)$ and combining equations (2) and (5), we obtain

$$\tau_1 = -\frac{\partial U}{\partial \theta_1}, \tau_1 = -\frac{\partial U}{\partial \theta_2} \text{ and } \tau_3 = -\frac{\partial U}{\partial \theta_3} \qquad ...(5)$$

Summarising we may write

$$F_x = -\left(\frac{\partial U}{\partial x}\right)_q \text{ and } \tau_1 = -\left(\frac{\partial U}{\partial \theta_1}\right)_q, \qquad ...(7)$$

where the subscript q denotes that the system is isolated and hence its total charge remains constant during the displacement $d\mathbf{r}$ or $\vec{d\theta}$. This may be used only, if U is expressed in analytic form and the specific dependence of U on the co-ordinates x and θ_1 is given.

Equation (7) do not, however, cover all cases of interest because it was asssumed in their derivation that the system is isolated in which the charge in the system remains constant. Another important case is when all of the free charge exists on the surface of conductors and these are maintained at fixed potentials by means of external sources of energy (*e.g.* by means of batteries). Here again let us consider that one of the parts of the system moves under the influence of electrical forces acting supplied by the batteries, then energy conservation equation becomes

$$dU + dW_m = dU_p. \qquad ...(8)$$

The electrostatic energy of charged conductors is given by equation (12) of section 3·2 viz.

$$U = \tfrac{1}{2} \sum_i q_i \phi_i. \qquad ...(9)$$

If some part of the system is displaced such that the potentials of all conductors remain fixed, then equation (9) gives

$$dU = \tfrac{1}{2} \sum_i \phi_i \, dq_i. \qquad ...(10)$$

Furthermore the energy supplied by the batteries dU_b is the work required to move each of the charge increments dq from zero potential to the potential of appropriate conductor. Hence by eqn. (1) of section 3·1, we may write

$$dU_b = \sum_i \phi_i \, dq_i. \qquad ...(11)$$

Comparing equations (10) and (11), we get

$$dU_b = 2dU. \qquad ...(12)$$

Using this equation to eliminate dU_b from equation (8), we get

$$dU + dW_m = 2dU$$

or

$$dU = dW_m, \qquad ...(13)$$

using equation (1), we get

$$dU = F_x \, dx + F_y \, dy + F_z \, dz$$

which implies,

$$F_x = \left(\frac{\partial U}{\partial x}\right)_\phi, F_y = \left(\frac{\partial U}{\partial y}\right)_\phi \text{ and } F_z = \left(\frac{\partial U}{\partial z}\right)_\phi \qquad \text{...(14)}$$

where the subscript denotes the fact that all potentials are maintained constant during the virtual displacement *dr*.

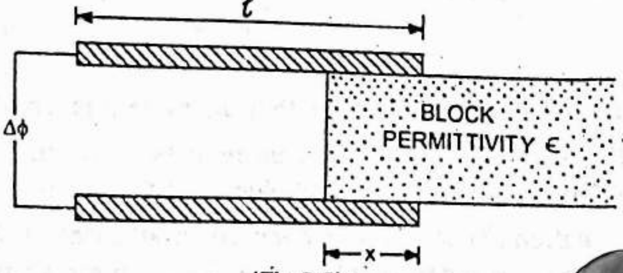
In a similar manner, we may derive

$$\tau_1 = \left(\frac{dU}{\partial \theta_1}\right)_\phi, \tau_2 = \left(\frac{dU}{\partial \theta_2}\right)_\phi \text{ and } \tau_3 = \left(\frac{dU}{\partial \theta_3}\right)_\phi \qquad \text{...(15)}$$

Ex. 2. *A parallel plate capacitor of plate separation d has the region between the plates field by a block of solid dielectric of permittivity ε. The length and width of each plate are l and y respectively. If the plates are maintained at constant potential difference Δϕ and the dielectric block is withdrawn along the l-dimension until only the length x remains between the plates (Fig. 3·2), calculate he forces tending to pull the block back into plate.*

Solution. The energy of the sytem is

$$U = \tfrac{1}{2}\int_V \varepsilon E^2\, dv$$



(Fig.3.2)

where volume integration extends over the entire region of system where $E \neq 0$. Neglecting fringing effects over the edge of capacitor and remembering that $E = \frac{\Delta\phi}{d}$, we get,

$$U = \tfrac{1}{2}\int \varepsilon\left(\frac{\Delta\phi}{d}\right)^2 x\,.\,d\,.\,dy + \tfrac{1}{2}\int \varepsilon\left(\frac{\Delta\phi}{d}\right)^2 (l-x)\,.\,d\,.\,dy$$

where first integral extends over volume of block still inside the capacitor and the second integral extends over block free region of capacitor, *i.e.*,

$$U = \tfrac{1}{2}\varepsilon\left(\frac{\Delta\phi}{d}\right)^2 xy\,.\,d + \tfrac{1}{2}\varepsilon_0\left(\frac{\Delta\phi}{d}\right)^2 (l-x)\,yd. \qquad \text{...(16)}$$

From equation (14), we have

$$F_x = \left(\frac{\partial U}{\partial x}\right)_\phi$$

Therefore from (16), we get

$$F_x = \left(\frac{\partial U}{\partial x}\right)_\phi = \tfrac{1}{2}\varepsilon\left(\frac{\Delta\phi}{d}\right)^2 y\,.\,d + \tfrac{1}{2}\varepsilon_0\left(\frac{\Delta\phi}{d}\right)^2 (0-1)\,y\,.\,d.$$

$$= \tfrac{1}{2}\varepsilon\left(\frac{\Delta\phi}{d}\right)^2 yd - \tfrac{1}{2}\varepsilon_0\left(\frac{\Delta\phi}{d}\right)^2 yd = \tfrac{1}{2}(\varepsilon-\varepsilon_0)\left(\frac{\Delta\phi}{d}\right)^2 yd.$$

This force is along the direction of increasing *x*.

3.9. Mechanical Forces in an Electrostatic Field

The force per unit volume that acts on a dielectric when it is under the influence of an external electrostatic field may be derived from the *free energy U* given by

$$U = \tfrac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, dv\,. \qquad \text{...(1)}$$

Let us consider a virtual process in which the physical coordinates of charge and dielectric are given a virtual displacement $\delta\mathbf{r}$ at each point in space, but where no external charges are added so that equation (1) holds. If $\mathbf{f}$ is the force per unit volume, then the variation in free energy δU when a unit volume of dielectric undergoes a virtual displacement $\delta\mathbf{r}$ is given by

$$\delta U = -\int \mathbf{f} \cdot \delta\,\mathbf{r}\, dv. \qquad \text{...(2)}$$

Keeping in mind that the magnitude of virtual displacement $\delta\mathbf{r}$ is quite arbitary, we can identify $\mathbf{f}$ with the true *volume force*. This information may be put in a different way as follows; if $\mathbf{u}$ is an arbitrary velocity field within a dielectric, then rate at which energy is lost by the field is given by

$$\frac{\partial U}{\partial t} = -\int \mathbf{f} \cdot \mathbf{u}\, dv, \qquad \text{...(3)}$$

where $\mathbf{f}$ again represents the volume force.

As a point of fact we do not known $\mathbf{f}$ to begin with; the real problem before us is to transform the rate of change of function U defined by (1), so that it may assume the form (3). When this has been done, we shall be justified in regarding the factor $\mathbf{f}$ so found as a *force density* compatible with the principle of energy.

The field is uniquely defined when the *charge density* ρ and the *dielectric constant* K are everywhere given. Hence U only varies with time in so far as ρ and K vary. Now let us consider the energy change due to both a change $\delta\rho$ in true charge distribution and a change δK in dielectric constant caused by displacements. Hence from equation (1)

$$\begin{aligned}\delta U &= \tfrac{1}{2}\delta \int \mathbf{D} \cdot \mathbf{E}\, dv \\ &= \tfrac{1}{2}\int \mathbf{E} \cdot \delta\mathbf{E}\ dv + \tfrac{1}{2}\int \delta\,\mathbf{D} \cdot \mathbf{E}\, dv. \end{aligned} \qquad \text{...(4)}$$

But $\mathbf{D} = \varepsilon\,\mathbf{E} = \varepsilon_0 K\,\mathbf{E}$ or $\mathbf{E} = \dfrac{\mathbf{D}}{\varepsilon_0 K}$ ...(5)

Substituting this in integrand of first integral of (4), we get

$$\begin{aligned}\delta U &= \tfrac{1}{2}\int \mathbf{D} \cdot \delta\left(\frac{\mathbf{D}}{\varepsilon_0 K}\right) dv + \tfrac{1}{2}\int \mathbf{E} \cdot \delta\mathbf{D}\, dv \\ &= \tfrac{1}{2}\int \left(\frac{\mathbf{D}}{\varepsilon_0 K}\right) \cdot \delta\mathbf{D}\, dv + \frac{1}{2}\int \frac{\mathbf{D} \cdot \mathbf{D}}{\varepsilon_0}\,\delta\left(\frac{1}{K}\right) dv + \tfrac{1}{2}\int \mathbf{E} \cdot \delta\mathbf{D}\, dv \\ &= \tfrac{1}{2}\int \mathbf{E} \cdot \delta\mathbf{D}\, dv + \frac{1}{2\varepsilon_0}\int \mathbf{D} \cdot \mathbf{D}\left(-\frac{1}{K}\,\delta K\right) dv + \tfrac{1}{2}\int \mathbf{E} \cdot \delta\mathbf{D}\, dv.\end{aligned}$$

Again using (5) and rearranging, we get

$$\delta E = -\frac{\varepsilon_0}{2}\int E^2\,\delta K\, dv + \int \mathbf{E} \cdot \delta\mathbf{D}\, dv \qquad \text{...(6)}$$

In this equation the first term represents the energy change $(\delta_K U)$ used by the change in dielectric constant due to virtual displacement and the second term corresponds to the energy change $(\delta_p U)$ caused by displacement of sources (or charge density).

The second term can be simplified as follows :

$$\delta_p U = \int \mathbf{E} \cdot \delta\mathbf{D}\, dv = -\int \nabla\phi \cdot\ \delta\mathbf{D}\, dv \text{ (since } \mathbf{E} = -\nabla\phi)$$

we have vector identity

$$\nabla \cdot (\phi\,\delta\mathbf{D}) = \phi\,\nabla \cdot \delta\mathbf{D} + \delta\mathbf{D} \cdot \nabla\phi$$

This gives

$$\nabla\phi \cdot \delta\mathbf{D} = \nabla \cdot (\phi\,\delta\mathbf{D}) - \phi\,\nabla \cdot \delta\mathbf{D}$$

Substituting this in (7), we get

$$\int \mathbf{E} \cdot \delta \mathbf{D}\, dv = \int_V \phi\, \nabla \cdot \delta \mathbf{D}\, dv - \int_V \nabla \cdot (\phi\, \delta \mathbf{D})\, dv \qquad ...(7)$$

But div $\mathbf{D} = \rho$ implies $\nabla \cdot \delta\mathbf{D} = \delta\rho$. Using this and remembering that second integral of (7) vanishes as in section 3·5 ; equation (7) takes the form

$$\delta_p U = \int \mathbf{E} \cdot \delta\mathbf{D}\, dv = \int \phi\, \delta\rho\, dv \qquad ...(8)$$

This result includes as a special case of the theorem that in small displacements of charges on an isolated conductor, the field energy is not changed; in fact in the state of electrostatic equilibrium thus specified ϕ is constant and $\int \delta\rho\, dv = 0$. This is of course is simply a special case of Thomson's theorem, that the electrostatic energy is a minimum is equilibrium. This special case is of importance here for this reason, that it shows that part of $\delta\rho$ which consists of a motion of the charges on metallic conductors, contributes nothing to the time rate of charge of U, so long as the electrostatic equilibrium continues to be maintained.

Using (8) and rearrainging, equation (6) becomes

$$\delta U = \int \phi\, \delta\rho\, dv - \frac{\varepsilon_0}{2} \int \mathbf{E}^2\, \delta K\, dv \qquad ...(9)$$

Therefore the total rate of change of the field energy is given by

$$\frac{dU}{dt} = \int \phi \frac{\partial \rho}{\partial t}\, dv - \frac{\varepsilon_0}{2} \int \mathbf{E}^2 \frac{\partial K}{t}\, dv . \qquad ...(10)$$

To arrive at the expression for **f**, we much express the time derivatives $\partial\rho/\partial t$ and $\partial K/\partial t$, in terms of arbitrary velocity field **u**. With respect to $\partial\rho/\partial t$, by Thomson;s theorem we need not take into account the motion of electricity on a conducting surface. We may therefore make the calculations as if the charge were everywhere rigidly fixed to the matter. In that case, however, $\rho\mathbf{u}$ is the density of convection current, so that by the hydrodynamic equation of continuity

$$\nabla \cdot (\rho\, \mathbf{u}) + \frac{\partial \rho}{\partial t} = 0$$

This equation represents the *conservation of charge.* If σ is the mass-density, then similar formula conservation of mass is

$$\nabla \cdot (\sigma \mathbf{u}) + \frac{\partial \sigma}{\partial t} = 0. \qquad ...(12)$$

To calculate $\partial K/\partial t$ we must associate the charge in dielectric constant with the velocity flow. Since there is net transport of material in a velocity field, the change in dielectric constant can be associated with changes in geometry only if we consider the time history of a volume element that is moving with velocity **u** *i.e.* for calculation of $\partial K/\partial t$, we have to compare the value of K at time t at the point (x, y, z) with the corresponding value at time $t + dt$ at the point $(x + u_x\, dt, y + u_y\, dt, z + u_z, dt)$. Then the total derivative, also known as *substantial derivative,* of K is given by

$$\begin{aligned} \frac{dK}{dt} &= \frac{K(t + dt, x + u_x dt, y + u_y\, dt, z + u_z dt) - K(t, x, y, z)}{t + dt - t} \\ &= \frac{\partial K}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial K}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial K}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial K}{\partial t} \\ &= (\nabla K) \cdot \mathbf{u} + \frac{\partial K}{\partial t} \end{aligned} \qquad ...(13)$$

and therefore

$$\frac{\partial K}{\partial t} = \frac{dK}{dt} - (\nabla K) \cdot \mathbf{u} \qquad ...(14)$$

Again the substantial derivative of mass-density σ is

$$\frac{d\sigma}{dt} = (\nabla\sigma)\cdot\mathbf{u} + \frac{\partial\sigma}{\partial t} \qquad ...(15)$$

therefore

$$\frac{\partial\sigma}{\partial t} = \frac{\partial\sigma}{\partial t} - (\nabla\sigma)\cdot\mathbf{u} \qquad ...(16)$$

If there is dielectric equation of state *i.e.* a relation which gives the dependence of dielectric constant on the density, such as Clausius Mossotti relation; then we may assume *that K is a single-valued function of density* σ. (This assumption includes the assumption that virtual processes are isothermal) and express substantial derivative of dielectric constant in terms of substantial derivative of density. In then follows that

$$\frac{dK}{dt} = \frac{dK}{d\sigma}\cdot\frac{d\sigma}{dt} \qquad ...(17)$$

where $dK/d\sigma$ is a known property of the dielectric. Using (15), equation (17) gives

$$\begin{aligned}\frac{dK}{dt} &= \frac{dK}{d\sigma}\left[(\nabla\sigma)\cdot\mathbf{u} + \frac{\partial\sigma}{\partial t}\right]\\ &= \frac{\partial K}{d\sigma}\left[(\nabla\sigma)\cdot\mathbf{u} - \nabla\cdot(\sigma\mathbf{u})\right] \qquad \text{using (12)}\\ &= -\frac{dK}{d\sigma}\sigma(\nabla\cdot\mathbf{u}) \qquad ...(18)\end{aligned}$$

Hence from (14) [since $\nabla\cdot(\sigma\mathbf{u}) = \sigma\nabla\cdot\mathbf{u} + (\nabla\sigma)\cdot\mathbf{u}$]

$$\frac{dK}{\partial t} = -\frac{dK}{d\sigma}\sigma(\nabla\cdot\mathbf{u}) - (\nabla K)\cdot\mathbf{u} \qquad ...(19)$$

using (11) and (19) in (10), we have

$$\frac{dU}{dt} = -\int\phi\,\nabla\cdot(\rho\mathbf{u})\,dv + \frac{\varepsilon_0}{2}\int\mathbf{E}^2\left(\nabla K\cdot\mathbf{u} + \sigma\frac{dK}{d\sigma}\nabla\cdot\mathbf{u}\right)dv \qquad ...(20)$$

Now by vector identity $\text{div}(\phi\mathbf{A}) = \phi\,\text{div}\,\mathbf{A} + \mathbf{A}\cdot\nabla\phi$

Replacing $\mathbf{A}$ by $\rho\mathbf{u}$, we note that

$$\nabla\cdot(\phi\,\rho\mathbf{u}) = \phi\,\nabla\cdot(\rho\mathbf{u}) + (\rho\,\mathbf{u})\cdot\nabla\phi$$

i.e.,

$$\nabla\phi\cdot(\rho\mathbf{u}) = \nabla\cdot(\phi\,\rho\mathbf{u}) - (\rho\mathbf{u})\cdot\nabla\phi \qquad ...(21)$$

Similarly

$$\mathbf{E}^2\sigma\frac{dK}{d\sigma}\nabla\cdot\mathbf{u} = \nabla\cdot\left(\mathbf{E}^2\sigma\frac{dK}{d\sigma}\mathbf{u}\right) - \mathbf{u}\cdot\nabla\left(\mathbf{E}^2\frac{dK}{d\sigma}\right) \qquad ...(22)$$

Using (21) the first term in R.H.S. of (20) may be written as

$$-\int\phi\,\nabla\cdot(\rho\mathbf{u})\,dv = \int\rho\mathbf{u}\cdot\nabla\phi\,dv - \int\nabla\cdot(\phi\,\rho\mathbf{u})\,dv \qquad ...(23)$$

Applying Gauss divergence theorem to change second volume integral on R.H.S. into surface integral and keeping in mind that surface integral vanishes over infinitely distant surface, we may write (23) as

$$\begin{aligned}-\int\phi\nabla\cdot(\rho\mathbf{u})\,dv &= \int\rho\mathbf{u}\cdot\nabla\phi\,d\rho\\ &= -\int\rho\mathbf{u}\cdot\mathbf{E}\,dv\ (\text{since }\mathbf{E} = -\nabla\phi); \qquad ...(24)\end{aligned}$$

Similarly, the second term in second volume integral in R.H.S. of (20) may be put into the desired form as

$$\int\mathbf{E}^2\sigma\frac{dK}{d\sigma}\nabla\cdot\mathbf{u}\,dv = \int\nabla\cdot\left(\mathbf{E}^2\sigma\frac{dK}{dv}\mathbf{u}\right)dv - \int\mathbf{u}\cdot\nabla\left(\mathbf{E}^2\sigma\frac{dK}{d\sigma}\right)dv \qquad \text{[using (22)]} ...(25)$$

Again applying Gauss divergence theorem to the first integral on R.H.S. and assuming that the surface boundary is too distant *i.e.* outside of dielectric and therefore outside the region of charge density. Hence dropping out the surface integral term which vanishes; we may write (25) as

$$\int \mathbf{E}^2 \sigma \frac{dK}{d\sigma} \nabla \cdot \mathbf{u}\, dv = -\int \mathbf{u} \cdot \nabla \left(\mathbf{E}^2 \sigma \frac{dK}{d\sigma} \right) dv \qquad ...(26)$$

using (24) and (26), equation (20) takes the final form as

$$\frac{dU}{dt} = -\int \rho \mathbf{u} \cdot \mathbf{E}\, dv + \frac{\varepsilon_0}{2} \int \mathbf{E}^2 \nabla K \cdot \mathbf{u}\, dv - \int \mathbf{u} \cdot \nabla \left(\mathbf{E}^2 \sigma \frac{dK}{d\sigma} \right) dv$$

$$= \int \left[-\rho \mathbf{E} + \frac{\varepsilon_0}{2} \mathbf{E}^2 \nabla K - \frac{\varepsilon_0}{2} \nabla \left(\mathbf{E}^2 \sigma \frac{dK}{d\sigma} \right) \right]. \qquad ...(27)$$

We have thus actually obtained an expression of the form (3). Comparing (3) and (27) we can at once deduce the force density **f**.

$$\mathbf{f} = \rho \mathbf{E} - \frac{\varepsilon_0}{2} \mathbf{E}^2 \nabla K + \frac{\varepsilon_0}{2} \nabla \left(\mathbf{E}^2 \sigma \frac{dK}{d\sigma} \right) \qquad ...(28)$$

It is obvious that the expression for force **f** in (28), consits of three terms :

(i) The first term $\rho\mathbf{E}$ gives the known electrostatic force on true charges.

(ii) The second term $\varepsilon_0 \mathbf{E}^2 \nabla K/2$, makes a contribution to the value of **f**, wherever K is variable from point to point *i.e.* the second term will appear only when there is an inhomogeneous dielectric in electric field. In particular, at the interface between an insulator and free space it gives a force at right angles to the surface of the insulator, tending to pull the insulator into free space (vacuum).

(iii) The last and the third terms is knwon as *electrostriction term* and gives a volume force on a dielectric in an inhomogeneous electric field. It may be noted that the magnitude of electrostriction term depends explicitly through $dK/d\sigma$ on the electrical equation of state of material. In ordinary field the required changes are quite low; hence the third term is frequently omitted.

3.10. Maxwell Stress Tensor

The Faraday Maxwell field theory has no dealings with forces acting at a distance, but regards all action as transmitted continuously from one body to another through the electromagnetic field. The conception of the continuous transmission of force through a body is familiar enough—we may instance the example of a stretched elastic spring. According to Maxwell the *electrostatic field is simply the stress transmitting medium.*

Now consider a given volume in electrostatic field bounded by a surface. If we consider that the force acting on this volume is transmitted across the elements of surface bounding the volume, then this transmitting force can be formulated in terms of a quantity known as the *stress tensor T* (the stress is an abstract concept and is independent of the presence of matter). The $\alpha\beta^{th}$ *component* $T_{\alpha\beta}$ *of stress tensor T* is so constituted that the α^{th} component dF_α of the force $d\mathbf{F}$ transmitted across the surface element dS whose component in the β^{th} direction is dS_β is given by

$$dF_\alpha = \sum_{\beta=1}^{3} T_{\alpha\beta}\, dS_\beta . \qquad ...(1)$$

The consideration of the rotational equilibrium of a rectangular solid under surface stresses indicates that the *stress tensor T must be symmetric.*

Keeping in mind the summation convention, equation (1) may be written as

$$dF_\alpha = T_{\alpha\beta}\, dS_\beta \qquad ...(2)$$

Integrating above equation, we get the α^{th} component of the total force acting on a given volume

$$F_\alpha = \int T_{\alpha\beta}\, dS_\beta . \qquad ...(3)$$

If this force is expressed in terms of a volume force **f**, whose α^{th} component is f_α, then

$$F_\alpha = \int T_{\alpha\beta}\, dS_\beta = \int f_\alpha\, dv. \qquad ...(4)$$

Using Gauss's divergence theorem in tensor notation

$$\int \frac{\partial T_{\alpha\beta}}{\partial x_\beta}\, dv = \int T_{\alpha\beta}\, dS_\beta \qquad ...(5)$$

equation may be written as

$$\int \frac{\partial T_{\alpha\beta}}{\partial x_\beta}\, dv = \int f_\alpha\, dv$$

i.e.

$$\int \left(f_\alpha - \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \right) dv = 0. \qquad ...(6)$$

As volume is arbitrary, hence equation (6), indicates that α^{th} component of volume force in terms of stress tensor is

$$f_\alpha = \frac{\partial T_{\alpha\beta}}{\partial x_\beta}. \qquad ...(7)$$

Thus, if we express a volume force as the tensor divergence of a certain quantity *T*, then the quantity *T* can be identified with the surface stress *T* giving the stress transmitted by the field across the surface bounding the given volume.

The tensor corresponding to a given volume force **f** is not unique; since any additional tensor of vanishing divergence can always be added without affecting the value of **f** in (7).

Let us now consider the case :

When dielectrics are absent. In the absence of dielectrics the volume force **f** is given by

$$\mathbf{f} = \rho \mathbf{E} = (\nabla \cdot \mathbf{D})\, \mathbf{E} \qquad ...(8)$$

This equation in tensor notation is expressed as

$$\mathbf{f}_\alpha = \varepsilon_0 E_\alpha \frac{\partial E_\beta}{\partial x_\beta}$$

$$= \varepsilon_0 \left[\frac{\partial}{\partial x_\beta} (E_\alpha E_\beta) - E_\beta \frac{\partial E_\alpha}{\partial x_\beta} \right] \qquad ...(9)$$

The first term of this equation is already in form (7). The second term in the bracket of (9), may be expressed in the same form as follows :

We know for *electrostatic field* $\nabla \times \mathbf{E} = 0$, *i.e.*

$$\frac{\partial E_\alpha}{\partial x_\beta} = \frac{\partial E_\beta}{\partial x_\alpha}.$$

Hence

$$E_\beta \frac{\partial E_\alpha}{\partial x_\beta} = E_\beta \frac{\partial E_\beta}{\partial x_\alpha} = \frac{1}{2} \frac{\partial}{\partial x_\alpha} (E_\beta E_\beta) = \frac{1}{2} \frac{\partial}{\partial x_\alpha} (E^2). \qquad ...(10)$$

where *E* is the magnitude of **E**. Equation (10) can also be expressed as

$$E_\beta \frac{\partial E_\alpha}{\partial x_\beta} = \frac{1}{2} \delta_{\alpha\beta} \frac{\partial}{\partial x_\beta} (E^2) = \frac{\partial}{\partial x_\beta} (\tfrac{1}{2} \delta_{\alpha\beta} E^2). \qquad ...(11)$$

Substituting this in (9), we get

$$\mathbf{f}_\alpha = \varepsilon_0 \left[\frac{\partial}{\partial x_\beta} (E_\alpha E_\beta) - \frac{\partial}{\partial x_\beta} (\tfrac{1}{2} \delta_{\alpha\beta} E^2) \right]$$

$$= \frac{\partial}{\partial x_\beta} \left[\varepsilon_0 (E_\alpha E_\beta - \tfrac{1}{2} \delta_{\alpha\beta} E^2) \right] \qquad ...(12)$$

Comparing (7) and (12), we get

$$T_{\alpha\beta} = \varepsilon_0 \left(E_\alpha E_\beta - \tfrac{1}{2}\delta_{\alpha\beta} E^2\right). \qquad \text{...(13)}$$

By this equation we can obtain the components of Maxwell stress tensor T:

$$T = \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix}$$

$$= \varepsilon_0 \begin{vmatrix} \frac{1}{2}(E_x^2 - E_y^2 - E_z^2) & E_xE_y & E_xE_z \\ E_xE_y & \frac{1}{2}(E_y^2 - E_x^2 - E_z^2) & E_yE_z \\ E_xE_z & E_yE_z & \frac{1}{2}(E_z^2 - E_x^2 - E_y^2) \end{vmatrix} \qquad \text{...(14)}$$

This is Maxwell electric stress tensor in the absence of dielectrics. It may be pointed out that the additional terms will appear in the stress tensor if the field is not irrotational.

i.e. $\nabla \times \mathbf{E} \neq 0.$

The Maxwell stress tensor is a symmetric tensor of the second rank and can therefore be reduced to three components by transformation to principal axes. The principal values of the matrix can be obtained by solving the secular determinant

$$| T_{\alpha\beta} - \delta_{\alpha\beta}\lambda | = 0. \qquad \text{...(15)}$$

This gives the principal values of the tensor as

$$\lambda_1 = \frac{\varepsilon_0 E^2}{2}, \lambda_2 = \lambda_3 = -\frac{\varepsilon_0 E^2}{2}. \qquad \text{...(16)}$$

In terms of principal values the Maxwell stress tensor takes the simple form as

$$T = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \frac{\varepsilon_0}{2} \begin{vmatrix} E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & -E^2 \end{vmatrix} \qquad \text{...(17)}$$

The principal axes are so oriented that the coordinate axes corresponding to the single root of secular determinant λ_1 is parallel to **E**, while the two axes corresponding to double roots λ_2 and λ_3 are perpendicular to **E**. The qualitative interpretation of this fact is that the electric field transmits a tension $(\varepsilon_0 E^2)/2$ parallel to electric field **E** and a transverse pressure of magnitude $(\varepsilon_2 E^2)/2$ transverse to the field direction.

Questions and Problems

1. Show that the energy density in electrostatic field is given as

$$u = \frac{1}{2}\, \mathbf{D} \cdot \mathbf{E}.$$

2. Obtain an expression for the total energy of an electrostatic system. Give thermodynamical interpretation of field energy.
3. State and prove Thomson's theorem regarding free energy of an electrostatic system
4. What do you mean by the capacitance of a capacitor ? On what factors does it depend ? Show that the energy density of electric field in the region between the plates of a parallel plate capacitor is given by

$$u = \frac{1}{2}\varepsilon\, \mathbf{E}^2.$$

5. Calculate the energy of interaction between the electron cloud of a hydrogen atoms and the proton (which is the nucleus of the hydrogen atom). The charge density inside the cloud is

$$\rho = \frac{e}{\pi a^3} e^{-2r/a}$$

where *a* is Bohr radius.

$$\left[\textbf{Ans.}\quad U = -\frac{e^2}{4\pi\varepsilon_0\, a}\right]$$

6. Assuming tha the total charge (Ze) of an atomic nuclear is uniformly distributed within a sphere of radius a, show that potential at a distance ($r \le a$) is

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{a}\left[\frac{3}{2}-\frac{1}{2}\left(\frac{r}{a}\right)^2\right]$$

and potential energy of such a nucleus is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\left(\frac{Ze^2}{a}\right)$$

7. Calculate the energy of interaction between two balls whose distribution of charge e_1 and e_2 is spherically symmetric. The distance between the centres of balls is a.

$$\left[\textbf{Ans}\quad \frac{1}{4\pi\varepsilon_0}\frac{e_1 e_2}{a}\right]$$

8. Consider a sphere that sinks into a liquid to a depth less than half its diameter when it is not charged. What should be the charge of the sphere so that it sinks to a depth exactly half its diameter ? The mass of the sphere is M, its radius is R and the liquid has a density ρ and dielectric constant K.

$$\left[\textbf{Ans.}\quad q = R(1+K)\sqrt{\left\{\frac{4\pi\varepsilon_0}{K-1}\left(\frac{2}{3}\pi R^3\,\sigma\, g - Mg\right)\right\}}\,\right]$$

9. Obtain an expression for the force in an electrostatic field in the presence of dielectrics. Show that the volume force density reduces to $f = \rho E$, when dielectrics are absent.

10. Obtain an expression for field energy density in an isotropic and linear dielectric medium as a function of electric field intensity and derive an expression for force on a dielectric body placed in an electric field.

(Rohilkhand 1984, 82)

11. What is the physical significance of 'Maxwell's stress tensor' ? Obtain Maxwell's stress tensor in matrix form in free space. Hence show that the electric field transmits a tension $\varepsilon_0 E^2/2$ parallel to field diirection and a transverse pressure of magnitude $\varepsilon_0 E^2/2$ transverse to field direction.

12. Show that the Maxwell stress tensor in the presence of dielectrics may be expressed as

$$T_{\alpha\beta} = E_\alpha D_\beta - \frac{\delta_{\alpha\beta}}{2}(E\cdot D)(1-\lambda)$$

where
$$\lambda = \frac{\sigma}{K}\left(\frac{dK}{d\sigma}\right)$$

where σ is mass density, K is dielectric constant and $dK/d\sigma$ represents substantial derivative of K with respect to σ.

13. (a) Prove that energy U within a dielectric in an electric field E is given by

$$U = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\,dv.$$

(Rohilkhand 1997, 95)

(b) What is Maxwell's stress tensor ? Derive an expression for this tensor.

(Rohilkhand 1996, 91, 85, 83; Agra 72)

14. Explain Maxwell's concept of stress arising on account of the existance of electrostatic field. Obtain the stress tensor representing an electrostatic field existing in free space. Discuss the interaction force across a given path of area dS when the dielectric field is (i) normal (ii) tangential to it. **(Rohilkhand 1980)**

15. Derive an expression for the energy of the electrostatic field in free space. Show that in the case of an anisotropic dielectric medium this expresion can be suitably modified to involve the dielectric constant tensor and then can be used to prove the symmetry of this tensor. **(Agra 1983)**

4

Solution of Electrostatic Problems

4.1. Introduction

In principle the electrostatic problems can be directly solved if all charge distributions are known : since in such cases the potential and electric field are given directly as integral over this charge distribution

$$\phi(\mathbf{r}) = \frac{1}{4\pi\,\varepsilon_0}\int \frac{dq'}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{...(4.1)}$$

$$E(\mathbf{r}) = \frac{1}{4\pi\,\varepsilon_0}\int \frac{(\mathbf{r}-\mathbf{r}')\,dq'}{|\mathbf{r}-\mathbf{r}'|^3} \qquad \text{...(4.2)}$$

But if the charge distribution is not specified in advance, it may be necessary to determine the electric field first, before the charge distribution can be calculated. For example if an electrostatic problem involves several conductros, with either the potential or total charge of each conductor given, out the distribution of surface charge is not known ; then this distribution of surface charge can not be obtained until the problem is solved completely.

In this chapter we shall derive the fundamental differential equations which must be satisfied by the potential ϕ and develop various alternative methods to solve electrostatic problems.

4.2. Poisson's and Laplace's equations

The Gauss' theorem in differential form is

$$\operatorname{div}\mathbf{E} = \frac{\rho}{\varepsilon_0} \text{ (in free space)} \qquad \text{...(4.3)}$$

Furthermore, in a purely electrostatic field

$$\mathbf{E} = -\operatorname{grad}\phi \qquad \text{...(4.4)}$$

Combining (4.3) and (4.4), we obtain

$$\operatorname{div}\operatorname{grad}\phi = \frac{\rho}{\varepsilon_0} \qquad \text{..(4.5a)}$$

i.e.

$$\nabla\cdot\nabla\phi = -\frac{\rho}{\varepsilon_0}$$

or

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \qquad \text{...(4.5b)}$$

The operator $\nabla\cdot\nabla = \nabla^2$ is known as the *Laplacian* and the equation (4.5b) is known as *Poisson's equation.* Evidently Laplacian is a pure scalar differential operator and hence Poisson's equation is a differential equation. As the operator ∇^2 (Laplacian) involves differentiation respect to more than one variable, hence Poisson's equation is a *partial differential equation.* If the functional dependance of charge density function $\rho\,(x, y, z)$ and the appropriate boundary conditions are known, then Poisson's equation may be solved for the electrostatic potential ϕ. In order to solve a specific problem we must write ∇^2 in terms of cartesian co-ordinates (x, y, z) or spherical polar coordinates (r, θ, ϕ) etc. The choice of a particular set of coordinates is arbitrary : but the problem is greatly simplified by the choosing a set compatible with the symmetry of electrostatic problem.

Laplace's Equation. In charge free region $\rho = 0$; hence Poisson's equation charge free region takes the form

$$\nabla^2 \phi = 0. \qquad ...(4.6)$$

This equation as is known as *Laplace's equation.*

We already is known that the scalar potential ϕ for an arbitrary charge distribution over a given volume is

$$\phi\,(\mathbf{r}) = \frac{1}{4\pi\,\varepsilon_0} \int \frac{\rho\,(\mathbf{r}')}{|\,\mathbf{r} - \mathbf{r}'\,|}\, dv' \qquad ...(4.7)$$

where $\rho\,(\mathbf{r}')$ is the charge density at $\mathbf{r}'$ and $\mathbf{r}$ is the point of observation.

The verify that equation (4.7) satisfies Poisson's equation (4.5b), we operator with the Laplacian on both the sides

$$\nabla^2 \phi = \frac{1}{4\pi\,\varepsilon_0}\, \nabla^2 \int \frac{\rho\,(\mathbf{r}')}{|\,\mathbf{r} - \mathbf{r}'\,|}\, dv'$$

$$= \frac{1}{4\pi\,\varepsilon_0} \int \rho\,(\mathbf{r}')\; \nabla^2 \left(\frac{1}{|\,\mathbf{r} - \mathbf{r}'\,|}\right) dv \qquad ...(4.8)$$

Let us now calculate $\nabla^2 \left(\frac{1}{|\,\mathbf{r} - \mathbf{r}'\,|}\right)$. For convenience let us translate the origin to $\mathbf{r}'$ (*i.e.* we assume that charge is placed at origin), then $\nabla^2 \left(\frac{1}{|\,\mathbf{r} - \mathbf{r}'\,|}\right)$ transforms to $\nabla^2 \left(\frac{1}{r}\right)$.

By direct calculation, we find that

$$\nabla^2 \left(\frac{1}{r}\right) = 0 \quad \text{for} \quad r \neq 0.$$

At $r = 0$, however the expression is undefined. Hence we must use a limiting process. Since we anticipate something like a Dirac delta function. we integrate $\nabla^2 \left(\frac{1}{r}\right)$ over a small volume V containing the origin. Then we use the divergence theorem to obtain a surface integral.

$$\int_V \nabla^2 \left(\frac{1}{r}\right) dv = \int_V \nabla \cdot \nabla \left(\frac{1}{r}\right) dv = \int_S \nabla \left(\frac{1}{r}\right) \cdot \mathbf{n}\, da$$

$$= \int_S \frac{\partial}{\partial r} \left(\frac{1}{r}\right) r^2\, d\Omega = -4\pi.$$

Thus we have $\nabla^2 (1/r) = 0$ for $r \neq 0$ and that its volume integral is -4π. Consequently we may write improper (but mathematically justifiable) equation. $\nabla^2 (1/r) = -4\pi\,\delta\,(\mathbf{r})$ or more generally

$$\nabla^2 \left(\frac{1}{|\,\mathbf{r} - \mathbf{r}'\,|}\right) = -\,4\pi\,\delta\,(\mathbf{r} - \mathbf{r}') \qquad ...(4.9)$$

using (4.9), equation (4.8) becomes

$$\nabla^2 \phi = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{r}) \, [-4\pi\delta(\mathbf{r}-\mathbf{r}')] \, dv' = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

which is exactly Poisson's equation (4.5b).

This verifies that the potential function (4.7) represent the solution of Poisson's equation.

Ex. 1. *Show that the potential function*

$$\phi = q(x^2+y^2+z^2)^{-1/2}$$

satisfies the Laplace's equations. **(Rohilkhand 1999)**

Solution. We have $\phi = q(x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}[q(x^2+y^2+z^2)^{-1/2}]$$

$$= -qx(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -qx\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} + 2x - q(x^2+y^2+z^2)^{-3/2}$$

$$= q\left[\frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right] \quad \text{...(1)}$$

Similarly

$$\frac{\partial^2 \phi}{\partial y^2} = q\left[\frac{3y^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right] \quad \text{...(2)}$$

and

$$\frac{\partial^2 \phi}{\partial z^2} = q\left[\frac{3z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right] \quad \text{...(3)}$$

Adding (1), (2) and (3)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = q\left[\frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= q\left[\frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}}\right]$$

i.e. $\nabla^2 \phi = 0$, which is Laplace's equation. Thus the function $\phi = q(x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation.

Ex. 2. *Does the potential function* $2x^2 - 2y^2 + 2z$ *satisfies Laplace's equation ?* **(Rohilkhand 1997)**

Solution. $\phi = 2x^2 - 2y^2 + 2z$

$$\frac{\partial \phi}{\partial x} = 4x \text{ and } \frac{\partial^2 \phi}{\partial x^2} = 4$$

$$\frac{\partial \phi}{\partial y} = -4y \text{ and } \frac{\partial^2 \phi}{\partial y^2} = -4$$

$$\frac{\partial \phi}{\partial z} = 2 \text{ and } \frac{\partial^2 \phi}{\partial z^2} = 0$$

$\therefore$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 4 - 4 + 0 = 0$$

or

$$\nabla^2 \phi = 0$$

Thus the given function satisfies Laplace's equation.

Ex. 3. *If $\phi_1, \phi_2, \ldots \phi_n$ are all solutions of Laplace's equation, then show that*

$$\phi = \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_n\phi_n$$

is also a solution. Here λ's are arbitrary constants.

Solution. We are given that

$$\nabla^2\phi_1 = 0,\ \nabla^2\phi_2 = 0, \ldots \text{ and } \nabla^2\phi_n = 0 \qquad \ldots(1)$$

Also
$$\phi = \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_n\phi_n . \qquad \ldots(2)$$

Therefore we have

$$\nabla^2\phi = \nabla^2(\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_n\phi_n)$$
$$= \nabla^2(\lambda_1\phi_1) + \nabla^2(\lambda_2\phi_2) + \ldots + \nabla^2(\lambda_n\phi_n)$$
$$\lambda_1\nabla^2\phi_1 + \lambda_2\nabla^2\phi_2 + \ldots + \lambda_n\nabla^2\phi_n = 0 \text{ using (1).}$$

Note. By the use of this example we may superimpose two or more solutions of Laplace's equation in such a way what the resulting solution satifies a given set of boundary conditions.

4.3. Uniqueness Theorem

Electrostatic problems can be stated in terms of the electrostatic potential ϕ, for if ϕ can be found we can at once get the electric field strength E from the relation $\mathbf{E} = -\operatorname{grad}\phi$. Consider electrostatic problem of a number of charged conductors $C_1, C_2, C_3, \ldots, C_k$ etc., having potentials $\phi_1, \phi_2, \phi_3, \ldots \phi_k$, etc. respectively. The electric potential ϕ every where outside the conductors, where charge density $\rho = 0$, must satisfy the Laplace's equation

$$\nabla^2\phi = 0. \qquad \ldots(1)$$

Thus the problem of finding the electric field due to a given set of charged conductors is same as one of solving the Laplace's equation satisfying the specifed conditions on the conducting surfaces. Now a general question of interest is that with given boudnary condition, does the problem have no solution, one solution or more than one solutions ?

Uniqueness theorem *states that Laplace's equation satisfying given boundary conditions have one and only one solution (i.e. unique solution).*

To prove this theorem let us consider a closed volume V_0 exterior to the surfaces $S_1, S_2, S_3 \ldots S_n$ of the various conductors $C_1, C_2, C_3 \ldots C_n$ and bounded on the outside by a surface S, latter being either a surface at infinity or a real physical surface which encloses V_0. Let us suppose that ϕ_1 and ϕ_2 are two different solutions of Laplace's equation in V_0 which satisfy the same boundary conditions on surface $S, S_1, S_2, \ldots S_n$. These boundary conditions may be specified by assigning values of either $\phi_1 = \phi_2$ or $(\nabla\phi_1)_n = (\nabla\phi_2)_n$. (The component of $\nabla\phi$ normal to a surface is often, called the *normal derivative* and may be designated by $\partial\phi/\partial n$).

If we substitute $\phi\nabla\phi$ into Gauss divergence theorem, we obtain

$$\int_{S+S_1+S_2+\ldots+S_n} \phi\nabla\phi\cdot d\mathbf{S} \int_{V_0} \nabla\cdot(\phi\nabla\phi)\,dv. \qquad \ldots(2)$$

From vector identity

$$\nabla\cdot(\phi\nabla\phi) = \phi\nabla^2\phi + (\nabla\phi)^2$$

we may write equation (2) as

$$\int_{S+S_1+S_2+\ldots+S_n} \phi\nabla\phi\cdot d\mathbf{S} = \int_{V_0} [\phi\nabla^2\phi + (\nabla\phi)^2]\,dv. \qquad \ldots(3)$$

But Laplace's equation indicates that $\nabla^2\phi = 0$ at all points in V_0, so that equation (3) takes the form

$$\int_{S+S_1+S_2+\ldots+S_n} \phi\, \nabla\phi \cdot d\mathbf{S} = \int_{V_0} (\nabla\,\phi)^2\, dv. \qquad \ldots(4)$$

As ϕ_1 and ϕ_2 are two solutions of Laplace's equation in V_0, hence from example 1, their linear combination $\psi = \phi_1 - \phi_2$ is also a solution of Laplace's equation satisfiying same boundary conditions. Therefore, substituting $(\phi_1 - \phi_2)$ for ϕ in equation (4), we get

$$\int (\phi_1 - \phi_2)\, \nabla\,(\phi_1 - \phi_2) \cdot d\mathbf{S} = \int [\nabla\,(\phi_1 - \phi_2)]^2\, dv. \qquad \ldots(5)$$

Either boundary condition equality of potentials $\phi_1 = \phi_2$ or their normal derivatives $[(\nabla\phi_1)_n = (\nabla\phi_2)_n]$ assures that L.H.S. of (5) vanishes: hence we have

$$\int_{V_0} [\nabla\,(\phi_1 - \phi_2)]\, dv = 0. \qquad \ldots(6)$$

Since the integrand of above equation is definitely positive, it must be zero in order for the integral to vanish, hence throughout the volume V_0

$$\nabla\,(\phi_1 - \phi_2) = 0 \text{ i.e } \nabla\phi_1 = \nabla\phi_2. \qquad \ldots(7a)$$

This implies

$$\phi_1 = \phi_2 + \text{constant}. \qquad \ldots(7b)$$

Thus the two potentials that were assumed different, yet satisfying the same boundary conditions can differ at most by an additive constant which makes no contribution to the gradient, therefore these potentials will give the same electric field distribution. In other words under given boundary conditions Laplace's equation has unique solution. Thus the theorem is proved.

In the system involves linear dielectrics, then equation (2) may be replaced by

$$\int \phi K\, \nabla\phi \cdot d\mathbf{S} = \int \nabla \cdot (\phi K\, \nabla\phi)\, dv$$

$$= \int [K\,(\nabla\phi)^2 + \phi\, \nabla \cdot (K\, \nabla\phi)]\, dv \qquad \ldots(8)$$

and Laplace's equation for dielectric is

$$\nabla \cdot (K\, \nabla\phi) = 0. \qquad \ldots(9)$$

Hence

$$\int \phi K\, \nabla\phi \cdot d\mathbf{S} = \int [K\,(\nabla\phi)^2]\, dv.$$

As L.H.S. vanishes by either boundary condition as before. Hence R.H.S. proves uniqueness theorem. Thus uniqueness theorem remains valid for linear dielectrics. If the system involves non-linear dielectrics, then the region may be divided into sub regions each having uniform polarisation density and for each sub-region uniqueness theorem holds separately.

4.4. Green's Reciprocity Theorem

This theorem is useful to transform the solution of a known problem into the solution of desired unknown problem.

Let us consider a set of n point charges q_i placed at points, where the potentials due to the other charges are given by a set of numbers ϕ_j. The potential at j^{th} point due to charges q_i at other points is given by

$$\phi_j = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n}{}' \frac{q_i}{r_{ij}} \qquad \ldots(1)$$

The prime on the summation indicates that $i \neq j$.

Now, if a different set of charges q_i' is placed at the same points giving rise to the corresponding potential ϕ_j', then

$$\phi_j' = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n}{}' \frac{q_i'}{r_{ij}} \qquad \ldots(2)$$

Multiplying equation (1) by q_j' and equation (2) by q_j, and summing each expression over the index j, we get

$$\sum_{j=1}^{n} \phi_j q_j' = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{q_i q_j'}{r_{ij}} \qquad ...(3)$$

and

$$\sum_{j=1}^{n} \phi_j' q_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i' q_j}{r_{ij}} \qquad ...(4)$$

Since i and j are summation indices, we may interchange them in one product of q's. Then we obtain

$$\sum_{j=1}^{n} \phi_j q_j' = \sum_{j=1}^{n} \phi_j' q_j$$

This is the required *Green's reciprocity theorem.* This theorem may be generalised from a set of n point charges to set of n conductors at potential ϕ_j and carrying charges q_j.

4.5. Formal Solution of Potential by Green's Function

A large number of potential problems may be solved by Green's function. The Green's function is simply the solution of potential problem for a given geometrical arrangement of grounded conducting boundaries when the only charge present is a unit point charge located at $\mathbf{r}'$. It may be shown with the aid of Green's reciprocity theorem that the Green's function for a particular geometrical arrangement is a symmetrical function of co-ordinates of a unit charge located at the point $\mathbf{r}'$ and the co-ordinates of the point of observation $\mathbf{r}$,

i.e.

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}). \qquad ...(1)$$

This symmetry property merely represents the physical interchangeability of the position of the point charge and the observation point.

We may solve two general types of problems by the use of Green's function : (1) *The problems where the potential distribution* ϕs *over a certain boundary is given.* (2) *The problems in which the charge distribution is given in a region with a conducting boundary.*

The solution of above both types of problem can be given by means of usual Green's theorem

$$\int (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dv = \int (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{a} \qquad ...(2)$$

where ψ and ϕ are arbitrary functions of position with condition that they are non-singular throughout the volume V.

Let ϕ be the desired solution of the problems and let $\psi = G$ be the Green's function for the geometry of problems, *i.e.* the solution of the problem of a unit point charge with grounded surface S. Then Green's function $G(\mathbf{r}, \mathbf{r}')$ will be of the form

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') \qquad ...(3)$$

where the function $F(\mathbf{r}, \mathbf{r}')$ represents the potential due to the induced charge on S and satisfies Laplace's equation inside the volume V,

i.e.

$$\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0 \qquad ...(4)$$

From equation (4.9) it follows that

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \qquad ...(5)$$

Hence
$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \nabla^2\left[\frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}')\right]$$
using (4) and (5), we immediately get
$$\nabla^2 G(\mathbf{r}+\mathbf{r}') = -\frac{\delta(\mathbf{r}-\mathbf{r}')}{\varepsilon_0} \qquad \text{...(6)}$$
Now substituting $\psi = G$ in Green's theorem (2) we get
$$\int(\phi\,\nabla^2 G - G\nabla^2\phi)\,dv = \int(\phi\,\nabla G - G\,\nabla\phi)\cdot da$$
i.e.
$$\int_S [\phi(\mathbf{r}')\,\nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\,\nabla^2\phi(\mathbf{r}')]\,dv'$$
$$= \int_V [\phi(\mathbf{r}')\,\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\,\nabla\phi(\mathbf{r}')]\cdot d\mathbf{a}'. \qquad \text{...(7)}$$
Since Green's function has singularity only $\mathbf{r} = \mathbf{r}'$ (refer equation (3)) which may be handled by means of the δ-function given by (6). Thus using (6) we get
$$\int_V\left[-\phi(\mathbf{r}')\frac{\delta(\mathbf{r}-\mathbf{r}')}{\varepsilon_0} - G(\mathbf{r}, \mathbf{r}')\,\nabla^2\phi(\mathbf{r}')\right]dv'$$
$$= \int_S [\phi(\mathbf{r}')\,\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\,\nabla\phi(\mathbf{r}')]\cdot d\mathbf{a}'$$
i.e.
$$\frac{\phi(\mathbf{r})}{\varepsilon_0} - \int G(\mathbf{r}, \mathbf{r}')\,\nabla^2\phi(\mathbf{r}')\,dv$$
$$= \int_S [\phi(\mathbf{r}')\,\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')]\,\nabla\phi(\mathbf{r}')\cdot d\mathbf{a}'$$
i.e.
$$\phi(\mathbf{r}) = -\varepsilon_0\int_V G(\mathbf{r}, \mathbf{r}')\,\nabla^2\phi(\mathbf{r}')\,dv' + \varepsilon_0\int_S [G(\mathbf{r}, \mathbf{r}')]\,\nabla\phi(\mathbf{r}') - \phi(\mathbf{r}')\,\nabla G(\mathbf{r}, \mathbf{r}')]\cdot d\mathbf{a}'. \qquad \text{...(8)}$$
The freedom available in the definition of G given by (3), means that we can make the surface integral depend only on the chosen type of boundary conditions. For uniqueness of potential on surface S, for *Dirichlet boundary condition,* we demand
$$G(\mathbf{r}, \mathbf{r}') = 0 \text{ for } \mathbf{r}' \text{ on } S.$$
Hence on collecting the non-vanishing terms, we get
$$\phi(\mathbf{r}) = -\varepsilon_0\int_V G(\mathbf{r}, \mathbf{r}')\,\nabla^2\phi(\mathbf{r}')\,dv' + \int_S \phi(\mathbf{r}')\,\nabla G(\mathbf{r}, \mathbf{r}')\cdot d\mathbf{a} \qquad \text{...(9)}$$

Let us now consider the two cases mentioned earlier :

Case (i) *The surface surrounding the point* **r'** *is grounded, making*
$$\phi(\mathbf{r}') = 0 \text{ and } \nabla^2\phi(\mathbf{r}') = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$
due to charge distribution $\rho(\mathbf{r}')$ throughout the volume V. In this case equation (9) reduces to
$$\phi(\mathbf{r}) = -\varepsilon_0\int G(\mathbf{r}, \mathbf{r}')\,\nabla^2\phi(\mathbf{r})\,dv' \qquad \text{...[10(a)]}$$
$$= \int_V G(\mathbf{r}, \mathbf{r}')\,\rho(\mathbf{r}')\,dv'. \qquad \text{...[10(b)]}$$
The validity of the equation is obvious since it merely represents the principle of superposition applied to density of point source within the volume V.

Case (ii). *When there are no source of* ϕ *throughout the volume* V, *so that* $\nabla^2\phi = 0$; *but the potential function* $\phi(\mathbf{r}')$ *over the surface* S *is given.* In this case equation (9) reduces to

$$\phi(\mathbf{r}) = -\varepsilon_0 \int_S \phi(\mathbf{r}') \nabla G(\mathbf{r}, \mathbf{r}') \cdot d\,\mathbf{a}'. \qquad ...(11)$$

This equation gives the *potential* within a given region enclosed by a boundary such that its different parts are raised to a given set of potentials. It is obvious that the solution in this case expresses the potential within this boundary in terms of surface integral of the potential on the boundary multiplied by the normal derivative of the Green's function. Physically the normal derivative of Green's function represents the surface charge density that would be induced on the boundary by a unit charge placed at the position $\mathbf{r}'$ if the boundary were a grounded conductor. Thus equation (11) represents the solution of the potential problem corresponding to a given potential on the boundary in terms of the surface integral of this given potential multiplied by the induced charge on the grounded boundary by a unit charge placed at the field point $\mathbf{r}'$.

4.6. Method of Electrical Images

Lord Kelvin (1124-1207), invented the method of electrical images for solving many special problems in electrostatics with great simplicity. In other words, *complicated distributions of charge are replaced, for simplicity of calculations, either by a single point charge or by a set of point charges or by a much simplified charge distribution* without affecting the boundary condition of the problem. Thus *the method of electrical images is to locate either a point charge or a set of point charges or a simplified charge distribution, known as* **electrical image** *in place of complicated charge distributions such that by their electrical effects the boundary conditions of the problem remain the same as with the charge distributions.*

The general principle underlying the method of electrical images is given below :

(i) In a system of an n point charges $q_1, q_2, ..., q_n$ the potential at any point is given by

$$\phi = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} \qquad ...(i)$$

(ii) The surface of zero potential (*i.e.*, grounded conductor) is the locus of points for which

$$\phi = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} = 0 \qquad ...(2)$$

(iii) In any problems involving the system of charges $q_1, q_2, \ldots q_j$ and the grounded conductor, the latter may be replaced by the system of charges $q_{j+1}, q_{j+2}, \ldots q_n$ constituting the image if the image charge and the system of real charges combine to produce zero potential over the surface occupied by the grounded conductor. So equation (2) may be rewritten as

$$\phi = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{j} \frac{q_i}{r_i} + \frac{1}{4\pi\varepsilon_0} \sum_{i=j+1}^{n} \frac{q_i}{r_i} = 0 \qquad ...(3)$$

This equation represents the condition of potential function over the surface occupied by the grounded conductor replaced by electrical image constituted by the system of charges $q_1, q_2, ..., q_j$.

(iv) If the conductor under discussion is at a constant potential $\phi \neq 0$, the system of charges and its image must combine to produce the actual potential ϕ. Equation (3) then may be written as

$$\phi = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} + \frac{1}{4\pi\varepsilon_0} \sum_{i=j+1}^{n} \frac{q_i}{r_i} \qquad ...(4)$$

This equation represents the condition of potential function over the surface occupied by the conductor replaced by electrical image constituted by the system of charges $q_{j+1}, q_{j+2}, \ldots q_n$.

Here we shall illustrate the method by some simple, but significant cases.

4.7. A Point Charge Near an Infinite Conducting Plane

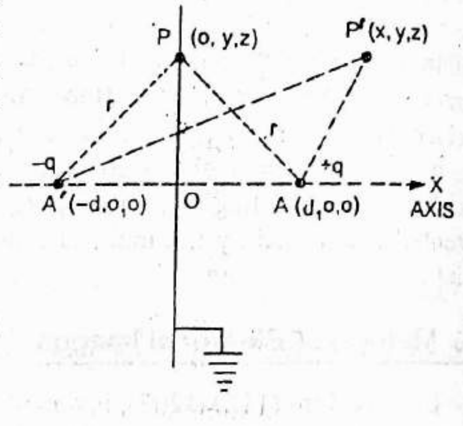


Fig.4.1 [Point (x, y, z) is P']

Let a point charge $+q$ be placed on X-axis at a distance d to the right of an infinite grounded conducting plane placed at $x = 0$ with its surface in X-Z plane. As the conducting plane is grounded, the electric potential at every point of the conductor is zero. Thus the boundary conditions are.

(i) The electric potential ϕ at every point of the conductor is zero, *i.e.*, $\phi = 0$ at $x = 0$.

(ii) The electric potential at infinitely far points is zero, *i.e.* $\phi = 0$ at $x = \infty$.

These conditions can be satisfied if we replace the grounded plane conductor by a point charge (image charge) $-q$ at a distance d to the left of the plane such that the line joining the image charge to the real charge interesects the plane of the conductor at right angle, (fig 4.1).

(i) At point P $(0, y, s)$ on the plane conductor, the electric potential due to real charge $+q$ and the image charge $-q$ (the conductor having been removed) is

$$\phi\,(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{PA} - \frac{q}{PA'}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} - \frac{q}{r}\right] = 0$$

which is the first boundary condition of the problem.

(i) The charge $-q$ placed at point A' does not affect the condition of electric potential at infinitely fa points.

Thus we see that the replacement of the grounded conducting plane by the charge $-q$ at A' fulfils the given boundary conditions and therefore according to uniqueness theorem, is the only solution of the problem.

The image charge $-q$ at A' and the real charge q at A will produce same field on the right hand side of the plane as produced due to real charge $+q$ and the charge induced on the grounded conductor. With the help of this image charge $-q$ (the conductor having been removed) we can really calculate the surface density of the induced charge on the conductor, the force between the real charge $+q$ and the grounded conducting plane, electric field and electric potential at any point due the real charge and the grounded conducting plane etc.

Electric potential and electric field strength at any point (x, y, z) due to the point charge q and the earthed plane conductor : The electric potential at point P' (x, y, z) due to the point charge $\pm q$ and the earthed conducting plane is same as that due to the point charge $+q$ at A and the image charge $-q$ at A'. Thus

$$\phi\,(P') = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{AP'} - \frac{q}{A'P'}\right]$$

But $$AP' = \sqrt{\{(x-d)^2 + y^2 + z^2\}} \text{ and } A'P' = \sqrt{\{(x+d)^2 + y^2 + z^2\}}$$

$\therefore$ $$\phi(P') = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\{(x-d)^2 + y^2 + z^2\}}} - \frac{1}{\sqrt{\{(x+d)^2 + y^2 + z^2\}}}\right] \qquad \text{...(1)}$$

If $x = \infty$, equation (1) gives, $\phi = 0$.

Thus the boundary condition $\phi = 0$ at $x = \infty$ is also satisfied if the grounded plane conductor is replaced by a point charge $-q$ at point A'.

The electric field components E_x, E_y and E_z at point $P'(x, y, z)$ can be obtained by differentiating equation (i) partially with respect to x, y, z respectively, *i.e.*

$$E_x = -\frac{\partial\phi}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{\{(x-d)^2+y^2+z^2}}-\frac{1}{\sqrt{\{(x+d)^2+y^2+z^2\}}}\right\}\right]$$

$$E_x = \frac{1}{4\pi\varepsilon_0}\left[\frac{x-d}{\{(x-d)^2+y^2+z^2\}^{3/2}}-\frac{x+d}{\{(x+d)^2+y^2+z^2\}^{3/2}}\right] \quad \text{...(2i)}$$

$$E_y = -\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}\left[\frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{\{(x+d)^2+y^2+z^2\}}}-\frac{1}{\sqrt{\{(x+d)^2+y^2+z^2\}}}\right\}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{y}{\{(x-d)^2+y^2+z^2\}^{3/2}}-\frac{y}{\{(x+d)^2+y^2+z^2\}^{3/2}}\right] \quad \text{...(2ii)}$$

and
$$E_z = -\frac{\partial\phi}{\partial z} = \frac{q}{4\pi\varepsilon_0}\left[\frac{z}{\{(x-d)^2+y^2+z^2\}^{3/2}}-\frac{z}{\{(x+d)^2+y^2+z^2\}^{3/2}}\right] \quad \text{...(2iii)}$$

The electric field strength

$$\mathbf{E} = \mathbf{i}\,E_x + \mathbf{j}\,E_y + \mathbf{k}\,E_z. \quad \text{...(3)}$$

Electric field on the conductor : From equations (2i), (2ii) and (2iii), the components of electric field at a point $P(o, y, z)$ on the conducting plane are given by

$$(E_x)_{x=0} = -\left(\frac{\partial\phi}{\partial x}\right)_{x=0}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-d}{(d^2+y^2+z^2)^{3/2}}-\frac{d}{(d^2+y^2+z^2)^{3/2}}\right]$$

or
$$(E_x)_{x=0} = -\frac{qd}{2\pi\varepsilon_0\,(d^2+y^2+z^2)^{3/2}}. \quad \text{...(4i)}$$

$$(E_y)_{x=0} = -\left(\frac{\partial\phi}{\partial y}\right)_{x=0}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{y}{(d^2+y^2+z^2)^{3/2}}-\frac{y}{(d^2+y^2+z^2)^{3/2}}\right] = 0 \quad \text{...(4ii)}$$

and
$$(E_z)_{x=0} = -\left(\frac{\partial\phi}{\partial z}\right)_{x=0} = \frac{q}{4\pi\varepsilon_0}\left[\frac{z}{(d^2+y^2+z^2)^{3/2}}-\frac{z}{(d^2+y^2+z^2)^{3/2}}\right] = 0 \quad \text{...(4iii)}$$

Obviously the electric field at any point $P(0, y, z)$ on the conductor has magnitude $\dfrac{qd}{2\pi\varepsilon_0\,(d^2+y^2+z^2)^{3/2}}$ and is directed normally towards the conductor. It may be represented as

$$\mathbf{E}(P) = -\frac{qd}{2\pi\varepsilon_0\,(d^2+y^2+z^2)^{3/2}}\,\mathbf{i}. \quad \text{...(5)}$$

Here **i** is unit vector along X-axis *i.e.* unit vector directed normally away from the conductor.

Surface charge density of the conductor : If σ is the surface density of induced charge on the conducting plane at point $P(0, y, z)$, the electric field at P can be written as

$$E = (E_x)_{x=0} = \frac{\sigma}{\varepsilon_0} \quad \text{...(6)}$$

so that surface density of induced charge at P, $\sigma = \varepsilon_0\,(E_x)_{x=0}$

$$= \varepsilon_0\left[\frac{-qd}{2\pi\varepsilon_0\,(d^2+y^2+z^2)^{3/2}}\right] = -\frac{qd}{2\pi\,(d^2+y^2+z^2)^{3/2}} \quad \text{...(7)}$$

As the quantity $(d^2+y^2+z^2)^{1/2}$ represents the distance of the point P on the conducting plane from the point charge q, it is obvious tnat the surface density of induced charge on the conducting plane varies as the cube of the distance from the point charge.

Force of attraction between the point charge $+q$ and the earthed plane conductor : The force F between the point charge $+q$ and the earthed plane conductor is same as that between the charge $+q$ at A and the image charge $-q$ at A', thus

$$F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q(-q)}{(AA')^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2} = -\frac{q^2}{16\pi\varepsilon_o d^2}\text{ Newton} \qquad ...(8)$$

negative sign indicates that the force is **attractive.**

As the force is directed normally, towards the conductor, we may write

$$\mathbf{F} = -\frac{q^2}{16\pi\varepsilon_0 d^2}\mathbf{i}. \qquad ...(9)$$

Work done in removing charge q to infinity : The force between the point charge $+q$ and the earthed conducting plane is given by

$$\mathbf{F} = -\frac{q^2}{16\pi\varepsilon_0 d^2}\mathbf{i}.$$

Therefore the work done against this force in removing the charge $+q$ from $x = d$ to $x = \infty$ is given by

$$W = -\int_{x=d}^{\infty}\mathbf{F}\cdot d\mathbf{r} = -\int_{x=d}^{\infty}\left(-\frac{q^2}{16\pi\varepsilon_0 x^2}\mathbf{i}\right)(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)$$

$$= \frac{q^2}{16\pi\varepsilon_0}\int_d^{\infty}\frac{1}{x^2}dx = \frac{-q^2}{16\pi\varepsilon_0 d}. \qquad ...(10)$$

This work done in bringing the charge from $x = \infty$ to $x = d$ is given by

$$W' = \int_{x=\infty}^{d}\mathbf{F}\cdot d\mathbf{r} = -\frac{q^2}{16\pi\varepsilon_0 d}. \qquad ...(11)$$

The total charge induced on the conducting plane. Consider a ring area on the plane conductor around O with radii r and $r+\delta r$. The charge δQ_i on this elementary area

$$\delta Q_i = 2\pi r\delta r.\sigma = 2\pi r.\delta r\left\{\frac{-qd}{2\pi(r^2+d^3)^{3/2}}\right\} \qquad (\text{Since } r^2 = y^2+z^2)$$

$$= \frac{-qdr.\delta r}{(r^2+d^2)^{3/2}}.$$

The total charge induced

$$Q_i = -\int_0^{\infty}\frac{q.dr.\delta r}{(r^2+a^2)^{3/2}} = -qd\int_0^{\infty}\frac{r\,\delta r}{(r^2+d^2)^{3/2}}$$

Putting $r = d\tan\theta$; So that $\delta r = d\sec^2\theta\,\delta\theta$, we get

$$Q_i = \int_0^{\pi/2}\frac{d\tan\theta.\,d\sec^2\theta\,\delta\theta}{(d^2\tan^2\theta+d^2)^{3/2}}$$

$$= -qd\int_0^{\pi/2}\frac{d^2\tan\theta\sec^2\theta\,\delta\theta}{d^3\sec^3\theta}$$

$$= -q\int_0^{\pi/2}\sin\theta\,\delta\theta = -q$$

Thus the total charge induced on earthed plane conductor is $-q$.

Ex. 4. *A proton is at a distance 10 Å from an infinite plane conductor. Calculate the force experienced by the proton and the work done in moving it to infinite distance away from the conductor.*

(Rohilkhand 1995, 87)

Solution. By the method of images the force on the point charge due to charge induced on the earth plane conductor

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d^2} \text{ (attractive)}$$

Here $q = +e = 1{\cdot}6 \times 10^{-19}$ C, $d = 100$ Å $= 10^{-9}$ m

$$\therefore \qquad F = 9 \times 10^9 \frac{(1{\cdot}6 \times 10^{-19})^2}{4 \times (10^{-9})^2} = \mathbf{5{\cdot}76 \times 10^{-11}\ N} \quad \text{(attractive)}$$

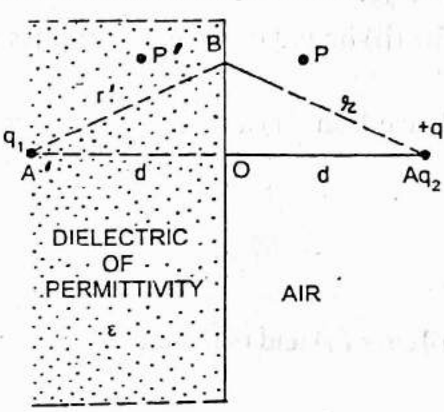
Work done in moving the proton to infinity

$$W = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

$$= 9 \times 10^9 \times \frac{(1{\cdot}6 \times 10^{-19})^2}{4 \times 10^{-9}} = \mathbf{5{\cdot}76 \times 10^{-20}\ J}$$

4.8. A Point Charge in Front of Semi-Infinite Dielectric

Let us consider that we have a point charge at point A in air, at a distance d from the plane surface of a dielectric. Now question arises what is the effect produced by the presence of the dielectric ? This problem corresponds to that in preceding section for the conducting plane. In that case, however, we had only to take the field in air into account, since on the far side of the conducting plane no field existed at all. But now we shall have to consider the field within the dielectric. Let us assume the dielectric to occupy all space beyond an infinite bounding plane. Let K be the dielectric constant of the dielectric medium. Let us now solve the problem by the method of images. As before let us take a point A' within the dielectric; where A' is the optical image of A in the bounding plane such that $AO = A'O = d$. Let r and r' be the distances of field point from A and A' respectively.



(Fig.4.2)

If we put for the potential at the field point (say P) in air space

$$\phi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} - \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r'}. \qquad \text{...(1)}$$

The field in air thus corresponds to the true charge q at A and the true charge $-q_1$ imagined at A'. This assumption satisfies the fundamental condition that sources of electric displacement must occur at A only; because the image point A' is outside the air space.

Let us try to represent the potential within the dielectric at any point (say P') by putting

$$\phi_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r} = \frac{1}{4\pi\varepsilon_0 K}\frac{q_2}{r}. \qquad \text{...(2)}$$

The field within the dielectric is thus to be the same as if the dielectric occupied all free space and a true charge q_2 were placed at A. This assumption is also in agreement with the condition that within the region of dielectric there are no sources or sinks of electrical displacement.

To Prove that the field is actually represented by these assumptions, we shall show that boundary conditions at the plane separating the two media can be satisfied by a suitable choice of quantities q_1, q_2 which so far have not been defined. The boundary conditions are :

(i) The normal component of electric displacement **D** is continuous. This condition requires $(D_n)_{\text{air}} = (D_n)_{\text{dielectric}}$ at the boundary where $r = r'$.

(ii) The tangential component of **E** on the two sides of the boundary must possess equal values. This condition requires that $\phi_1 = \phi_2$ all along the plane since $\mathbf{E} = -\text{grad}\,\phi$.

The *normal component of electric displacement at field point P* in air (assuming charge q at A and charge $= q_1$ at A') is given by

$$(D_n)_{\text{air}} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 d}{r'^3} \text{ along } \mathbf{AA'}.$$

and the normal component of electric displacement at any point P' in dielectric (assuming only charge q_2 at A) is given by

$$(D_n)_{\text{dielectric}} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 d}{r^3} \text{ along } \mathbf{AA'}$$

Hence from (i) boundary condition

viz $(\mathbf{D}_n)_{\text{air}} = (D_n)_{\text{dielectric}}$ at $r = r'$, we get

$$\frac{1}{4\pi\varepsilon_0}\frac{qd}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 d}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 d}{r^3}.$$

This gives

$$q + q_1 = +q_2 \qquad \text{...(3)}$$

The (ii) boundary equation implies that

$$\phi_1 = \phi_2 \text{ at } r = r'.$$

Hence from (1) and (2) at $r = r'$, we get

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{r} - \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r} = \frac{1}{4\pi\varepsilon_0 K}\frac{q_2}{r}.$$

i.e.

$$q - q_1 = \frac{q_2}{K}. \qquad \text{...(4)}$$

Solving (3) and (4), we get

$$q_1 = \frac{K-1}{K+1}q \qquad \text{...(5)}$$

and

$$q_2 = \frac{2K}{K+1}q. \qquad \text{...(6)}$$

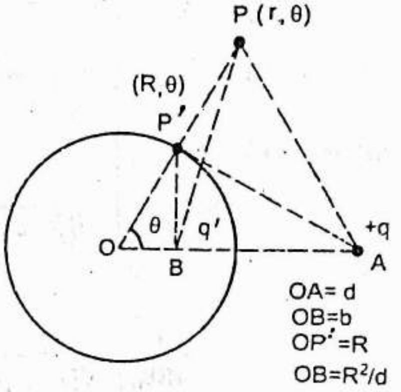
Thus, we have determined the assumed **'true'** charges $-q_1$ at A' and $+q_2$ at A. The line of force within the dielectric run so that they appear to diverge radially from the source A, while in air space the field is the resultant of the two fields arising from source A and the sink A'.

Force of Attraction. The force of attraction between the point charge and the dielectric is the same as that between the charge $+q$ at A and $-q_1$ at A' assuming all air space; hence

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q(-q_1)}{4d^2} = -\frac{1}{4\pi\varepsilon_0}\left(\frac{K-1}{K+1}\right)\frac{q^2}{4d^2}. \qquad \text{...(7)}$$

4.9. A Point Charge in Front of a Conducting Sphere

Consider a point charge $+q$ placed at a distance d from the centre O of the conducting sphere of radius R such that $R < d$ (fig. 4.3). We wish to find the distribution of the induced charge on the sphere by the external point charge $+q$ and the electric field as various points of space.



(Fig.4.3)

Case (i) When the conducting sphere is grounded. As the conducting sphere is grounded, the boundary conditions are :

(i) The electric potential ϕ at every point on the surface of sphere is zero, *i.e.* $\phi = 0$ when $r = R$.

(ii) The electric potential at infinitely far point is zero, *i.e.* $\phi = 0$ when $r = \infty$.

Let us suppose that the image charge q' placed at point B (distance $OB = b$), when combined with charge $+q$ (the conducting sphere being removed), leads to zero potential at any point P' on the surface of the conductor.

The potential at P', due to the real charge $+q$ and its image charge q' is given by

$$\phi\,(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{AP'} + \frac{q'}{BP'}\right] = 0.$$

This equation yields,

$$q' = -q.\frac{BP}{AP'} \qquad \text{...(1)}$$

we have from triangle OAP' is

$$(AP')^2 = (OP')^2 + (OA)^2 - 2\,OP'.OA\cos\theta$$

or

$$AP' = \sqrt{(R^2 + d^2 - 2Rd\cos\theta)}$$

and from triangle OBP' is

$$(BP')^2 = (OP')^2 + (OA)^2 - 2\,(OP').\,OA\cos\theta$$

or

$$BP' = \sqrt{(R^2 + b^2 - 2Rb\cos\theta)}.$$

Substituting values of AP' and BP' in equation (1), we get

$$q' = -q.\frac{\sqrt{(R^2 + b^2 - 2Rb\cos\theta)}}{\sqrt{(R^2 + d^2 - 2Rd\cos\theta)}} \qquad \text{...(2)}$$

Let us now choose $b = \dfrac{R^2}{d}$, then equation (2) becomes

$$q' = -q\,\frac{\sqrt{\left[R^2 + \left(\frac{R^2}{d}\right)^2 - 2R.\frac{R^2}{d}\cos\theta\right]}}{\sqrt{(R^2 + d^2 - 2Rd\cos\theta)}}$$

$$= -q'\,\frac{R}{d}\,\frac{\sqrt{(d^2 + R^2 + 2Rd\cos\theta)}}{\sqrt{(R^2 + d^2 - 2Rd\cos\theta)}} = -q\left(\frac{R}{d}\right) \qquad \text{...(3)}$$

Thus the *image charge* in this case is a point charge of magnitude $-q.\left(\frac{R}{d}\right)$ placed at a distance $\left(\frac{R^2}{d}\right)$ from the centre of the spherical conductor along the line joining the centre of the sphere to the real point charge $+q$.

The potential at any point $P(r, \theta)$ due to the point charge $+q$ placed near an earthed conductin sphere of radius R is given by

$$\phi(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{AP}+\frac{q'}{BP}\right]$$

But $$AP = \sqrt{(r^2+d^2-2rd\cos\theta)}$$

and $$BP = \sqrt{(r^2+b^2-2rb\cos\theta)}.$$

$\therefore$ $$\phi(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(r^2+d^2-2rd\cos\theta)}}+\frac{q}{\sqrt{(r^2+b^2-2rb\cos\theta)}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(r^2+d^2-2rd\cos\theta)}}-\frac{q\cdot\left(\frac{R}{d}\right)}{\sqrt{\left\{r^2+\left(\frac{R^2}{d}\right)^2-2r.\frac{R^2}{d}\cos\theta\right\}}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{[r^2+d^2-2rd\cos\theta]}}-\frac{q}{\sqrt{\left\{\left(\frac{r^2d^2}{R^2}+R^2-2rd\cos\theta\right)\right\}}}\right] \quad ...(4)$$

The components of electric field strength at point $P(r, \theta)$ along and normal to OP are given by

$$E_r = -\frac{\partial\phi}{\partial r}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{r-d\cos\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}}-\frac{r^2\frac{d^2}{R^2}-d\cos\theta}{\left(r^2\frac{d^2}{R^2}+R^2-2rd\cos\theta\right)^{3/2}}\right] \quad ...(5)$$

$$E_r = -\frac{1}{r}\frac{\partial\phi}{\partial r}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{d\sin\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}}-\frac{d\sin\theta}{\left(\frac{r^2d^2}{R^2}+R^2-2rd\cos\theta\right)^{3/2}}\right] \quad ...(6)$$

Electric field on the surface of the sphere. The components of electric field strength on any point $P'(R, \theta)$ on the surface of the sphere are given by

$$(E_r)_{r=R} = -\left(\frac{\partial\phi}{\partial r}\right)_{r=R} = \frac{q}{4\pi\varepsilon_0}\left[\frac{(R^2-d^2/R)}{(d^2+R^2-2Rd\cos\theta)^{3/2}}\right]$$

or $$(E_\theta)_{r=R} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{R}\cdot\frac{R^2-d^2}{(R^2+d^2-2Rd\cos\theta)^{3/2}} \quad ...(7)$$

$$(E_r)_{r=R} = 0. \quad ...(8)$$

The field at the surface of the sphere has magnitude

$$\frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{R}\frac{R^2-d^2}{(R^2+d^2-2Rd\cos\theta)^{3/2}}$$

and is directed along the line joining from the centre of the sphere to the point P on the surface.

Surface charge density of induced charge on the sphere. If σ is the surface charge density of the induced charge on the point P' on the sphere, the electric field at point P' on the surface of the sphere is given by

$$E = (E_r)_{r=R} = \frac{\sigma}{\varepsilon_0}$$

$\therefore$ Surface charge density $\sigma = \varepsilon_0 (E_r)_{r=R}$

$$= -\frac{q}{4\pi\varepsilon_0} \cdot \frac{d^2 - R^2}{(R^2 + d^2 - 2Rd\cos\theta)^{3/2}}, \qquad \text{...(9)}$$

The force between the sphere and the point charge. The force between the sphere of radius R and point charge $+q$ is the same as that between the point charges q and $q' = \left(-q.\frac{R}{d}\right)$; thus

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(AB)^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q.\left(-q\frac{R}{d}\right)}{\left(d - \frac{R^2}{d}\right)^2}$$

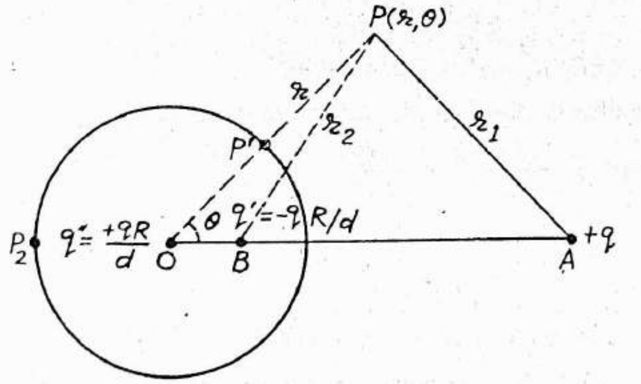
$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2 Rd}{d^2 - R^2} \text{ Newton} \qquad \text{...(10)}$$

negative sign indicates that the force is *attractive.*

Case (ii) When the conducting sphere is insulated. Let us now suppose that the conducting sphere is insulated and is uncharged before the induction of point charge. In this case the following two additional boundary conditions must be satisfied.

(i) The potential on the spherical surface is uniform throughout; since the sphere is conducting.

(ii) The net charge on the conductor remains zero througout



$OA = d$

$OB = b$

Radius $= R$

and $OB = \frac{R^2}{d}$

(Fig.4.4)

The charge $+q$ at A and the image $q' = -\left(\frac{qR}{d}\right)$ at B (fig. 4.4) satisfy the boundary condition outside the sphere and also that of potential at infinity. Now in order to satisfy the boundary conditions (i) and (ii) (*i.e.* constancy of potential on the surface of sphere and zero net charge on the conducting sphere) let us imagine a further charge of magnitude $q' = (+qR)/d$ to be situated at the centre O of sphere.

The charge at centre $(+qR)/d$ together with $(-qR)/d$ at B provides zero net charge and keeps the spherical surface at constant potential without disturbing other boundary conditions.

The potential on the spherical surface is

$$\phi = \frac{\text{Charge at centre}}{\text{Capacity of sphere}} = \frac{+qR/d}{4\pi\varepsilon_0 R} = \frac{1}{4\pi\varepsilon_0}\frac{q}{d}. \quad ...(11)$$

The potential at the field point P due to combination of point charge $+q$ and insulated uncharged sphere is then

$$\phi(P) = \frac{1}{4\phi\varepsilon_0}\frac{q}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad ...(12)$$

where r denotes its distance of the field point P from the centre O, r_1 the distance of the field point P from the charge q at A and r_2 the distance P from the image point charge q' at B

Force between the sphere and the point charge. The force between the conducting sphere due to the induced charge and the point charge q must be the resultant of those between the point charge q and $-qR/d$ at B and the point charge q and $+qR/d$ at O. Thus the *force of attraction is*

$$F = \frac{1}{4\,\phi\varepsilon_0}\frac{q.\frac{qR}{d}}{\left(d-\frac{R^2}{d}\right)^2} - \frac{1}{4\pi\varepsilon_0}\frac{q\left(\frac{qR}{d}\right)}{d^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2Rd}{(d^2-R^2)^2} - \frac{q^2R}{d^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^3}\left[\frac{1}{\left(1+\frac{R^2}{d^2}\right)} - 1\right] \quad ...(13)$$

Surface Density of Induced Charge. We have already found that surface density at point P' due to charge $+q$ and $-q/Rd$ is

$$\sigma_1 = \frac{q\,(d^2-r^2)}{4\pi R\,(R^2+d^2-2Rd\cos\theta)^{3/2}}$$

The surface charge density at P' due to the charge $+qR/d$ placed at the centre is

$$\sigma' = \frac{(QR/d)}{4\pi R^2} = \frac{Q}{4\pi Rd}$$

Hence *net surface charge density* at P' is

$$\sigma = \sigma_1 + \sigma_2 = -\frac{q\,(d^2-R^2)}{4\pi R\,(R^2+d^2-2Rd\cos\theta)^{1/2}} + \frac{Q}{4\pi Rd} \quad ...(14)$$

At P_1, the point nearest to $+q$, on the sphere; surface charge density is given by $[\theta = 0,\ \text{so}\ \cos\theta = \cos 0 = 1]$

$$(\sigma)_{p_1} = -\frac{q\,(d^2-R^2)}{4\pi R\,(R^2+d^2-2Rd)^{3/2}} + \frac{Q}{4\pi Rd}$$

$$= -\frac{q}{4\pi R}\left[\frac{(d^2-R^2)}{(d-R)^3} - \frac{1}{d}\right]$$

$$= -\frac{q}{4\pi R}\left[\frac{d+R}{(d-R)^2} - \frac{1}{d}\right] = \frac{q\,(3d-R)}{4\pi d\,(d-R)^2}$$

This shows that the surface charge density at nearest point P_1 is negative. Now the surface charge density at point P_2, the point farthest to $+q$, on the sphere is

$$[\theta = 180^\circ,\ \text{so that}\ \cos\theta = \cos 180^\circ = -1]$$

$$(\sigma)_{P_2} = -\frac{q\,(d^2-R^2)}{4\pi R\,(R^2+d^2+2Rd)^{3/2}} + \frac{Q}{4\pi Rd} = \frac{q\,(3d+R)}{4\pi d\,(d+R)^2}.$$

This shows that the surface charge density at farthest point P_2 is positive.

Circle of no electrification. Since the surface densities of induced charge on the sphere at point nearest and farthest from the point charge $+q$ are negative and positive respectively; therefore there must be some place where the surface charge density is zero. For that the region $\sigma = 0$. Thus from (14), we get

$$\frac{-q\,(d^2-R^2)}{4\pi R\,(R^2+d^2-2Rd\cos\theta)^{3/2}} + \frac{q}{4\pi Rd} = 0$$

i.e.

$$\frac{-q\,(d^2-R^2)}{4\pi R\,(AP')^3} + \frac{q}{4\pi Rd} = 0$$

since

$$(AP')^2 = R^2 + d^2 - 2\,Rd\cos\theta.$$

This gives

$$(AP')^3 = d\,(d^2-R^2). \qquad ...(15)$$

This value of AP' gives the circle of no electrification.

Case (iii) When the sphere is charged and insulated. Let us now suppose that the sphere is insulated and contains a net charge $+e$ before the introduction of the point charge. In this case the following boundary conditions must be satisfied.

(i) The potential on the surface of the sphere is uniform throughout.

(ii) The net charge on the conductor remains $+e$ throughout.

(iii) The potential at infinity is zero.

(iv) $\nabla^2\phi = 0$ in the external space except at A.

The above boundary conditions are satisfied if assume the given charge $+e$ to be concentrated at the centre O in addition to the charge of case (ii). The system consists of the charge $+q$ at point A, the image charge $q' = -qR/d$ at point B and the charge $q' = e + qR/d$ at the centre O of the sphere (fig. 4.5).

The charge at centre $e + qR/d$ together with $-qR/d$ at B provides net charge equal to $+e$ on the sphere.

The potential on the spherical surface.

$$\phi = \frac{e + \dfrac{qR}{d}}{4\pi\varepsilon_0 R} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{e}{R} + \frac{q}{d}\right\}. \qquad ...(16)$$

The potential at field point P due to *combination* of point charge $= q$ and insulated charged sphere is then

$$\phi(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1} - \frac{1}{4\pi\varepsilon_0}\frac{q'}{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q''}{r} \qquad ...(17)$$

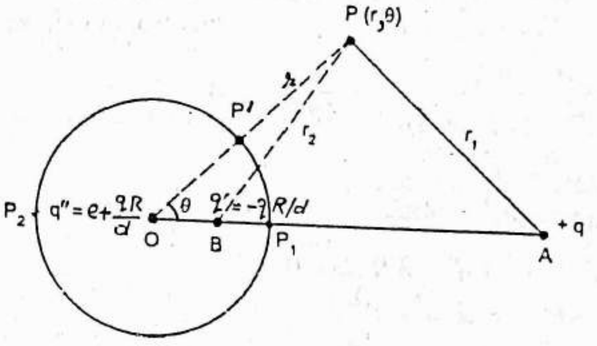
where

$$q' = -\frac{qR}{d}\ \text{and}\ q'' = e + \frac{qR}{d}.$$

Force of conductor. The force between the insulated charged sphere and the point charge $+q$ must be the resultant of force (i) between the point charge q and $-qR/d$ at B, and (ii) the point charge q and $e + (qR/d)$ at O. *The net repulsive force is given by*

$$F = \frac{1}{4\pi\varepsilon_0}\left[\frac{(q)\left(e+\dfrac{qR}{d}\right)}{d^2} - \frac{(q)\left(\dfrac{qR}{d}\right)}{\left(d-\dfrac{R^2}{d}\right)^2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{eq}{d^2} + \frac{q^2R}{d^2} - \frac{q^2Rd}{(d^2-R^2)^2}\right] \qquad \text{...(18)}$$



$OA = d$
$OB = b$
Radius $= R$
and $AB = d - \dfrac{R^2}{d}$

(Fig.4.5)

If A is very near to the spherical surface, then let us substitute $d = R + x$ where x is very small so

$$F = \frac{1}{4\pi\varepsilon_0}\left[\frac{eq}{(R+x)^2} + \frac{q^2R}{(R+x)^3} - \frac{q^3R\,(R+x)}{[(R+x)^2-R^2]^2}\right] \qquad \text{...(19)}$$

When x is negligibly small, then we may write

$$(R+x)^2 = x^2,\ (R+x)^3 = R^3$$

and $\quad [(R+x)^2 - R^2]^2 = [R^2 + x^2 + 2Rx - R^2]^2 = (2R+x)^2x^2.$

Hence equation (19) becomes

$$F = \frac{1}{4\pi\varepsilon_0}\left[\frac{eq}{R^2} + \frac{q^2}{R^2} - \frac{q^2R^2}{(2R+x)^2.x^2}\right]$$

$$\approx \frac{1}{4\pi\varepsilon_0}\left[\frac{q\,(q+e)}{R^2} - \frac{q^2}{4x^2}\right]. \qquad \text{...(20)}$$

Since x is very small, therefore $q^2/4x^2 \gg q\,(q+e)/R^2$; hence there is always a force of *attraction* if the point charge $+q$ is very near to the spherical surface. In order. In order that this force may be repulsive F must be positive *i.e.*

$$\frac{q\,(q+e)}{R^2} > \frac{q^2}{4x^2}$$

this gives

$$e > \frac{qR^2}{4x^2} - q$$

i.e.

$$e > q\left[\frac{R^2}{4x^2} - 1\right] \text{ (neglecting 1)}$$

i.e.
$$e > \frac{qR^2}{4x^2} \text{ i.e. } x > \frac{R}{2} \cdot \sqrt{\frac{q}{e}}. \qquad \text{...(21)}$$

Surface density of charge. At a point P' in the sphere the surface density of charge is

$$\sigma = -\frac{q\,(d^2 - R^2)}{4\pi R\,(R^2 + d^2 - 2Rd\cos\theta)^{3/2}} + \frac{\left(e + \dfrac{qR}{d}\right)}{4\pi R^2} \qquad \text{...(22)}$$

where $AP' = [R^2 + d^2 - 2Rd\cos\theta]^{1/2}$.

The first term in the above expression is contribution due to charges $+q$ and $-qR/d$; while the second term represents the contribution due to charge $\left(e + \dfrac{qR}{d}\right)$ at centre.

If σ_1 and σ_2 are the surface charge densities at the nearest and farthest points P_1 and P_2 on the sphere, then

$$\sigma_1 = \frac{-q\,(d^2 - R^2)}{4\pi R\,(R^2 + d^2 - 2Rd)^{3/2}} + \frac{\left(e + \dfrac{qR}{d}\right)}{4\pi R^2} \quad (\text{since } \theta = 0;\ \cos 0 = 1)$$

$$= \frac{q\,(d - R)}{4\pi d\,(d - R)^2} + \frac{e}{4\pi R^2} \qquad \text{...(23)}$$

and
$$\sigma_2 = \frac{-q\,(d^2 - R^2)}{4\pi R\,(R^2 + d^2 + 2Rd)^{3/2}} + \frac{\left(e + \dfrac{qR}{d}\right)}{4\pi R^2}$$

$$= \frac{q\,(3d + R)}{4\pi d\,(d + R)^2} + \frac{e}{4\pi R^2} \qquad \text{...(24)}$$

From equations (23) and (24) it is obvious that σ_2 is always positive, and but the sign of σ_1 depends on the magnitude of $+e$ if

$$\frac{e}{4\pi R^2} > \frac{q\,(3d + R)}{4\pi d\,(d + R)^2}$$

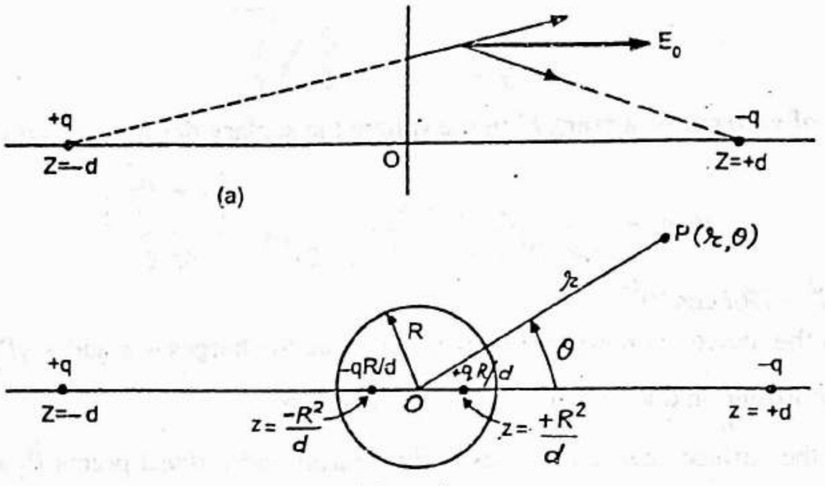
i.e.
$$e > \frac{qR^2\,(3d + R)}{d\,(d + R)^2}, \text{ then } \sigma_1 \text{ is never negative.}$$

4.10. Conducting Sphere in a Uniform Electric Field by the Method of Images

As a final of the method of images let us consider a conducting sphere of radius R in a uniform field E_0. A uniform field can be considered as being produced by appropriate positive and negative charges at infinity. For example if there are two charges $\pm q$, located at positions $z = \mp d$ as shown in fig. 4.6 (a) then in a region near the origin whose dimensions are much samaller than d, there is approximately constant electric field given by $E_0 \approx (2q/4\pi\varepsilon_0 d^2)$, parallel to z-axis. This approximation holds good if q/d^2 remains constant in the limit $q \to \infty$ and $d \to \infty$.

If now we assume a conducting sphere to be placed at the origin, then the *potential at any field point P* will be due to the charge $\pm q$ at $z = \mp i$ and their images $\mp \dfrac{bR}{d}$ and $z = \mp \dfrac{R^2}{d}$ *i.e.*

$$\phi(P) = \frac{1}{4\pi\varepsilon_0} \frac{q}{[r^2 + d^2 + 2rd\cos\theta]^{1/2}} - \frac{1}{4\pi\varepsilon_0} \frac{q}{[r^2 + d^2 - 2rd\cos\theta]^{1/2}}$$



(Fig.4.6)

$$-\frac{1}{4\pi\varepsilon_0}\frac{(qR/d)}{\left[r^2+\frac{R^4}{d^2}+\frac{2R^2r}{d}\cos\theta\right]^{1/2}}+\frac{1}{4\pi\varepsilon_0}\frac{(qR/d)}{\left[r^2+\frac{R^2}{d^2}-\frac{2R^2r}{d}\cos\theta\right]^{1/2}} \quad ...(1)$$

In the first two terms d is much larger than r by assumption; hence we can expand the radicals after factoring out d^2. Similarly, in the third and fourth terms, we can factor out r^2 and then expand. Thus we have

$$\frac{q}{(r^2+d^2+2rd\cos\theta)^{1/2}}=\frac{q}{d}\left[1+\frac{r^2}{d^2}+\frac{2r}{d}\cos\theta\right]^{1/2}$$

$$\approx\frac{q}{d}\left[1+\frac{r^2}{d^2}\cos\theta\right]$$

$$\frac{q}{(r^2+d^2-2rd\cos\theta)^{1/2}}=\frac{q}{d}\left[1+\frac{r^2}{d^2}-\frac{2r}{d}\cos\theta\right]^{-1/2}$$

$$\approx\frac{q}{d}\left[1+\frac{r}{d}\cos\theta\right]$$

Also

$$\frac{(qR/d)}{\left[r^2+\frac{R^4}{d^2}+\frac{2R^2r}{d}\cos\theta\right]^{1/2}}=\frac{qR}{rd}\left[1+\frac{R^4}{r^2d^2}+\frac{2R^2}{rd}\cos\theta\right]^{1/2}$$

$$=\frac{qR}{rd}\left[1+\frac{R^2}{rd}\cos\theta\right]$$

and

$$\frac{(qR/d)}{\left[r^2+\frac{R^4}{d^2}+\frac{2R^2r}{d}\cos\theta\right]^{1/2}}=\frac{qR}{rd}\left[1+\frac{R^4}{r^2d^2}-\frac{2R^2}{rd}\cos\theta\right]^{1/2}$$

$$\approx\frac{qR}{rd}\left[1+\frac{R^2}{rd}\cos\theta\right]$$

Using the approximations, equation (1) becomes

$$\phi(r, \theta) = \frac{1}{4\pi\varepsilon_0}\left[-\frac{2q}{d^2} r\cos\theta + \frac{2q}{d^2}\frac{R^3}{r^2}\cos\theta\right] \qquad \text{...(2)}$$

where the omitted terms vanish in the limit $d \rightarrow \infty$. In that case $\frac{1}{4\pi\varepsilon_0}\frac{2q}{d^2}$ becomes the applied uniform field E_0 *i.e.*

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{2q}{d^2}, \qquad \text{...(3)}$$

so that equation (2) becomes

$$\phi(r, \theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \qquad \text{...(4)}$$

As $z = r\cos\theta$, so equation (4) may be written as

$$\phi(P) = -E_0 z + E_0\frac{R^3 z}{r^3}. \qquad \text{...(5)}$$

The first term $(-E_0 z)$ is of course, just the potential of a uniform field E_0 which may be written down directly instead of first two terms in equation (1). The second term is the potential due to the induced surface charge density or equivalently, the image charges. It may be noted that the image charges form a dipole of strength

$$p = \frac{qR}{d}\times\frac{2R^2}{d}\, 4\pi\varepsilon_0 \mathrm{E}_0 R^3 \qquad \text{...(6)}$$

The *induced surface charge density is*

$$\sigma = \varepsilon_0 (E)_{r=R} = -\varepsilon_0 \left.\frac{\partial\phi}{\partial r}\right|_{r=R} = 3\varepsilon_0 E_0\cos\theta \ \text{[using (4)]} \qquad \text{...(7)]}$$

It may be pointed out that the surface integral of this charge density vanishes, so that there is no difference between a grounded and an insulated sphere.

4.11. Solution of Laplace's Equation in Cartesian Coordinates

The Laplace's equation is

$$\nabla^2\phi = 0. \qquad \text{...(1)}$$

In cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

therefore Laplace's equation in cartesian coordinates is given by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{...(2)}$$

Let us solve this equation by the method of separation of variable. Obviously ϕ is a function (x, y, z). Let this be denoted by

$$\phi = X(x)\, Y(y)\, Z(z), \qquad \text{...(3)}$$

where $X(x)$ is a function of x only, $Y(y)$ is a function of y only and $Z(z)$ is a function of z only.

Subsituting value of ϕ from equation (3) in equation (2), we get

$$\frac{\partial^2}{\partial x^2}(XYZ) + \frac{\partial^2}{\partial y^2}(XYZ) + \frac{\partial^2}{\partial z^2}(XYZ) = 0$$

i.e.

$$YZ\frac{\partial^2 X}{\partial x^2}+XZ\frac{\partial^2 Y}{\partial y^2}+XY\frac{\partial^2 Z}{\partial z^2} = 0.$$

Dividing this equation by *XYZ*, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Z}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0 \qquad ...(4)$$

i.e.

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\frac{1}{X}\frac{\partial^2 X}{\partial x^2} \qquad ...(5)$$

In this equation L.H.S. is a function of y and z only, while R.H.S. is a function of x only; Therefore if this equation is to be satisfied both sides must be equal to the same constant say α^2. Thus

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 z} = \alpha^2 \qquad ...(6)$$

and

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\alpha^2 \qquad ...(7)$$

Equation (7) leads to

$$\frac{\partial^2 X}{\partial x^2}+\alpha^2 X = 0 \qquad ...(8)$$

and equation (6) may be written as

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-\alpha^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}. \qquad ...(9)$$

In this equation L.H.S. is a function of z only, while R.H.S. is a function of y only, hence each side must be equal to the same constant say β^2. Thus

$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \beta^2 \qquad ...(10)$$

and

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-\alpha^2 = \beta^2 \qquad ...(11)$$

Equation (10), leads to

$$\frac{\partial^2 Y}{\partial y^2}+\beta^2 Y = 0 \qquad ...(12)$$

and equation (11), may be expressed as

$$\frac{\partial^2 Z}{dz^2}-(\alpha^2+\beta^2)\,Z = 0. \qquad ...(13)$$

The solution of equation (8), (12) and (13) are written as

$$X = C_\alpha e^{\pm i\alpha x} \qquad ...(14)$$

$$Y = C_\beta e^{\pm i\beta y} \qquad ...(15)$$

and

$$Z = C_{\alpha\beta}\, e^{\pm\sqrt{((\alpha^2+\beta^2)\,z)}} \qquad ...(16)$$

where C_α, C_β and $C_{\alpha\beta}$ are arbitrary constants. Hence from (3) the required potential, *i.e.* solution of Laplace's equation is given by

$$\begin{aligned}\phi &= X(x)\,Y(y)\,Z(z)\\ &= C_\alpha C_\beta C_{\alpha\beta}\, e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\sqrt{\{(\alpha^2+\beta^2)z\}}}\\ &= C\, e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\sqrt{\{(a^2+\beta^2)z\}}} \qquad \text{...(17)}\end{aligned}$$

where $C = C_\alpha C_\beta C_{\alpha\beta}$ is arbitrary constant. As this stage α and β are quite arbitrary. Hence equation (17) represents the general solution of Laplace's equation.

Ex. 5. *Consider a rectangular box, located as show in fig. 4.7 with dimensions (a, b, c) in the (x, y, z) direction. If all surfaces of the box are at zero potential except the surface $z = c$ which is at potential $\phi_0(x, y)$, then find the potential every where inside the box.*

Solution. Laplace's equation is

$$\nabla^2\phi = 0. \qquad \text{...(1)}$$

Its general solution is

$$\phi = X(x)\,Y(y)\,Z(z). \qquad \text{...(2)}$$

Using boundary condition that $\phi = 0$ for $x = 0, y = 0, z = 0$ it is easy to note that

$$\left.\begin{aligned} X(x) &= \sin\alpha x\\ Y(y) &= \sin\beta y\\ \text{and}\quad Z(z) &= \sinh\sqrt{\{(\alpha^2+\beta^2)\,z\}}\end{aligned}\right\} \qquad \text{...(3)}$$

In order that $\phi = 0$ at $x = a$ and $y = b$, it is necessary that

$$\alpha_n A = n\pi \text{ and } \beta_m b = m\pi$$

$$\left.\begin{aligned} \text{i.e.}\quad \alpha_n &= \frac{n\pi}{a},\ \beta_m = \frac{m\pi}{b}.\\ \text{and so}\quad \gamma_{nm} &= \sqrt{(\alpha^2+\beta^2)} = \pi\sqrt{\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)}\end{aligned}\right\} \qquad \text{...(4)}$$

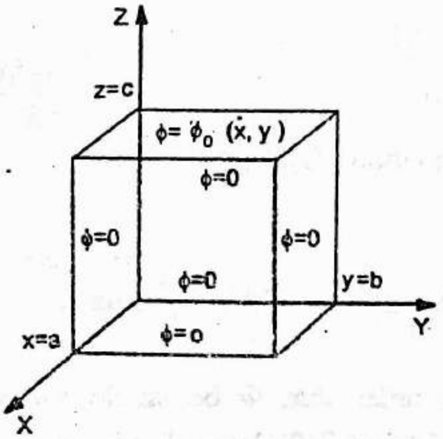
Now we can write the partial potential ϕ_{nm}, satisfy all boundary conditions except one

$$\phi_{nm} = \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z). \qquad \text{...(5)}$$

The potential can be expanded in terms of these ϕ_{nm} with initially arbitrary coefficients to be chosen to satisfy the final boundary condition,

$$\phi(x, y, z) = \sum_{nm} A_{nm}\,\phi_{nm}$$

$$= \sum_{n, m-1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z), \qquad \text{...(6)}$$



(Fig.4.7)

The final boundary condition is $\phi = \phi_0(x, y)$ at $z = c$;

Thus $$\phi_0(x, y) = \sum_{n, m=1}^{\infty}\left[A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)\right]. \qquad \text{...(7)}$$

This is just a double. Fourier series for the function $\phi_0(x, y)$. Consequently the coefficients A_{nm} are given by

$$A_{nm} = \frac{4}{ab\sinh(\gamma_{nm} c)}\int_0^a dx\int_0^b dy\,\phi_0(x, y)\sin(\alpha_m x)\sin(\beta_m y). \qquad \text{...(8)}$$

4.12. Solution of Laplace's Equation in Spherical Co-ordinates

Laplace's equation *for the potential function V** is

$$\nabla^2 V = 0. \qquad \text{...(1)}$$

In spherical co-ordinates (r, θ, ϕ) Laplace's equation is written in the form [refer equation (10) of section 1.12]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0. \qquad \text{...(2)}$$

It is obvious from above equation that the potential function V is a function of (r, θ, ϕ); hence using method of separation of variables, we may write

$$V(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi).$$

Substituting this in equation (2), weget

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(R\Theta\Phi)\right)+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\theta}{\partial\theta}(R\Theta\Phi)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi^2}(R\Theta\Phi) = 0 \qquad \text{...(4)}$$

or

$$\Theta\Phi\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial R}{\partial r}\right\}+\frac{R\Phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{R\Theta}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

Multiplying above equation by $r^2\sin^2\theta$ and dividing by $R\Theta\Phi$, we get

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \qquad \text{...(5)}$$

This equation may be expressed as

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left\{r^2\frac{\partial R}{\partial r}\right\}+\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\Theta}{\partial\theta}\right\} = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}. \qquad \text{...(6)}$$

In this equation L.H.S. is a function of r and θ, while R.H.S. is a function of Φ alone. Consequently each term must be equal to *same constant* m^2 (say) *i.e.*

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = m^2. \qquad \text{...(7)}$$

and

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = m^2. \qquad \text{...(8)}$$

Equation (7) may be written as

$$\frac{\partial^2\Phi}{\partial\phi^2}+m^2\Phi = 0. \qquad \text{...(9)}$$

The solution of this equation is

$$\Phi = C_1 e^{\pm im\phi}. \qquad \text{...(10)}$$

In order that Φ be single valued, m must be an integer. Now in order to separate equations for $R(r)$ and $\Theta(\theta)$, let us divide equation (8) by $\sin^2\theta$

i.e.

$$\frac{1}{R}\frac{\partial R}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = \frac{m^2}{\sin^2\theta}$$

---

*Hence the potential has been denoted by V in order to avoid confusion with the co-ordinate ϕ.

i.e.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{m^2}{\sin^2\theta} - \frac{1}{\Theta\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) \qquad \text{...(11)}$$

In this equation L.H.S. is a function of r alone, while R.H.S. is a function of θ alone. Hence writing each side equal to same constant $l\,(l+1)$ (say), we get

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l\,(l+x) \qquad \text{...(12)}$$

and

$$\frac{m^2}{\sin^2\theta} - \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = l\,(l+1) \qquad \text{...(13)}$$

Equation (13) may be written as

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \left[l\,(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0. \qquad \text{...(14)}$$

Let $a = \cos\theta$; then $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial\theta} = -\frac{\partial}{\partial x}(\sin\theta) = -\sin\theta\frac{\partial}{\partial x}$.

Using these substitutions equation (14) may be expressed as

$$\frac{\partial}{\partial x}\left(\sin^2\theta\frac{\partial\Theta}{\partial x}\right) + \left[l\,(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0.$$

i.e.

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\Theta}{\partial x}\right] - \left[l\,(l+1) - \frac{m^2}{(1-x^2)}\right]\Theta = 0. \qquad \text{...(15)}$$

This equation is *associated Legendre's equation.* It has physical significance only for values of x between the limits ± 1, since $x = \cos\theta$. The solution of above equation is *associated Legendre's function* which may be defined as

$$\Theta = P_l^m(x) = (1-x^2)^m\frac{e^m P_l(x)}{dx^m} \qquad \text{...(16a)}$$

where $P_l(x)$ is *Legendre polynomial* of degree l given by the *Rodridue's formula*

$$P_l(x) = \frac{l}{2^l l\,!}\frac{d^l}{dx^l}(x^2-1)^l \qquad \text{...(17)}$$

Since $x = \cos\theta$, $(1-x^2) = \sin^2\theta$; therefore equation (16a) may also be expressed as

$$\Theta = P_k^m(\cos\theta) = (\sin\theta)^m\frac{d^m P_l(\cos\theta)}{d(\cos\theta)^m} \qquad \text{...(16b)}$$

Now it remains to find the solution of radial equation (12); which may be expressed as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - l\,(l+1)\,R = 0 \qquad \text{...(18)}$$

Let us substitute

$$R\,(r) = \frac{F\,(r)}{r} \qquad \text{...(19)}$$

Then equation (18) becomes

$$\frac{\partial^2 F\,(r)}{\partial r^2} - \frac{l\,(l+1)}{r^2}F\,(r) = 0 \qquad \text{...(20)}$$

From the form of above equation it is apparent that a single power of r (rather than of power series) will satisfy it. The solution is found to be

$$F(r) = C_2 r^{l+1} + C_3 r^{-1} \qquad ...(21)$$

where l is yet undermind. Using (21), we note that

$$\mathbf{R}(r) = \frac{F(r)}{r} = C_2 r^{1} + C_3 r^{-(l+1)} \qquad ...(22)$$

Therefore the complete solution of Laplace's equation in spherical coordinates is

$$V = (r, \theta, \phi) = R(r)\, \Theta(\theta)\, \Phi(\phi)$$

$$= [C_2 r^{l} + C_3 r^{-(l+1)}]\, P_l^m (\cos\theta).\, C_1 e^{\pm im\phi} \qquad ...(23)$$

For a problem possessing *azimuthal symmetry* $m = 0$ *i.e.* where the potential is independent of angle ϕ, then equation (2) is of the form

$$V(r, \theta) = \sum_{l=0}^{\infty} [A_l r_l + B_l r^{-(l+1)}]\, P_l (\cos\theta) \qquad ...(24)$$

where the coefficients A_l and B_l are quite arbitray and can be determined from the boundary conditions.

4.13. Use of Laplace's Equation in Spherical Co-ordinates in the Solution of Electrostatic Problems.

The use of Laplace's equation in spherical coordinates is demonstrated below with reference to certain special problem.

1. Potential outside a charged conducting sphere. Let a be the radius of the given conducting sphere and V_a the potential at the surface of the sphere. The potential at infinity is assumed to be zero as usual, *i.e.* the boundary conditions are

$$\left.\begin{aligned} V &= V_a \text{ at } r = a \\ &= 0 \text{ at } r = \infty \end{aligned}\right\} \qquad ...(1)$$

The Laplace's equation in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta\, \partial r}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial\phi^2} = 0. \qquad ...(2)$$

Because of spherical symmetry, the potential V is a function of r only; hence equation (2) can be rewritten as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_r}{\partial r}\right) = 0. \qquad ...(3)$$

This equation is to be solved in such way that it satisfies boundary conditions (1).

Integration of equation (3) yields

$$r^2 \frac{\partial V}{\partial r} = \text{constant, } C_1 \text{ (say)} \qquad ...(4)$$

i.e.

$$\frac{\partial V}{\partial r} = \frac{C_1}{r^2}$$

Integrating above equation, we get

$$V = -\frac{C_1}{r} + C_2 \qquad ...(5)$$

where C_2 is constant of integration. Equation (5) represents the general expression for the potential.

Since $V = 0$ at $r = \infty$, it follows that $C_2 = 0$, hence equation (5) becomes

$$V = -\frac{C_1}{r}. \qquad ...(6)$$

Further $V = V_a$ at $r = a$, hence

$$V_a = -\frac{C_1}{a} \text{ i.e. } C_1 = -aV_a \quad ...(7)$$

Using (7) we note that from (6) the *potential in the region surrounding the charged metal sphere is given as*

$$V = \frac{aV_a}{r} \quad ...(8)$$

If a charge q is given to a metal sphere of radius a, then we are familiar that

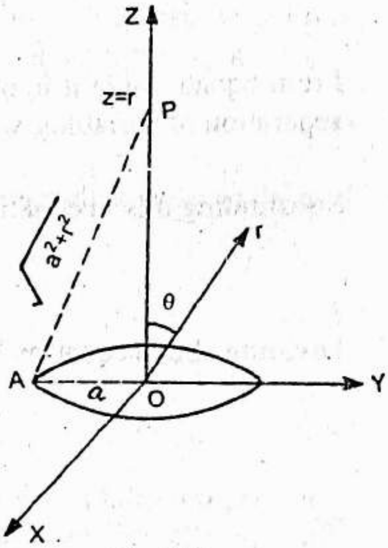
$$V_a = q/4\pi\varepsilon a. \quad ...(9)$$

Substituting this value of V_a in equation (8), we get

$$V = \frac{a}{r}\left(\frac{1}{4\pi\varepsilon}\frac{q}{a}\right) = \frac{1}{4\pi\varepsilon}\frac{q}{r} \quad ...(10)$$

which is a known relation

2. The potential of a ring. Let us consider a circular ring of radius a carrying total charge q. which is uniformly distributed. Let the ring be located as shown in fig. 4.8 with its axis along Z direction and its centre at origin O.



(Fig.4.8)

Evidently the potential V is symmetric about Z-axis and is independent of ϕ. Therefore, we know from equation (24) of section 4.12, that potential is of the form

$$V = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+l)} P_l(\cos\theta)] \quad ...(11)$$

The problem is to determine the unknown coefficients A_l and B_l. We may determine the coefficients by realising that at any point P on the axis of symmetry with $z = r$, the potential is given by

$$V(z = r) = \frac{1}{4\pi\varepsilon}\frac{q}{AP} = \frac{1}{4\pi\varepsilon}\frac{q}{\sqrt{(a^2+r^2)}} \quad ...(12)$$

where $AP = \sqrt{(a^2 + r^2)}$ is the distance of point P from all points of the ring.

By the binomial theorem we have

$$\frac{1}{4\pi\varepsilon}\frac{q}{\sqrt{(a^2+r^2)}} \begin{cases} = \dfrac{1}{4\pi\varepsilon}\dfrac{q}{a}\left(1 - \dfrac{r^2}{2a^2} + \dfrac{1\times 3r^3}{2\times 4a^4} - \ldots\right) \text{where } r < a & ...(13a) \\ = \dfrac{1}{4\pi\varepsilon}\dfrac{q}{a}\left(\dfrac{a}{r} - \dfrac{1}{a}\dfrac{a^3}{r^3} + \dfrac{1\times 3a^5}{2\times 4r^5} - \ldots\right) \text{where } r < a & ...(13b) \end{cases}$$

The general solution (11) must reduce to either (13a) or (13b) for a point on Z-axis where $\theta = 0$. From the property of Legendre polynomials, we have

$$P_l(\cos 0) = P_l(1) = 1. \quad ...(14)$$

Hence for points on Z -axis, equation (11) becomes

$$V = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l-l)}].$$

Comparing this equation with (13a) and (13b), we have

$B_i = 0$ for $r < a$ and P_l are the coefficients of (13a), while $A_l = 0$ for $r > a$ and B_l are the coefficients of (13b). Hence we have the solution

$$V = \frac{1}{4\pi\varepsilon}\frac{q}{a}\left[P_0 \cos(\theta) - \frac{1}{2}\frac{r^2}{a^2} P_2(\cos\theta) + \frac{1 \times 3r^4}{2 \times 4a^2} P_4(r\cos\theta) - \ldots\right] \qquad \text{for } r < a$$

and
$$V = \frac{1}{4\pi\varepsilon}\frac{q}{a}\left[\frac{a}{r} P_0(\cos\theta) - \frac{1}{2}\frac{a^3}{r^3} P_2(\cos\theta) + \frac{1 \times 3a^5}{2 \times 4r^5} P_4(\cos\theta) - \ldots\right] \qquad \text{for } r > a$$

4.14. Solution of Laplace's Equation in Cylindrical Coordinates

The Laplace's equation's for the potential V is

$$\nabla^2 V = 0 \qquad \ldots(1)$$

In cylindrical coordinates (r, θ, z). Laplace's equation is written in the form (refer equation 18 of 1.12)

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}\frac{\partial^2 V}{\partial^2 z} = 0. \qquad \ldots(2)$$

From equation (2) it is obvious that the potential function V is a function (r, θ, z), hence using method of separation of variables, we may write

$$V(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z). \qquad \ldots(3)$$

Substituting this in equation (2), we get

$$\frac{\partial^2 (R\Theta Z)}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}(R\Theta Z) + \frac{1}{r^2}\frac{\partial^2 (R\Theta Z)}{\partial\theta^2} + \frac{\partial^2 (R\Theta Z)}{\partial z^2} = 0.$$

Dividing above equation by $R\Theta Z$, we get

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Rr}\frac{\partial R}{\partial r} + \frac{1}{\Theta r^2}\frac{\partial^2\Theta}{\partial\theta^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

or
$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Rr}\frac{\partial R}{\partial r} + \frac{1}{\Theta r^2}\frac{\partial^2\Theta}{r\theta^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \qquad \ldots(4)$$

In this equation L.H.S. is a function of r and θ and is independent of z, while R.H.S. is a function of z alone, hence each side must be equal to the same constant $-k^2$ say, thus,

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k^2 \qquad \ldots(5)$$

and
$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Rr}\frac{\partial R}{\partial r} + \frac{1}{Rr^2}\frac{\partial^2\Theta}{\partial\theta^2} = -k^2. \qquad \ldots(6)$$

Equation (5) may be written as

$$\frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0. \qquad \ldots(7)$$

The solution of this equation is

$$Z = C_1 e^{\pm kz} \qquad \ldots(8)$$

where C_1 is any arbitrary constant.

Now multiplying equation (6) by r^2 and rearranging, we get

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + k^2 r^2 = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \qquad \ldots(9)$$

In this equation L.H.S. is a function of r alone while R.H.S. is a function of θ alone. This equation is only satisfied if each side is equal to the same constant m^2 (say), so that

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+k^2r^2=+m^2 \qquad ...(10)$$

and

$$-\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}=m^2. \qquad ...(11)$$

Equation (11) may be written as

$$\frac{\partial^2 \Theta}{\partial \theta^2}+m^2\Theta = 0.$$

The solution of above equation is

$$\Theta = C\,e^{\pm im\theta}. \qquad ...(13)$$

In order that the potential be single valued, m must be an integer. The radial equation (10) may be put in a standard form by the change of variable from r to x through the relation $x = kr$, then it takes the form

$$\frac{\partial^2 R}{\partial x^2}+\frac{1}{x}\frac{\partial R}{\partial x}+\left(1-\frac{m}{x^2}\right)R=0. \qquad ...(14)$$

This equation is known as *Bessel's equation* and its solution is known as Bessel function of order m defined as

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j\,!\,\Gamma(j+m+1)}\left(\frac{x}{2}\right)^{2j} \qquad ...(15)$$

and

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j\,!\,\Gamma(j-m+1)}\left(\frac{x}{2}\right)^{2j}. \qquad ...(16)$$

These two solutions are called Bessel's functions of the first kind of order $\pm m$. *If m is not an integer,* these two solutions $J_{\pm m}(x)$ from a pair of linearly independent solutions to the second order Bessel's equation. However *if m is an integer,* it is well known that the solutions are linearly dependent, since then

$$J_{-m}(x) = (-1)^m J_m(x). \qquad ...(17)$$

Consequently it is necessary to find another linearly independent solution when m is integer. It is customary, even if m is not an integer, to replace that pair $J_{\pm m}(x)$ by $J_m(x)$ and $N_m(x)$, the *Neumann function* (or Bessel's function of second kind) defined as

$$N_m(x)\,\frac{J_m(x)\cos mx - J_{-m}(x)}{\sin mx} \qquad ...(18)$$

For m not an integer, $N_m(x)$ is clearly independent of $J_m(x)$. In the limit m tends to an integer, it can be seen that $N_m(x)$ is still linearly inpependent of $J_m(x)$. Hence the general solution of radial equation is

$$R(r) = C_3J_m(x) + C_4N_m(x) \qquad ...(19a)$$

$$= C_3J_m(kr) + C_4N_m(kr) \qquad ...(19b)$$

Thus the complete potential function $V(r, \theta, z)$ may be written as

$$V(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z)$$

$$= [C_3I_m(kr) + C_4N_m(kr)]\,C_2\,e^{\pm im\theta}\,C_1\,e^{\pm kz}. \qquad ...(20)$$

4.15. Use of Laplace Equation in Cylindrical Coordinates in the Solution of Electrostatic Problems

The use of Laplace equation in cylindrical coordinates is demonstrated below with reference to certain special problems.

1. Potential between co-axial conducting cylinders. Let us consider a cylinder of radius 'a', surrounded by another co-axial cylinder of radius 'b'. Let inner cylinder be charged to a potential V_a and outer cylinder by grounded. Hence in this case infinity is not one of the boundaries. Instead the space under consideration is bounded by two given co-axial cylinders. The arrangement is similar to that of the cylindrical capacitor. Thus in this case the boundary conditions are

$$\left.\begin{array}{ll} & V = 0 \text{ when } r = b \\ \text{and} & V = V_a \text{ when } r = a \end{array}\right\} \quad \text{...(1)}$$

where r is the distance from the axis. In this case of cylindrical co-axial cylinders, it is convenient to use Laplace's equation in cylindrical coordinates *i.e.*

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{...(2)}$$

It is assumed that the cylinders are long; hence end effects may be neglected. If so, the potential is independent of both θ and z, so that equation (2) takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \quad \text{..(3)}$$

Integration of above equation yields

$$r\frac{\partial V}{\partial r} = \text{constant, } C_1 \text{ (say)}$$

i.e.

$$\frac{\partial V}{\partial r} = \frac{C_1}{r}$$

Integrating again we obtain

$$V = C_1 \log_e r + C_2 \quad \text{...(4)}$$

where C_2 is another constant of integration. This equation represents general expression for the potential and indicates that the potential V varies logarithmically. The constants C_1 and C_2 may be evaluated by applying boundary conditions (1).

Substituting first boundary equation *i.e.* $V = 0$ at $r = b$ in (4), we get

$$0 = C_1 \log_e a + C_2 \quad \text{...(5)}$$

Now substituting second boundary condition *i.e.* $V = V_a$ at $r = a$ in (4), weget

$$V_a = C_1 \log_e a + C_2 \quad \text{...(6)}$$

Solving equations (5) and (6) for C_1 and C_2, we get

$$C_1 = \frac{V_a}{\log_e\left[\frac{a}{b}\right]} \text{ and } C_2 = \frac{-V_a \log_e b}{\log_e\left[\frac{a}{b}\right]} \quad \text{...(7)}$$

Substituting values of C_1 and C_2 in equation (4), we get

$$V = \frac{V_a}{\log_e\left(\frac{a}{b}\right)}\log_e r - \frac{V_a \log_e b}{\log_e\left(\frac{a}{b}\right)}$$

i.e.

$$V = \frac{V_a}{\log_e\left(\frac{a}{b}\right)}\log_e\left(\frac{r}{b}\right) \quad \text{...(8)}$$

The equation gives the desired expression for potential in the space between the co-axial conducting cylinders.

2. As another example let us consider a boundary value problem shown in fig. 4.9. The cylinder has radius 'a', and height h, the top and bottom surfaces being at $z = h$ and $z = 0$. The boundary conditions are

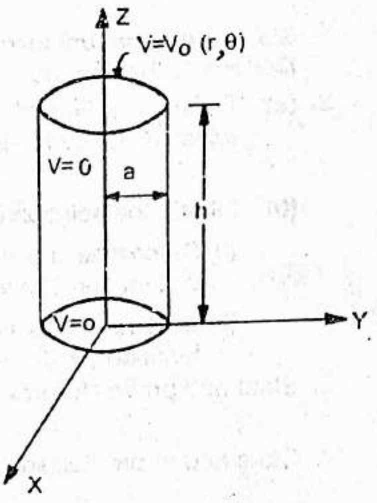
(i) The potential is zero on the side and the bottom of the cylinder;

(ii) The potential is $V = V_0 (r, \theta)$ at the top.

It is required to find the potential at any point inside the cylinder.

The Laplace's equation in cylinderical co-ordinates is

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2} = 0. \quad ...(9)$$



(Fig.4.9)

The solution of above equation by the method of separation of variables is

$$V(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z), \quad ...(10)$$

where

$$R(r) = AJ_m(kr) + BN_m(kr) \quad ...(11)$$

$$\Theta(\theta) = C\,e^{\pm im\theta} \quad ...(12)$$

and

$$Z(r) = d^{\pm kz}. \quad ...(13)$$

In order that the *potential V be single valued and vanish at* $z = 0$ then equation (12) may be expressed as

$$\Theta(\theta) = C \sin m\theta + D \cos m\theta \quad ...(14)$$

$$Z(z) = \sinh kz, \quad ...(15)$$

where m is an integer and k is a constant to be determined.

If the *potential is finite* at $r = 0$, *then* $B = 0$ in equation (11). Hence radial solution takes the form

$$R(r) = AJ_m(kr). \quad ...(16)$$

The requirement that the *potential vanish at* $r = a$ means that k can take only special values

$$k_{mn} = \frac{x_{mn}}{a}, \quad n = 1, 2, 3 \quad ...(17)$$

where x_{mn} are the roots of $J_{mn}(x_{mn}) = 0$.

Combining equations (10), (14), (15), (16) and (17), we find that the general form of the solution is

$$V(r, \theta, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_n(k_{rm}r) \sinh(k_{mn}z)\,[C_{mn} \sin m\theta + D_{mn} \cos m\theta]. \quad ...(18)$$

At $z = h$, $V(r, \theta, z) = V_0 (r, \theta)$; therefore we have

$$V_0(r, \theta) = \sum_{mn} \sinh(k_{mn}h)\,J_m(k_{mn}r)\,[C_{mn} \sin m\theta + D_{mn} \cos m\theta] \quad ...(19)$$

This is a Fourier series in θ and a Bessel Fourier series in r. The coefficients C_{mn} and D_{mn} are given by

$$\left.\begin{aligned} C_{mn} &= \frac{2\,\mathrm{cosech}\,(k_{mn}h)}{\pi a^2 J_{m+1}^2 (k_{mn}a)} \int_0^{2\pi} d\theta \int_0^{a} V_0 (r, \theta)\, J_m(k_{mn}r) \sin m\theta . r dr \\ \text{and} \quad Dmn &= \frac{2\,\mathrm{cosech}\,(k_{mn}h)}{\pi a^2 J_{m+1}^2 (k_{mn}a)} \int_0^{2\pi} d\theta \int_0^{a} V_0 (r,\theta)\, J_{mn} (k_{mn}r) \cos m\theta . r dr \end{aligned}\right\}$$

QUESTIONS AND PROBLEMS

1. State and prove uniqueness theory of electrostatics regarding electric potential. Use it to prove that the electric field inside the conductor in zero.
2. (a) Define the Green's function with reference to electrostatic potential problems. Prove that this function satisfies the symmetry

$$G(x, x') = G(x', x).$$

   (b) Obtain the solution of any one of the following two potential problems in terms of Green's function :

   (i) Determine the potential at a point inside a closed grounded conducting surface enclosing a charge distribution of a volume density ρ.

   (ii) Determine the potential at a point inside a closed surface on which the potential is prescribed as a function ϕ_S. **(Meerut 1976)**

3. State and prove Green's theorem. How is Green's function applied on the solution of potential problems ? **(Meerut 1991)**
4. State and prove Poisson's equation regarding electric potential. Show that the potential function is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'$$

   represents the solution of Poisson's equation.

5. Explain the method of images for the solution of electrostatic problems. Obtain the image system for a point charge sitting at a distance *c* from the centre of a conducting sphere of radius *b* (< *c*) at zero potential. **(Meerut 1973)**
6. (a) What is an electrical image ? A point charge is situated near an infinite plane earthed conductor. Apply the method of images to calculate :

   (i) Surface charge density induced on the plane.

   (ii) The force between the plane and the charge.

   (b) An electron is distance 10 Å from a plane conductor calculate the force experienced by the electron an the work done in moving it to infinite distance from conductor. **(Rohilkhand 1997, 95)**

   [**Ans.** (b) $F = -5{\cdot}75 \times 10^{-11}$ N , $W = 5{\cdot}75 \times 10^{-20}$ J.]

7. What is meant by an electrical image ? Describe the method of images for solving electrostatic problems.

   A point charge is placed outside a conducting sphere. Find by the method of electrical images the intensity of electrical field outside the sphere and also the surface density of induced charge on the sphere when the sphere is at zero potential. **(Meerut 1989, 82)**

8. (a) Write down Laplace's equation in spherical co-ordinates and obtain its solution. What will be the form of solution for spherically symmetric problems. **(Rohilkhand 1997; Agra 95, 90, 75, 73, 72)**
9. (a) Deduce the solution of Laplace's equation for a problem having azimuthal symmetry.

   (b) Use it to obtain potentials inside and outside a dielectric sphere in a uniform field. **(Rohilkhand 1977)**

   (c) A circular ring has charge *q* per unit length on its circumference. Find the electric potential all point not lying on the ring. **(Lucknow 1998; Agra 94)**

10. Give briefly the general methods for the solution of potential problems in electrostatics and compare their merits and limitation. **(Kanpur 1997, Meerut 94)**
11. What are Zontal Harmonics ? Use them to find out the field due to a charged sphere having a charge *q*. Calculate also the stress on the surface of the sphere. **(Kanpur 86)**

5

Steady Currents

5.1. Electric Current and Current Density

The *electric current* through any cross-sectional area of a conductor is defined as the net charge transferred through it per second. If a net charge q passes through any cross-sectional area of the conductor in time t, the electric current I is given by

$$I = \frac{q}{t}. \qquad \text{...(1)}$$

The M.K.S. unit of current is *ampere* and represents *the passage of* 1 *coloumb of charge per second.* Conventionally the direction of electric current is taken along the direction of motion of positive charges. When the current is caused by the motion of electrons (as in metallic conductors), the direction of current is opposite to the direction of electron-flow. Under steady state conditions, the value of electric current is same for all cross-sections of a conductor, even through the cross-sectional area may have different values at different places. This constancy of electric current in steady state follows from the law of conservation of charge. When the direction of motion of the charge does not change with time, the current is said to be *direct;* but when the direction of charge motion reverses periodically, the current is said to be *alternating.*

If the magnitude of electric current is the function of time the electric current is given by the equation,

$$I = \frac{dq}{dt}. \qquad \text{...(2)}$$

The *current density* at any point inside the conductor is defined as a vector quantity whose magnitude is equal to the current (*i.e.*, quantity of charge passing per second) through an infinitesimal area at the point; the area being normal to the direction of flow of charge and whose direction is along the direction of flow of positive charge at the point. It is denoted by the symbol **J**.

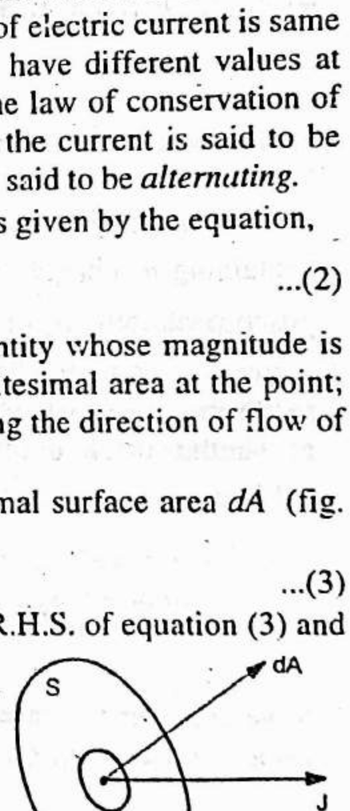
In terms of current density, the current dI through any element of infinitesimal surface area dA (fig. 5.1) is given by

$$dI = \mathbf{J}.d\,\mathbf{A}. \qquad \text{...(3)}$$

The total current I through the surface S is obtained by taking surface integral of R.H.S. of equation (3) and is given by,

$$I = \int_S \mathbf{J}.d\mathbf{A} \qquad \text{...(4)}$$

From this equation we may define *the current I as the flux of the current density vector* **J** *through a given area.*



(Fig. 5.1)

If the value of the current density vector **J** is constant over the entire cross-section area A of the current carrying conductor and is every where normal to it (*i.e.* along A), equation (4) may be written as

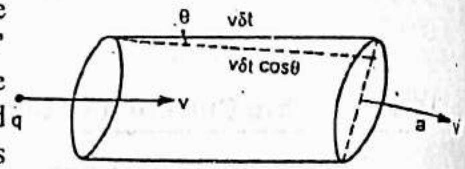
$$I = \mathbf{J}.\mathbf{A} = JA$$

or the magnitude of current density, $$|\mathbf{J}| = \frac{I}{A}. \quad ...(5)$$

Obviously the unit of current density is that of Current/Area, *i.e.*, ampere/metre^2.

5.2. Expression for Current Density

To deduce an expression for the current density in terms of the motion of the individual charge carriers, let us consider a conductor in which there are n charge carriers per cubic metre carrying the same charge q and travelling with the same velocity vector $\mathbf{v}$. Now consider a small circular elementary area 'a' placed in certain orientation through which the charge carriers are passing continuously. In a small time interval δt, the distance travelled by each charge carrier $l = v.\delta t$ and so in time δt all the charge carriers located within the cylinder of slant height '$v\delta t$' and base area 'a' will pass through the elementrary area. The volume of this cylinder will be (= base × altitude) $= a\,(v\delta t \cos\theta) = \mathbf{a}.v\delta t$.



(Fig. 5.2)

Therefore the number of charge carriers passing through area 'a' in time δt

$$= n\,\mathbf{a}.\mathbf{v}\,dt$$

So the total charge passing through the area 'a' in time δt

$$= q\,n\,.\,\mathbf{a}\,\mathbf{v}\,t\,.$$

Therefore the current flowing through the assumed area is,

$$\mathbf{I} = \text{Rate of flow of charge} = \frac{q\,n\,\mathbf{a}.\mathbf{v}\,\delta t}{\delta t} = n\,q\,\mathbf{v}\,.\,\mathbf{a} \quad ...(1)$$

Hence the current density $\mathbf{J}$, defined as $I = \mathbf{J}.\mathbf{a}$, is given by

$$\mathbf{J} = nq\,\mathbf{v}. \quad ...(2)$$

Now let us consider a case in which there are various groups of charge carriers per cubic metre : First group containing n_1 charge carriers per metre^3 each carrying charge q_1 and travelling with velocity $\mathbf{v}_1$, second group containing n_2 charge carriers per metre^3 each carrying charge q_2 and travelling with velocity $\mathbf{v}_2$, k^{th} group containing n_k charge carriers per metre^3 each carrying charge q_k and travelling with velocity $\mathbf{v}_k$ and so on. In this case each group of charge carriers contributes towards the current through the assumed area 'a', so that the current in this case is,

$$\begin{aligned}\mathbf{I} &= n_1q_1\,\mathbf{v}_1.\mathbf{a} + n_2q_2\,\mathbf{v}_2\mathbf{a} + \ldots + n_k\,q_k\,\mathbf{v}_k\,\mathbf{a} + \ldots \\ &= (n_1q_1\,\mathbf{v}_1 + n_2q_2\,\mathbf{v}_2 + \ldots + n_k\,q_k\,\mathbf{v}_k + \ldots)\,.\,\mathbf{a} \\ &= (\Sigma\, n_k\,q_k\,\mathbf{v}_k)\,.\,\mathbf{a} \quad ...(3)\end{aligned}$$

Hence the current density $\mathbf{J}$ (given by $\mathbf{I} = \mathbf{J}.\mathbf{a}$) is given as,

$$\mathbf{J} = \Sigma\, n_k\,q_k\,\mathbf{v}_k. \quad ...(4)$$

If we consider the case where the charge carriers are free electrons travelling with different velocities, the value of charges $q_1, q_2, \ldots. q_k \ldots$ will be identical, *i.e.*,

$$q_1 = q_2 = \ldots = q_k = \ldots = -e \text{ (electronic charge)}$$

and so the current density

$$\mathbf{J} = -\sum_k n_k\,e\,\mathbf{v}_k = -e\sum_k n_k\,\mathbf{v}_k \quad ...(5)$$

In the case of a metallic conductor the charge carriers are free electrons whose velocities differ widely in magnitude and direction. However these free electrons may be divided into a number of groups such that the free electrons of nearly same speed and direction correspond to one particular group. If $\mathbf{v}_{av}$ represents the average velocity of all the free electrons, then

$$\mathbf{v}_{av} = \frac{n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + \ldots + n_k\mathbf{v}_k + \ldots}{n_1 + n_2 + \ldots. + n_k + \ldots} = \frac{\sum_k n_k\mathbf{v}_k}{\sum_k n_k} = \frac{\sum_k n_k\mathbf{v}_k}{n_k} \qquad \ldots(6)$$

where $n = \sum_k n_k =$ total number of free electrons per metre^3 ...(7)

$$\therefore \qquad \sum_k n_k\,\mathbf{v}_k = n\,\mathbf{v}_{av}$$

Using this result and equation (5), the current density in the case of a metallic conductor is given by

$$\mathbf{J} = -en\,\mathbf{v}_{av} \qquad \ldots(8)$$

The negative sign here indicates that the direction of current density **J** is opposite to the velocity vector $\mathbf{v}_{av}$.

It should be noted that in the absence of external electric field, the free electrons in metallic conductor move in all possible directions with equal probability, so in such a case average velocity, $\mathbf{v}_{av} = 0$. Therefore in the absence of external electric field, the free electrons in a metallic conductor do not contribute any thing towards the current I or current density **J**.

In an electrolytic solution both positive and negative ions are set in motion under the action of an external electric field (the positive ions moving along the field and the negative ions moving against it), the total current density is the sum of two parallel vectors, viz,

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = n_1q_1\,\mathbf{v}_1 + n_2q_2\,\mathbf{v}_2 \qquad \ldots(9)$$

where the first term represents the contribution to current density **J** due to charge of one sign; while the second term represents the contribution to **J** due to charges of opposite sign.

5.3. Stationary Current; Equation of Continuity

The current I (*i.e.*, rate of flow of net charge) through any surface S of a conductor (from equation 4 of § 5.1) is the flux of current density vector **J** through the given surface area, *i.e.*,

$$I = \int_S \mathbf{J}\,.\,d\mathbf{A} \qquad \ldots(1)$$

If the current density vector **J** remains unchanged with time everywhere, then the current is said to be *steady* or *stationary*.

Now consider a closed surface S enclosing a region of volume V. For generality we first consider the case in which the current is not steady, but is a function of time t. Let at any instant ρ be the charge density of an infinitesimal element of volume δV. Then the volume integral $\int_V \rho\, dV$ will represent the value of total charge q inside the volume V at that instant. From the law of conservation of charge, "the rate of charge flowing away from the enclosed volume is the same as the rate of decrease of the charge inside it." *i.e.*

$$I = \oint_S \mathbf{J}.\mathbf{A} = -\frac{dq}{dt} = -\frac{d}{dt}\int_V \rho dV = -\int_V \frac{d\rho}{dt}\, dV. \qquad \ldots(2)$$

Also according to Gauss divergence theorem

$$\oint_S \mathbf{J}.d\,\mathbf{A} = \int_V \operatorname{div} \mathbf{J} dV \qquad \ldots(3)$$

Comparing equations (2) and (3), we get

$$\int_V \operatorname{div} \mathbf{J} dV = -\int_V \frac{d\rho}{dt}\, dV$$

or
$$\int_V \left(\operatorname{div} \mathbf{J} + \frac{d\rho}{dt} \right) dV = 0$$

As the volume is arbitrary, we have $\operatorname{div} \mathbf{J} + \frac{d\rho}{dt} = 0$

This equation is called the equation of continuity. It simply expresses the fact that the charge is conserved.

If the value of charge density ρ is independent of time, then $d\rho/dt = 0$ and hence for *steady currents or everywhere inside the time independent charge distribution*

$$\operatorname{div} \mathbf{J} = 0 \qquad ...(5)$$

5.4. Ohm's Law and Electrical Conductivity

George Simon Ohm (1787-1854) experimentally studied that the magnitude of electric current I through a conductor depends upon the value of the potential difference V applied between its ends. He simply plotted I as a function of V for a number of different conductors and discovered that in each case ***"the current passing through a conductor is directly proportional to the potential difference between its two ends, provided the physical conditions (temperature etc.,) of the conductor remain unchanged."*** This statement is recognised as Ohm's Law. Thus

$$I \propto V$$

or
$$I = kV \qquad ...(1)$$

where k is a constant of proportionality and is known as **conductance** of the conductor. Usually k is written as I/R, the constant R being known as the **resistance** of the conductor, so

$$I = \frac{V}{R} \text{ or } R = \frac{V}{I} \qquad ...(2)$$

The practical unit of resistance is *ohm*. As conductance is reciprocal of resistance, the unit of conductance is $(\text{ohm})^{-1}$ or mho.

If $I = 1$ ampere, $V = 1$ volt, then $R = 1$ ohm.

Thus 1 ohm is the resistance of that linear conductor through which 1 ampere current will be produc by application of 1 volt potential difference between its ends.

The C.G.S. unit of resistance is stat ohm

$$1 \text{ stat ohm or e.s.u. of resistance} = \frac{1}{9 \times 10^{11}} \text{ ohm.}$$

The conductors which obey Ohm's law are called *Ohmic conductors or linear conductors*. The examples of Ohmic conductors are metallic conductors Ag, Cu, Al etc. The resistances of such conductors are independent of I or V and are known as *linear resistances*. The I versus V graph for such conductors is a straight line.

The conductors which do not obey Ohm's law are called **non-ohmic conductors.** The examples of non-ohmic conductors are vacuum tubes, crystal rectifiers, transistors, thermistors etc. The I versus V graph for such conductors is non-linear. The resistances of such conductors are $V - I$ dependent and are called *non-linear resistances.*

Specific resistance (or resistivity) and Electrical conductivity (or specific conductance) : The resistance of an ohmic conductor depends upon

(i) length (ii) area of cross-section (iii) material and (iv) temperature, of the conductor.

For a given conductor maintained at constant temperature, the resistance R of a conductor is directly proportional to its length l and inversely proportional to its area of cross-section A.

i.e. $$R \propto \frac{l}{A} \quad \text{or} \quad R = \rho \cdot \frac{l}{A} \qquad ...(3)$$

where ρ is a constant of proportionality and is known as *specific* resistance or ***resistivity*** of the material. It is independent of shape and size of the conductor, but depends upon the nature and the temperature of the conductors.

In equation (3) if $A = 1, l = 1$, then $\rho = R$

i.e. specific resistance (or resistivity) *of the material of a conductor is defined as the resistance of the conductor of unit length and unit area of cross-section.*

From equation (3) it is obvious that in M.K.S. system the unit of resistivity is ohm-metre.

The reciprocal of resistivity ρ called the *specific conductance or electrical conductivity* σ, *i.e.*

$$\sigma = \frac{1}{\rho} = \frac{l}{RA}.$$

The unit of electrical conductivity is obviously the reciprocal of resistivity. Hence in M.K.S. system the unit of electrical conductivity is $(\text{ohm-metre})^{-1}$ or mho/metre.

So far Ohm's law has been expressed in terms of electric current I, potential difference V and resistance R of the conductor, for homogeneous materials it may also be expressed in the form of current density **J** and electric field **E**.

If I is the current flowing through a linear conductor of cross-sectional area A, then we have

$$I = JA \qquad ...(5)$$

If V is the **potential difference** between the ends of the conductor of length l, then electric field **E** inside the conductor has magnitude

$$E = \frac{V}{l} \quad \text{or} \quad V = El \qquad ...(6)$$

Substituting values of I and V from equations (5) and (6) in equation (2), we get

$$R = \frac{El}{JA}$$

or $$J = \frac{l}{RA}\, E = \sigma E \qquad \text{[from equation (4)]} \qquad ...(7)$$

If **J** and **E** are parallel to each other, we may write in vector from

$$\mathbf{J} = \sigma \mathbf{E} \qquad ...(8)$$

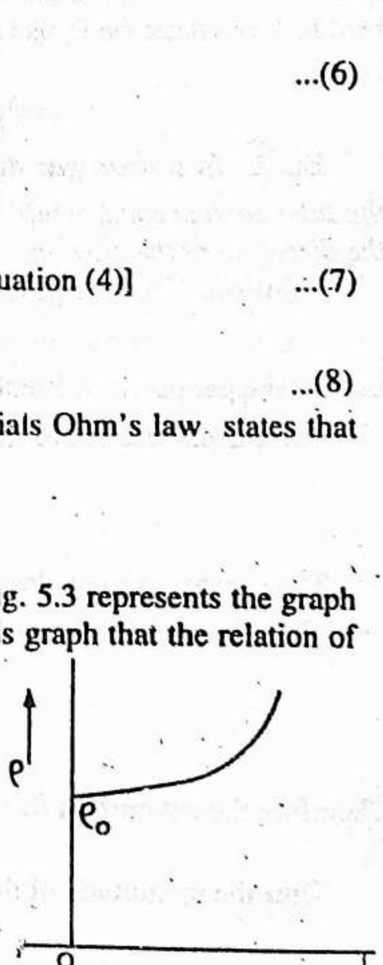
This result has been verified experimentally. Thus for homogeneous materials Ohm's law states that the current density **J** at any point is proportional to the electric field **E**.

5.5. Variation of Resistance and Resistivity with Temperature

The resistance and resistivity of a conductor varies with the temperature. Fig. 5.3 represents the graph of variation of resistivity of pure metals with temperature. It is obvious from this graph that the relation of resistivity and temperature is not linear; but is expressed by the relation

$$\rho_t = \rho_0 + at + bt^2 + ... + \qquad ...(8)$$

where ρ_0 is the resistivity at 0°C and $a, b, ...$ etc. are constants in descending order of their values (*i.e.* $a > b > c$ ....). The constants have different values for different metals. If the temperature t is not sufficiently large, the terms containing t^2 and other higher powers of t may be neglected. This means that the initial small part in the graph may be assumed as a straight line. So the effect of temperature upon resistivity may be expressed in simplified form as,



(Fig. 5.3)

$$\rho_t = \rho_0 + at = \rho_0\left(1 + \frac{a}{\rho_0}t\right) = \rho_0(1 + \alpha t) \qquad \text{...(9)}$$

or

$$\alpha = \frac{\rho_t - \rho_0}{\rho_0 . t} \qquad \text{...(10)}$$

Here the constant α is called **temperature coefficient of resistivity** of the conductor and is defined as the fractional increase in resistivity of the conductor per unit increase with temperature.

As the resistance of a conductor is directly proportional to its resistivity; the variation of resistance with temperature may be expressed as

$$R_t = R_0(1 + \alpha t) \qquad \text{...(11)}$$

or

$$\alpha = \frac{R_t - R_0}{R_0 . t} \qquad \text{...(12)}$$

Here α is the **temperature coefficient of resistance of the conductor** and may be defined as the *fractional increase in the resistance of the conductor per unit increase in its temperature.*

The resistance and resistivity of non-metals and electrolytes decrease with increase in temperature. For such substances the temperature coefficient of resistivity and resistance is negative.

For some alloys like manganin the resistivity and resistance are independent of temperature. Manganin is an alloy of 84% Copper, 12% Nickel and 4% Manganese.

The resistance and resistivity of a metal, besides temperature is also affected by light, magnetic field and traces of impurities. For example when light falls on selenium its resistivity decreases. When some impurity like arsenic is added to germanium, its resistivity decreases. On the other hand when bismuth is kept in a magnetic field, its resistivity increases sufficiently.

Solved Examples

Ex. 1. *In a neon gas discharge tube* $2{\cdot}9 \times 10^{18}$ Ne^+ *ions move to the right through a cross-section o the tube each second, while* $1{\cdot}2 \times 10^{10}$ *electrons move to the left in the same time. Find the magnitude and the direction of the current.* **(Agra 1991)**

Solution. The charge on each neon ion, $q_1 = e$

$$= 1{\cdot}6 \times 10^{-19} \text{ coul}$$

Let us take the positive direction from left to right and negative from right to left.

The current due to Ne^+ ions $= (n_1 q_1)/t_1$

$$= \frac{2{\cdot}9 \times 10^{18} \times 1{\cdot}6 \times 10^{-19}}{1} \text{ coul/sec.} = 0{\cdot}464 \text{ amp.}$$

The charge on each electron, $q_2 = -1{\cdot}6 \times 10^{-19}$ coul.

The current due to electrons

$$= -\frac{n_2 q_2}{t_2} = -\frac{(1{\cdot}2 \times 10^{19})(-1{\cdot}6 \times 10^{-19})}{1}$$

$$= 1{\cdot}2 \times 1{\cdot}6 \text{ coul sec} = \mathbf{1{\cdot}92 \text{ amp.}}$$

Therefore the net current through the discharge tube

$$0{\cdot}464 + 1{\cdot}92 = \mathbf{2{\cdot}384 \text{ amp.}}$$

Thus the magnitude of the current is 2·384 amp. and its *direction is from left to right.*

Ex. 2. *A silver wire 1 m.m. in diameter carries a charge of 90 coulombs in 1 hour and 15 minutes. Silver contains* $5{\cdot}8 \times 10^{22}$ *free electrons per* cm^3, *calculate*

(i) the current in the wire in amperes

(ii) the drift velocity of the electrons in metre/sec. **(Kanpur 1996, Agra 1990)**

Solution. (i) The Current in the wire

$$I = \frac{\text{total charge}}{\text{total time}}$$

$$= \frac{90 \text{ coul.}}{1 \text{ hour } 15 \text{ minutes}} = \frac{90 \text{ coul.}}{4500 \text{ sec.}} = \mathbf{0{\cdot}02 \text{ amp.}}$$

(ii) The current density $J = nqv$

or
$$\frac{I}{A} = nqv$$

or drift velocity,
$$v = \frac{A}{nqA}$$

Here current $I = 0{\cdot}02$ amp. $q = -1{\cdot}6 \times 10^{-19}$ coul.

$$n = 5{\cdot}8 \times 10^{22} \text{ electrons/cm}^3 = 5{\cdot}8 \times 10^{28} \text{ electrons/m}^3$$

and cross-sectional area, $A = \pi r^2 = 3{\cdot}14 \times \left(\frac{0{\cdot}05}{100}\right)^2 \text{ metre}^2$

$$\therefore \qquad v = \frac{1}{nqA}$$

$$= \frac{0{\cdot}02}{(5{\cdot}8 \times 10^{28})(-1{\cdot}6 \times 10^{-19})\left[3{\cdot}14 \times \left(\frac{0{\cdot}05}{100}\right)^2\right]} \text{ m/sec}$$

$$= \mathbf{-2{\cdot}746 \times 10^{-5} \text{ m/sec.}}$$

Ex. 3. *In a cloud of electrons having* 6×10^{29} *electrons per metre*^{3}, *40% of the electrons have a drift velocity 1 mm. per second in X-direction, 40% have drift velocity of 2 mm per second in Y-direction and the remaining 20% have a drift velocity of 3 mm. per second in Z-direction. Calculate the current density* **J**.

Solution. If n_x, n_y and n_z are the number of electrons per m^3 along X, Y and Z directions respectively and v_x, v_y and v_z are the drift velocities of the particles along the corresponding directions, then current density

$$\mathbf{J} = n_x q v_x \hat{i} + n_y q v_y \hat{j} + n_z q v_z \hat{k}$$

where q is the charge on each electron.

Given
$$n_x = 6 \times 10^{29} \times \frac{40}{100} = 2{\cdot}4 \times 10^{29} \text{ electrons/m}^3$$

$$n_y = 6 \times 10^{29} \times \frac{40}{100} = 2{\cdot}4 \times 10^{29} \text{ electrons/m}^3$$

$$n_z = 6 \times 10^{29} \times \frac{20}{100} = 1{\cdot}2 \times 10^{29} \text{ electrons/m}^3$$

$$v_x = 1 \text{ mm/sec} = 10^{-3} \text{ m/sec}$$

$$v_y = 2 \text{ mm/sec} = 2 \times 10^{-3} \text{ m/sec}$$

$$v_z = 3 \text{ mm/sec} = 3 \times 10^{-3} \text{ m/sec}$$

$$q = -1{\cdot}6 \times 10^{-19} \text{ coulomb}$$

$$\therefore \quad \mathbf{J} = q\,[n_x v_x \mathbf{i} + n_y v_y \mathbf{j} + n_z v_z \mathbf{k}] \text{ amp/m}^2$$

$$= -1{\cdot}6 \times 10^{-19}\,[2{\cdot}4 \times 10^{29} \times 10^{-3}\,\mathbf{i} + 2{\cdot}4 \times 10^{29} \times 2 \times 10^{-3}\,\mathbf{j} + 1{\cdot}2 \times 10^{29} \times 3 \times 10^{-3}\,\mathbf{k}]$$

$$= -1{\cdot}6 \times 10^{-19} \times 1{\cdot}2 \times 10^{29} \times 10^{-3}\,[2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}] \text{ amp/m}^2$$

$$= -1{\cdot}92 \times 10^{7}\,[2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}] \text{ amp. m}^2$$

The magnitude of current density $J = |\mathbf{J}|$

$$= 1{\cdot}92 \times 10^{7} \times \sqrt{[(4+16+9)]} = 1{\cdot}92 \times 10^{7} \times \sqrt{(29)}$$

$$= \mathbf{1{\cdot}034 \times 10^{8} \text{ amp/m}^2}.$$

If θ is the angle which the current density **J** makes with X-axis then

$$\cos\theta = \frac{J_x}{|\mathbf{J}|} = \frac{-1{\cdot}92 \times 10^{7} \times 2}{1{\cdot}034 \times 10^{8}} = (-0{\cdot}3714)$$

$$\therefore \quad \theta = \cos^{-1}(-0{\cdot}3714) = \mathbf{111^\circ 48'}$$

Ex. 4. *An aluminium wire whose diameter is* 0·245 *cm is welded end to end to a copper wire with diameter* 0·1626 *cm. The composite wire carries a steady current of* 10 *amp.*

(a) What is the current density in each wire ?

(b) Find the drift speed of the electrons in copper wire.

Assume one free electron per atom in copper.

Given Avogadro Number = 6×10^{23} *atoms per gm-mole*

Molecular weight of copper = 64

Density of copper = 9 *gm/c.c.*

(Meerut 1979; Agra 72)

Solution. (a) The current density in any wire of cross-sectional area A and carrying current I is given by

$$J = (I/A) \qquad \text{...(1)}$$

The area of Aluminium wire, $A_1 = \pi r_1^2$

$$= 3{\cdot}14 \times \left(\frac{0{\cdot}245 \times 10^{-2}}{2}\right)^2 = 4{\cdot}713 \times 10^{-6} \text{ m}^2$$

The area of copper wire, $A_2 = \pi r_2^2$

$$= 3{\cdot}14 \times \left(\frac{0{\cdot}1626 \times 10^{-2}}{2}\right)^2 = 2{\cdot}075 \times 10^{-6} \text{ m}^2.$$

The current density in Aluminium wire,

$$J_{Al} = \frac{I}{A_1} = \frac{10}{4{\cdot}713 \times 10^{-6}} = \mathbf{2{\cdot}121 \times 10^{-6} \text{ amp/m}^2}$$

The current density in copper wire,

$$J_{Cu} = \frac{I}{A_2} = \frac{10}{2{\cdot}075 \times 10^{-6}} = \mathbf{4{\cdot}818 \times 10^{6} \text{ amp/m}^2}$$

(b) The number of atoms per unit volume $= \rho N/M$ where ρ is the density of copper, N is Avogadro number and M is atomic weight of copper.

As there is one free electron per atom in copper, the number of free electrons per unit volume, $n = \rho N/M$

$$= \frac{9 \times (6 \times 10^{23})}{64} \text{ electrons/cm}^3$$

$$= 8{\cdot}437 \times 10^{22} \text{ electron/cm}^3$$
$$= 8{\cdot}437 \times 10^{28} \text{ electron/m}^3$$

But $$J = nqv$$

Therefore the drift speed of electrons in copper wire,

$$v = \frac{J_{Cu}}{nq} = \frac{4{\cdot}818 \times 10^6}{8{\cdot}437 \times 10^{28} \times 1{\cdot}6 \times 10^{-29}}$$
$$= \mathbf{2{\cdot}834 \times 10^{-1}\ m/sec.}$$

Ex. 5. *In copper there are* 10^{22} *free electrons per c.c., all of which contribute to a current of* 1 *ampere in a wire of copper of* 0·01 cm^2 *cross-sectional area :*

(i) What is the average speed of electrons in the copper ?

(ii) What is the electric field in the wire ?

Given specific resistance of copper $1{\cdot}6 \times 10^{-3}$ *ohm-metre.* **(Meerut 1976; Agra 71)**

Solution. (i) The current density,

$$J = \frac{I}{A} = \frac{1 \text{ amp.}}{0{\cdot}01 \times 10^{-4}\,\text{m}^2} = \mathbf{10^6\ amp/m^2}$$

The average drift velocity, $v = J/nq$

$$= \frac{10^6 \text{ amp/m}^2}{(10^{22} \times 10^6 \text{ electrons/m}^3) \times (-1{\cdot}6 \times 10^{-19} \text{ coul.})}$$
$$= \frac{10^6}{10^{22} \times 10^6 \times 1{\cdot}6 \times 10^{-19}} = -6{\cdot}25 \times 10^{-4} \text{ m/sec.}$$

$\therefore$ Average drift speed = $\mathbf{6{\cdot}25 \times 10^{-4}\ m/sec.}$

(i) We have $$J = \sigma E$$

$$\text{Here } \sigma = \frac{1}{\rho} = \frac{1}{1{\cdot}6 \times 10^{-8} \text{ ohm-metre}}$$
$$= \frac{1}{1{\cdot}6 \times 10^{-8}} \text{ mho/metre}$$

$\therefore$ Electric field in the wire

$$E = \frac{J}{\rho} = \frac{10^6 \text{ amp/m}^2}{(1/1{\cdot}6 \times 10^{-8} \text{ mho/metre})} = \mathbf{1{\cdot}6 \times 10^{-2}\ volt/m.}$$

Ex. 6. *The potential difference across a wire of* 10^{-3} cm^2 *cross-sectional area and* 50 *cm. length is* 2 *volt, when a current of* 0·25 *amp. exists in the wire. Calculate*

(i) the field strength in the wire

(ii) the current density and

(iii).the conductivity of the wire

Solution. (i) The field strength in the wire

$$E = \frac{V}{l} = \frac{2 \text{ volt}}{0{\cdot}5 \text{ metre}} = \mathbf{4\ volt/metre.}$$

(ii) The current density

$$J = \frac{I}{A} = \frac{0{\cdot}25 \text{ amp.}}{10^{-3} \text{ cm}^2}$$

$$= \frac{0{\cdot}25 \text{ amp.}}{13^{-3} \times 10^{-4} \text{ m}^2}$$

$$= \mathbf{2{\cdot}5 \times 10^6 \text{ amp./m}^2.}$$

(iii) The electrical conductivity σ of metal is given by

$$J = \sigma E$$

or

$$\sigma = \frac{J}{E} = \frac{(0{\cdot}25 \times 10^6 \text{ amp./m}^2)}{(4 \text{ volts/metre})}$$

$$= \mathbf{6{\cdot}25 \times 10^5 \text{ ohm/metre.}}$$

Ex. 7. *A current of 1 amp. is passing through a copper wire 10 metre in length and 0·08 mm. in diameter. If the resistivity of copper is $1{\cdot}65 \times 10^{-8}$ ohm-metre, calculate*

(i) the resistance of copper wire, and

(ii) the potential difference between the ends of the wire.

Solution. (i) The resistance of copper wire,

$$R = \frac{\rho l}{A} = \frac{\rho l}{\pi r^2}$$

$$= \frac{(1{\cdot}65 \times 10^{-8} \text{ ohm-metre})\,(10 \text{ metre})}{3{\cdot}14 \times (0{\cdot}04 \times 10^{-3} \text{ metre})^2}$$

$$= \mathbf{32{\cdot}85 \text{ ohm.}}$$

(ii) The potential difference between the ends of wire,

$$V = RI = (32{\cdot}85 \text{ ohm}) \times (1 \text{ amp}) = \mathbf{32{\cdot}85 \text{ volts}}$$

Ex. 8. *The resistance of a 240 volts, 230 watts electric bulb when hot is 10 times the resistance when cold. Find its resistance at room temperature. If the working temperature of the filament is 2000°C, find its approximate coefficient of resistance.*

Solution. If V is the voltage and I the current, the wattage of the bulb,

$$W = VI \qquad \text{...(1)}$$

The resistance of the bulb at working temperature t is

$$R_t = \frac{V}{I} = \frac{V}{W/V} = \frac{V^2}{W} \qquad \text{...(2)}$$

Given $V = 240$ volt and $W = 200$ watt

$$\therefore \qquad R_t = \frac{(240)^2}{200} = \mathbf{288 \text{ ohm.}}$$

If R is the resistance of the bulb at room temperature, then

$$\therefore \qquad R = \frac{R_t}{10} \text{ (given)} = \frac{288}{10} = \mathbf{28{\cdot}8 \text{ ohm.}}$$

The variation of resistance with temperature is given by

$$R_t = R_0 (1 + \alpha t)$$

where R_0 is the resistance of the filament of bulb at 0°C and α is the temperature coefficient of resistance.

$$\therefore \qquad \alpha = \frac{R_t - R_0}{R_0 t}$$

The approximate temperature coefficient of resistance is given by

$$\alpha = \frac{R_t - R_0}{R_0 t} = \frac{288 - 28{\cdot}8}{28{\cdot}8 \times 200}$$

$$= \mathbf{4{\cdot}5 \times 10^{-3}} \textbf{ per degree C.}$$

Ex. 9. *A metallic wire carries a current of* 100 *amp. its area of cross-section is* 1·0 *cm*^{2}. *Calculate*

(a) the electric field strength in the copper. The resistivity of copper is $1{\cdot}7 \times 10^{-8}$ *ohm-m.*

(b) the potential difference between points 10 *metre apart along the length of this conductor.*

(Kanpur 1998; Rohilkhand 1983)

Given area is uniform and normal to E and resistivity of metal is $1{\cdot}7 \times 10^{-8}$ ohm-cm.

Solution. (a) We have $J = \sigma E$

or $$E = \frac{J}{\sigma} = \frac{I/A}{I/\rho} = \frac{\rho I}{A}$$

Here $\rho = 1{\cdot}7 \times 10^{-8}$ ohm-metre,

$$I = 100 \text{ amp. and } A = 1 \text{ cm}^2 = 10^{-4} \text{ metre}^2$$

$$\therefore \quad E = \frac{\rho I}{A} = \frac{1{\cdot}7 \times 10^{-8} \times 100}{10^{-4}} = \mathbf{1{\cdot}7 \times 10^{-2}} \textbf{ volt/meter.}$$

(b) The required potential difference, $V = El$

$$= 1{\cdot}7 \times 10^{-2} \times 10 = \mathbf{0{\cdot}17} \textbf{ volt.}$$

5.6. Lorentz Drude Theory of Electrical Conduction

The phenomenon of electrical and thermal conduction in metals can be very well explained on the basis of electron gas model according to which the free electrons in the metals behave like the molecules of a perfect gas in a container. Drude, in 1900, put forward a simple theoretical explanation of electrical conductivity of metals based on electron gas model. Later on Drude's theory of electrical conductivity of metals was modified by Lorentz. The basic assumptions of Lorentz-Drude theory are :

1. There are a large number of free electrons in a metal. These electrons are free to move about the whole volume of the metal like the molecules of a perfect gas in a container.

2. The motion of free electrons in a metal is similar to the thermal agitation of a perfect gas. The assembly of free electrons in a metal is called *electron gas.* The electrical or thermal conductivity of metals is solely due to these free electrons.

3. The free electrons in metals move randomly in all possible directions with widely different velocities like the molecules of a perfect gas. The distribution of velocities is in accordance with Maxwellian distribution for a gas. The random speed is a function of temperature, the average kinetic energy of a free electron is equal to that of a gas molecule at the same temperature. Thus the average kinetic energy of an electron is $3/2\ kT$ where k is Boltzmann's constant and T is absolute temperature.

4. The free electrons make collisions from time to time with fixed positive ions in the lattice and also among themselves. The collisions among the electrons themselves have no practical contribution to the conductivity of metals.

5. In the absence of external electric field, the random motion of free electrons is equally probable along all directions. As a result the current density vector is zero in the absence of electric field.

6. When an external electric field is applied, the electrons drift slowly with some average velocity, known *as average drift velocity,* in the direction opposite to that of electric field. The drift velocity of free electrons is superimposed over their random velocity. Thus the random motion of free electrons is modified

by the application of external electric field. The drift velocity is much smaller in magnitude than average random velocity of free electrons.

7. In the presence of external electric field, the free electrons being accelerated, gain some additional kinetic energy. The accelerated free electron on collision with the positive ion fixed in the lattice loses its additional kinetic energy. Thus there is loss of kinetic energy in the collision of free electron with the positive ion in the presence of external electric field. Such a collision is called *inelastic collision*. Just after the inelastic collision, the electron momentarily will have random motion. Due to external electric field the electron is again accelerated and gains some drift velocity before suffering the next collision with the positive ion of the lattice. Thus in the presence of electric field the process of inelastic collision of free electrons with the positive ions of the lattice continues.

8. The average distance traversed by a free electron between two successive collisions with the positive ions is known as *mean free path*, represented by λ.

Let there be n free electrons per cubic meter in a metal. When an external electric field **E** volts/meter is applied, the force experienced by a free electron is $-e\mathbf{E}$ newtons and so the gain in momentum after time t (Force × time)

$$= -e\mathbf{E}\, t \text{ Newton-sec.}$$

If m is the mass of the free electron, v_i the random velocity of ith electron just after its last collision and t_i the time elapsed after its last collision, the momentum of ith electron at that instant.

$$= (m\mathbf{v}_i - e\,\mathbf{E}t_i)$$

Then the average momentum of electrons at that instant is expressed as

$$m\,\mathbf{v}_{\text{av}} = \frac{\sum_{i=1}^{n} (m\mathbf{v}_i - e\,\mathbf{E}.\,t_i)}{n}$$

or average drift velocity of electrons at that instant

$$m\mathbf{v}_{av} = \frac{m}{n}\sum_{i=1}^{n}\left(\mathbf{v}_i - \frac{e\,\mathbf{E}}{m}\,t_i\right)$$

As the velocities of free electrons are randomly distributed

$$\sum_{i=1}^{n} \mathbf{v}_i = 0$$

Hence

$$\mathbf{v}_{av} = \frac{e\,\mathbf{E}\sum_{i=1}^{n} t_i}{m\,n} = \frac{e\,\mathbf{E}}{m}\,t \qquad \text{...(2)}$$

where t is the average time elapsed after last collision.

The average time will be equal to the average time from this instant to the next collision. As the collisions occur at regular intervals, then the average time τ between collisions will be

$$\tau = t + t = 2t \quad \text{or} \quad t = \tau/2. \qquad \text{...(3)}$$

Therefore from equation (3),

$$\mathbf{v}_{av} = -\frac{e\,\mathbf{E}}{m}\frac{\tau}{2}. \qquad \text{...(4)}$$

Thus it is obvious that the average drift velocity of free electrons is proportional to force ($-e$ **E**). Therefore current density **J** ,

$$\therefore \qquad \mathbf{J} = -n\,e\,\mathbf{v}_{av} = \frac{ne^2\tau}{2m}\mathbf{E} \qquad ...(5)$$

But $\mathbf{J} = \sigma\,\mathbf{E}$.

Comparing equations (5) and (6), we get

Electrical conductivity,

$$\sigma = \frac{ne^2\tau}{2m} \qquad ...(7)$$

If λ is the mean free path and $\mathbf{v}$ the average thermal speed of free electrons, then average time τ between collisions is given by

$$\tau = \frac{\lambda}{v}\text{, therefore, } \sigma = \frac{ne^2\lambda}{2m\,v}$$

As free electrons in metals obey Maxwellian distribution, the kinetic energy of a free electron, $1/2\,mv^2 = 3/2\,kT$.

Therefore,

Electrical conductivity,

$$\sigma = \frac{ne^2\lambda}{2m\,v}$$

$$\sigma = \frac{ne^2\lambda\,v}{2m\,v^2} = \frac{ne^2\lambda\,v}{6\,kT} \qquad ...(9)$$

From expressions (8) and (9) it is obvious that at a given temperature the only factor which varies from one metal to another is n (number of free electrons per unit volume). Therefore we may conclude that *the great difference in the conductivities of different metals are due to the differences in the number of free electrons per unit volume.*

5.7. Deduction of Ohm's Law from Lorentz-Drude Theory

From Lorentz Drude theory,

$$\mathbf{J} = \frac{ne^2\lambda}{2mv}\mathbf{E}$$

At a particular temperature, λ and v are constant, so we have

$$\mathbf{J} \propto \mathbf{E}$$

If I is the current flowing through a conductor of length l, cross-sectional area A and V is the potential difference across it, thenwe have

$$\frac{I}{A} = \frac{V}{l}$$

If the physical conditions do not change, then for a given conductor l and A are constants, so

$$I \propto V$$

i.e. at given physical conditions the current flowing through a conductor is proportional to the potential difference across it, which is Ohm's law.

Questions and Problems

(a) Questions

1 Define electric current and current density. Derive an expression for the current density in terms of the motion of individual charge carriers.

2. What is meant by stationary current ? Prove that for stationary current div **J** = 0.

3. (a) Prove the equation of continuity, div $\mathbf{J} + \frac{d\rho}{dt} = 0$.

   (b) Is there any electric field near the surface of a conductor carrying direct current ? Explain **(Rohilkhand 1986)**

4. Define specific resistance and electrical conductivity. Using Ohm's law derive the relation $\mathbf{J} = \sigma$ **E**.

5. (a) Derive an expression for the electrical conductivity of a metal on the basis of Lorentz-Drude's theory and obtain from it Ohm's law. What relation does it have with thermal conductivity ?

   (b) What are the short-comings of Lorentz-Drude theory ? **(Rohilkhand 1993; Agra 94, 89)**

6. What is Wiedmann Franz law ? Derive it on the basis of Lorentz-Drude free electron theory. **(Rohilkhand 1986)**

7. On the assumptions of Lorentz-Drude theory derive an expression for the drift velocity of electrons and current density in a metal when it is subjected to an external electric field.

   (Kanpur 1998; Meerut 1996; Rohilkhand 97, Agra 95)

8. Derive Ohm's law on the basis of Lorentz-Drude theory. Discuss the variation of resistivity with temperature.

9. State and prove Weidmann-Franz law. **(Rohilkhand 1996)**

(b) Problems

1. A stream of electrons of cross-section 1 mm^2 constitutes a current of 0·1 milliampere.The energy of the electrons during the last part of their flight is equal to 18 kiloelectron volts. Assume the beam to be uniform, calculate the density of electrons in the beam.

   [**Ans.** $78\cdot1 \times 10^{12}$ electrons/m^2]

2. Calculate the drift velocity of electrons in a copper wire of cross-sectional area 10^{-6} m^2, when a current of 1·0 amp. flows through it a 27°C. What is the r.m.s. speed of free electrons ? Assume one free electron per atom of copper.

   (**Ans.** $-7\cdot8 \times 10^{-5}$ m/sec, $1\cdot17 \times 10^5$ m/sec]

3. When a sufficiently high potential difference is applied between two electrodes in a gas, the gas ionises, electrons start moving towards the positive electrode and positive ions towards the negative electrode.

   (a) What is the current in a hydrogen discharge tube if in each second 4×10^{18} electrons and $1{\cdot}0 \times 10^{18}$ protons move in opposite direction through the cross-section of the tube ?

   (b) What is the direction of the current ?

   [**Ans.** (a) 0·88 amp (b) direction of positive ions]

4. In a region 5×10^{16} protons per m^3 are moving with a speed of 10^5 m/sec towards west. In the same reg there are 10^{17} electrons per m^3 moving with speed 10^4 m/sec along north east. Calculate the magnitude and direction of **J**.

   [**Ans.** $1\cdot66 \times 10^4$ amp./m^2, 43° north of east]

5. A current of 2 amp. is flowing in a wire of copper of cross-sectional area 0·01 cm^2. If the number of free electrons in copper is 8×10^{22} per c.c. calculate the drift velocity of free electrons.

   [**Ans.** $1\cdot56 \times 10^{-4}$ m/s] **(Rohilkhand 1984)**

6. A copper conductor of cross-sectional area 1 cm^2 carries a current of 200 amp. In copper there are about $8\cdot5 \times 10^{28}$ free electrons per cubic meter and the resistivity of copper is $1\cdot72 \times 10^{-8}$ (ohm-meter) $^{-1}$. Calculate

   (i) the drift velocity of free electrons in the conductor,

   (ii) the average electric field strength in copper conductor,

   (iii) the potential difference between two points on the above conductor 100 metre apart.

   [**Ans.** (i) $1\cdot5 \times 10^{-4}$ m/sec (ii) 0·0344 volt (iii) 3·54 volt]

7. The conductivity of sulphur is about 10^{-15} mho/metre. Find the current density in sulphur when it is subjected to an electric field of 2000 volt/cm.

   [**Ans.** 2×10^{-10} amp./m^2]

8. A current of 300 amp. exists in a copper wire whose diameter is 1 cm. Find the current density and calculate the electron drift speed. Given that there are $8{\cdot}5 \times 10^{28}$ free electrons per cubic meter of the metal.

   [**Ans.** $254{\cdot}78 \times 10^4$ amp/m^2, $1{\cdot}78 \times 10^{-4}$ m/sec.] **(Agra 1972)**

9. Given an aluminium wire whose conductivity is $3{\cdot}5 \times 10^7$ mho/metre carrying a current of uniform density 8×10^5 amp/m^2. Find

   (i) the electric field in the wire.

   (ii) the potential difference per meter of the wire. If there is no source of e.m.f. in the wire.

   (iii) under what conditions, if at all, could the potential difference vanish ?

   [**Ans.** (i) 2·287 volt/m (ii) 2·287 volt]

10. A rectangular carbon block has dimensions 1·0 cm. × 1·0 cm. × 50 cm.

    What is the resistance measured

    (a) between two square ends

    (b) between two opposite rectangular faces

    The resistivity of carbon at 20°C is $3{\cdot}5 \times 10^{-5}$ ohm-meter.

    [**Ans.** 0·175 ohm, $7{\cdot}0 \times 10^{-5}$ ohm]

11. For a metal the ratio of thermal and electrical conductivity at 27°C is 75×10^{-12} erg/cm°C. If the electrical conductivity of the metal is $5{\cdot}2 \times 10^{17}$ sec^{-1}, calculate its thermal conductivity in calorie sec^{-1} (°C)^1 m^{-1} at this temp ($J = 4{\cdot}2$ joules/cal)

    [**Ans.** 93 cal sec^{-1} (°C)$^{-1}$ meter^{-1}]

12. Calculate the mean free time and mean free path between collisions of the electron inside the copper at room temperature 27°C from the following data : Resistivity of copper at room temperature = $1{\cdot}7 \times 10^{-6}$ ohm metre.

    Boltzmann's constant $k = 1{\cdot}38 \times 10^{-23}$ joule deg^{-1}.
    Avogadro Number $N = 6 \times 10^{23}$ per gm. mole.
    Charge on an electron = $-1{\cdot}6 \times 10^{-19}$ coul.
    Mass of an electron = 9×10^{-31} kg.
    Density of copper = $8{\cdot}9$ g/cm^3
    Molecular weight of copper = 63·5

    [**Ans.** 5×10^{-14} sec, 6×10^{-9} meter]

6

Magnetostatics

6.1. Introduction

In this chapter we shall study the subject of magnetism, keeping in mind the fact that *all magnetic effects are all associated with the motion of electric charges or current elements.* Since moving charge anc current elements posses magnitude and direction and so they are vector quantities whereas a static charg has only scalar properties. That is why the discussion of magnetostatics is more complicated than that o electrostatics. However, in each case we start with an experimental law and develop our theory logicall from it; the two procedures are much common in method of study; but the resulting expressions ar different. Let us now outline the steps which we shall take in the present chapter by comparing them wit the corresponding ones taken in the development of electrostatics.

1. We start with an experimental law, the law of force between current elements (Ampere's law). Thi corresponds to the law of force between static charges (Coulomb's law) in electrostatics.

2. In terms of the (magnetic) force postulated by Ampere's law we define the magnetic induction **B** This corresponds to the definition of electric field **E** in terms of the force postulated by Coulomb's law.

3. Using definition of **B** together with Ampere's law, we compute the magnetic field due to a give distribution of current elements or current circuits. This corresponds to the computation of the electrostati field due to a given distribution of charges by using definition of **E** and Coulomb's law.

4. With the aid of vector **B** we find an *integral statement* of Ampere's laws; just as we found th Gauss's law which is an integral statement of Coulomb's law.

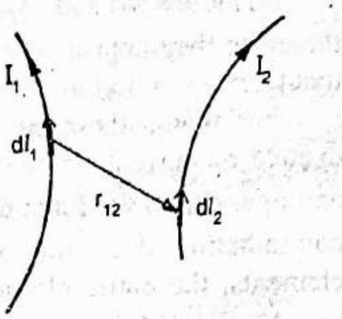
5. We examine the properties of magnetic induction **B**; find expression for its divergence and curl jus as we found

$$\operatorname{div} \mathbf{E} = \nabla \cdot \mathbf{E} = -\frac{\rho}{\varepsilon_0} \text{ and curl } \mathbf{E} = \nabla \times \mathbf{E} = 0.$$

6. We look for the scalar potential function from which **B** may be derived, just as we found that **E** could be derived from the electrostatic potential function ϕ through the relation$\mathbf{E} = -\operatorname{grad} \phi$. Keeping these steps in mind let us now develop the theory of steady magnetic fields.

6.2. Magnetic Interactions

Ampere's law of force between current elements : Andre Marie Ampere (1775–1836), a French physicist and mathematician performed a series of experiments to find the force between current elements. We shall now give the general statement of Ampere's law without proof. In fact there is no proof of it. Its correctness lies in the fact that it is supported by experimental evidence.



(Fig. 6.1)

Consider two current elements $d\,l_1$ and $d\,l_2$ (fig. 6.1), carrying the steady currents I_1 and I_2 respectively. The force between these two current elements depends upon the following facts :

(i) It varies directly as the magnitude of each current.

(ii) It varies inversely as the square of distance between the two current elements.

(iii) It depends upon the lengths and orientations of the two current elements.

(iv) It depends upon the nature of the medium.

The mathematical expression for the force exerted by $d\,l_1$ upon $d\,l_2$ is given by

$$d\,\mathbf{F}_{12} = C\ (I_1\,I_2)\left[\frac{1}{r_{12}^2}\right]\left[d\,l_2 \times \left(d\,l_1 \times \frac{\mathbf{r}_{12}}{r_{12}}\right)\right] \qquad ...(1)$$

Let us make term-by-term examination of this expression. The proportionality factor C depends on the system of units employed and in addition in incorporates the effect of medium, mentioned above as fact (iv). The factor (i) appears as the product $(I_1 I_2)$. The factor (ii) appears as the term $1/r_{12}^2$. The final term $\left[d\,l_2 \times \left\{d\,l_1 + \frac{\mathbf{r}_{12}}{\mathbf{r}_{12}}\right\}\right]$ denotes factor (iii). The vector $\left\{\frac{\mathbf{r}_{12}}{r_{12}}\right\}$ represents simply unit vector along $\mathbf{r}_{12}$ direction. In M.K.S.A. system the constant C is chosen

$$C = \frac{\mu}{4\pi} \qquad ...(2)$$

where μ is known as absolute magnetic permittivity (*or permeability*) of the medium. So equation (1) takes the form

$$d\,\mathbf{F}_{12} = \frac{\mu}{4\pi}(I_1\,I_2)\left[\frac{1}{r_{12}^2}\right]\left[\left\{d\,l_2 \times \frac{\mathbf{r}_{12}}{r_{12}}\right\}\right] \qquad ...(3)$$

Equation (3) may be arranged to express the force exerted by current elements $I_1\,d\,\mathbf{l}_1$ on $I_2\,d\,\mathbf{l}_2$ as

$$d\,\mathbf{F}_{12} = \frac{\mu}{4\pi}\frac{(I_2\,d\,l_2) \times (I_1\,d\,l_1 \times \mathbf{r}_{12})}{r_{12}^3} \qquad ...(4)$$

If $d\,\mathbf{F}_{21}$ is the force exerted by current element $I_2\,d\,l_2$ on current element $I_1\,d\,l_1$, and $\mathbf{r}_{21}$ is the position vector of element $I_1\,d\,l_1$ relative to element $I_2\,d\,l_2$, then

$$d\,\mathbf{F}_{21} = \frac{\mu}{4\pi}\frac{(I_1\,d\,l_1) \times (I_2\,d\,l_2 \times \mathbf{r}_{21})}{\mathbf{r}_{21}^3} \qquad ...(5)$$

Either of the two equation (4) and (5) is known as Ampere's law of force between two *elements*.

In this form the law of force between current elements has much in common with the law of force (Coulomb's law) between the point charges, although the direction of force, as expressed by (4) or (5) is not easy to visualise as was in the case of Coulomb's law.

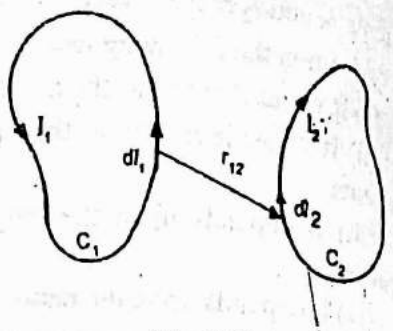
For *free space* $\mu = \mu_0$ and has the value

$$\mu_0 = 4\pi \times 10^{-7} \text{ weber/ampere–metre} \qquad ...(6)$$

This is not an arbitrary assigned value, but follows form the definition of ampere. The ratio $K_m = \mu/\mu_0$ is known as the relative permeability or the magnetic coefficient of the medium. Thus we may write

$$\mu = K_m\,\mu_0. \qquad ...(7)$$

Equations (4) and (5) are not symmetrical in $I_1\, d\,l_1$ and $I_2\, d\,l_2$; therefore they appear to contradict Newton's III law, which states that to every action there must exist an equal and opposite reaction. In other words these equations appear to indicate that the force exerted on current element $I_2\, d\,l_2$ by $I_1\, d\,l_1$ is not necessarily equal and opposite to the force exerted on elements $I_1\, d\,l_1$ by $I_2\, d\,l_2$. This contradiction does not arise if instead of differential current elements, the entire closed loops C_1 and C_2 are considered; since then Newton's law is satisfied as shown below.



(Fig. 6.2)

The force ($\mathbf{F}_{12}$) exerted by current loop C_1 on current loop C_2 may be obtained by integrating equation (4) over entire current loops *i.e.*

$$\mathbf{F}_{12} = \frac{\mu}{4\pi} \oint_{C_1} \oint_{C_2} \frac{I_1\, d\,l_1 \times (I_1\, d\,l_2 \times \mathbf{r}_{12})}{r_{12}^3} \quad ...(8)$$

Similarly the force exerted by current loop C_2 on current loop C_1 is written as

$$\mathbf{F}_{21} = \frac{\mu}{4\pi} \oint_{C_1} \oint_{C_2} \frac{I_1\, d\,l_1 \times (I_2\, d\,l_2 \times \mathbf{r}_{21})}{r_{21}^3} \quad ...(9)$$

Now using vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$; equation (8) may be rewitten as

$$\mathbf{F}_{12} = \frac{\mu}{4\pi} \oint \oint \frac{I_1 I_2}{r_{12}^3} [(d\,\mathbf{l}_2 \cdot \mathbf{r}_{12})\, d\,l_1 - (d\,l_2 \cdot d\,l_1)\, \mathbf{r}_{12}] \quad ...(10)$$

It may be easily seen that

$$\frac{\mathbf{r}_{12}}{r_{12}^3} = -\nabla_2 \left[\frac{1}{r_{12}} \right] \quad ...(11)$$

where ∇_2 means gradient with respect to point (x_2, y_2, z_2). Using (11), the first term in equation (10) takes the form

$$\frac{\mu\, I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(d\,l_2 \cdot \mathbf{r}_{12})\, d\,l_1}{r_{12}^3} = -\frac{\mu\, I_1 I_2}{4\pi} \oint_{C_1} dl_1 \oint_{C_2} \nabla_2 \left[\frac{1}{r_{12}} \right] \cdot d\,l_2 \quad ...(12)$$

The total differential of $\left[\frac{1}{r_{12}} \right]$ at any point on the circuit C_2 can be expressed as

$$d_2 \left[\frac{1}{r_{12}} \right] = \frac{\partial}{\partial x_2} \left[\frac{1}{r_{12}} \right] dx_2 + \frac{\partial}{\partial y_2} \left[\frac{1}{r_{12}} \right] dy_2 + \frac{\partial}{\partial z_2} \left[\frac{1}{r_{12}} \right] dz_2$$
$$= \nabla_2 \left[\frac{1}{r_{12}} \right] \cdot d\,l_2$$

Hence $\nabla_2 \left[\frac{1}{r_{12}} \right] \cdot d\,l_2$ is the total directional derivative of $\left[\frac{1}{r_{12}} \right]$ along C_2. Using the result, equation (12) becomes

$$\frac{\mu\, I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(d\,l_2 \cdot \mathbf{r}_{12})\, d\,l_1}{r_{12}^3} = -\frac{\mu\, I_1 I_2}{4\pi} \oint_{C_1} dl_1 \oint_{C_2} d_2 \left[\frac{1}{r_{12}} \right] = 0. \quad ...(14)$$

Since the integrand $d_2 \left[\frac{1}{r_{12}} \right]$ when integrated gives $\left[\frac{1}{r_{12}} \right]$; which being single valued function vanishes when evaluated around the closed path C_2. Therefore integral of first term on R.H.S. of equation (10) is zero; hence equation (10) takes the form

$$\mathbf{F}_{12} = -\frac{\mu I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{(d\mathbf{l}_2\cdot d\mathbf{l}_1)\,\mathbf{r}_{12}}{r_{12}^3} \quad ...(15)$$

This equation represents an equivalent form of Ampere's law of force acting on current loop C_2 due to current loop C_1. This form of Ampere's law is symmetrical with respect to loops 1 and 2. The force acting on current loop C_1 due to current loop C_2 may be obtained by interchanging indices 1 and 2 in equation (15) *i.e.*

$$\mathbf{F}_{21} = -\frac{\mu I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{(d\mathbf{l}_2\cdot d\mathbf{l}_1)\,\mathbf{r}_{21}}{r_{21}^3}$$

Since $|\mathbf{r}_{21}| = |\mathbf{r}_{12}|$ and $\mathbf{r}_{21} = -\mathbf{r}_{12}$; this implies

$$\mathbf{F}_{21} = +\frac{\mu I_1 I_2}{2\pi}\oint_{C_1}\oint_{C_2}\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)}{r_{12}^3}\,\mathbf{r}_{12}. \quad ...(16)$$

As dot product of two vectors is commutative, comparing (15) and (16), we note that

$$\mathbf{F}_{21} = -\mathbf{F}_{12}. \quad ...(17)$$

This shows that *Newton's III law holds for the force between two closed current loops given by Ampere's law.* Thus in order that Newton's III law is satisfied, we must state Ampere's law for the force between two current loops instead of current elements.

6.3. Biot-Savart Law

We know that the electric force exerted by a charge configuration upon any other charge q_0 can be computed by the relation $\mathbf{F} = q_0\,\mathbf{E}$ provided the electric field $\mathbf{E}$ due to charge configuration is known. Similarly in calculating magnetic effects, rather than computing directly the force exerted by one current element upon the other using equation (4) of section 6·2, it is usually convenient to employ the intermediate of the *magnetic field.* With this in mind let us rewrite equation (4) of section 6·2 as

$$d\,\mathbf{F}_{12} = I_2\,d\mathbf{l}_2 \times d\,\mathbf{B}_1 \quad ...(1)$$

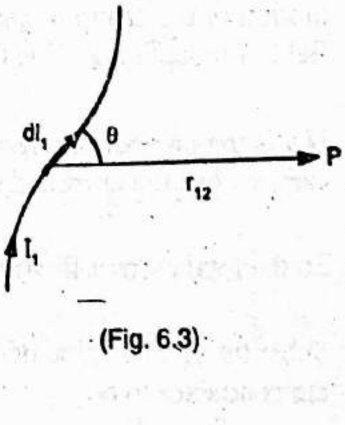
where

$$d\,\mathbf{B}_1 = \frac{\mu}{4\pi}\frac{I_1\,d\mathbf{l}_1 \times \mathbf{r}_{12}}{r_{12}^3} \quad ...(2)$$

is the *magnetic induction* or *magnetic flux density* of the current element $d\mathbf{l}_1$ at a vector distance $\mathbf{r}_{12}$. Equation (2) represents the *magnetic effect* or *field* or *field* that is produced by the current element $d\mathbf{l}_1$ and is called the *Biot-Savart law.* This law is sometimes also known as *Ampere's law.* It is obvious from equation (2) that the magnitude of magnetic induction $d\mathbf{B}_1$ produced by a small current element $d\mathbf{l}_1$ carrying a current I_1 at a point P having position vector $\mathbf{r}_1$, with respect to this current element (fig. 6·3) is

$$dB_1 = \frac{\mu}{4\pi}\frac{I_1\,d\,l_1\sin\theta}{r_{12}^2}\ \text{weber/m}^2$$

and its direction is that of vector $d\mathbf{l}_1 \times \mathbf{r}_{12}$. The resultant magnetic induction at point P is found by integrating expression (2) *i.e.*



(Fig. 6.3)

$$\mathbf{B}_1 = \int dB_1 = \frac{\mu}{4\pi}\int\frac{I_1\,d\mathbf{l}_1 \times \mathbf{r}_{12}}{r_{12}^3}$$

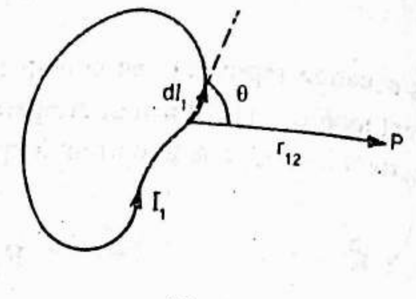
The integration extends over the whole length of the current element

For a closed loop of current as in fig. 6·4, the resultant magnetic induction at P is

$$\mathbf{B}_1 = \frac{\mu}{4\pi}\oint \frac{I_1\, d\boldsymbol{l}_1 \times \mathbf{r}_{12}}{r_{12}^3} \qquad \text{...(3)}$$

where the integration is taken over the entire closed loop circuit. Sometimes for practical purposes it is useful to express the magnetic induction in terms of current densities in the sysem.

If $\mathbf{J}_1$ is the current density at a point on the element of length $d\boldsymbol{l}_1$ and dV is the infinitesimal volume surrounding that point, then we have



(Fig. 6.4)

$$I_1\, d\boldsymbol{l}_1 = \mathbf{J}_1\, dV \left(\text{since } J_1 = \frac{I_1}{da_1} \text{ and } da_1\, dl_1 = dV \right)$$

Substituting this in equation (3), we get

$$\mathbf{B}_1 = \frac{\mu}{4\pi}\oint \frac{\mathbf{J}_1 \times \mathbf{r}_{12}}{r_{12}^3}\, dV. \qquad \text{...(4)}$$

Biot-Savart law is very convenient to evaluate the magnetic induction **B** in different cases of current elements.

6.4. Force on a Current Carrying Conductor and Lorentz Force

From equation (1) section 6·3 it obvious that the force experienced by a current element $\delta\mathbf{l}$ carrying current I situated in a magnetic field of induction **B** is given by

$$\delta\mathbf{F}_m = I\,\delta\boldsymbol{l} \times \mathbf{B}. \qquad \text{...(1)}$$

The expression holds even if the magnetic field **B** is produced due to some other electric current or due to a number of permanent magnets.

In many practical cases, the whole of the conductor of length l is situated in a uniform field of induction **B** and each element of it is right angles to **B**. The *magnetic force on the conductor* in such a case is given by

$$\mathbf{F}_m = \Sigma\,\delta\mathbf{F}_m = \Sigma\, I\delta\, l \times B = I\, l\, B \sin 90^\circ\, \hat{n} = I\, l\, B\, \hat{n}. \qquad \text{...(2)}$$

Thus the force has magnitude $I\,l\,B$ and its direction is normal to the plane containing **l** and B (Here $\hat{n}$ is unit vector normal to the plane containing **l** and **B**).

Lorentz Force. Fig. 6·5 represents a conductor of cross-sectional area **A** carrying current I due to the motion of the group of charge carriers, each charge carrier moving with velocity **v** and placed in a magnetic field of induction **B**. The force experienced by an element of length $\delta\, l$ of the conductor is given by

$$\delta\mathbf{F}_m = I\,\delta l \times \mathbf{B}.$$

If n is the number of charges carriers per unit volume, and q the charge carried by each charge carrier, then the current density in the conductor is given by

$$\mathbf{J} = nq\mathbf{v}. \qquad \text{...(3)}$$

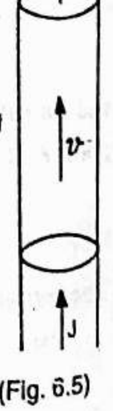
So the total current flowing in the conductor is given by

$$I = \mathbf{J}\cdot\mathbf{A} = nq\mathbf{v}\cdot\mathbf{A}. \qquad \text{...(4)}$$

Substituting this value of I in equation (1), we find the force experienced by the element $\delta\, l$ of the conductor to be

$$\delta\mathbf{F}_m = (nq\mathbf{v}\cdot\mathbf{A})\,\delta\, l \times \mathbf{B}. \qquad \text{...(5)}$$

$$\delta\mathbf{F}_m = (nq\,\delta l\cdot\mathbf{A})\,\mathbf{v} \times \mathbf{B}. \qquad \text{...(6)}$$



(Fig. 6.5)

Here $\delta \mathbf{l} \cdot \mathbf{A}$ is the volume of the current element, so $(n\, \delta \mathbf{l} \cdot \mathbf{A})$ is the number of charge carriers contained within it. Equation (6) thus, represents the force experienced by $(n\delta \mathbf{l} \cdot \mathbf{A})$ charge carriers. Hence the *magnetic force known as the Lorentz force* experienced by a single charge carrier carrying the charge q and moving with velocity $\mathbf{v}$ in a magnetic field of induction $\mathbf{B}$ is given by

$$\mathbf{F}_m = \frac{\delta \mathbf{F}}{(n\delta \mathbf{l} \cdot \mathbf{A})} = \frac{(nq\delta \mathbf{l} \cdot \mathbf{A})\, \mathbf{v} \times \mathbf{B}}{n\delta \mathbf{l} \cdot \mathbf{A}} = q\, (\mathbf{v} \times \mathbf{B}). \qquad ...(7)$$

Clearly the force is perpendicular to both the directions of $\mathbf{v}$ and $\mathbf{B}$. We know that the force experienced by a charge q in an electric field of intensity $\mathbf{E}$ is $\mathbf{F}_\theta = q\mathbf{E}$. ...(8)

Therefore the *total force* acting on a charge in a region pervaded by electric and magnetic fields is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\, (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \qquad ...(9)$$

6.5 The Definition of B Without Using the Concept of Magnetic Poles and Some Other Useful Information

Let us consider a case in which the electric field is absent and a test charge q moves with an arbitrary velcoity $\mathbf{v}$ through a point P. Neglecting the force of gravity if the test charges q experiences a *sideways deflecting force* $\mathbf{F}_m$, dependent on velocity $\mathbf{v}$, we assert that a magnetic field exists at the point P and the magnetic induction $\mathbf{B}$ of this field is defined in terms of magnetic force $\mathbf{F}_m$ and other measured quantities. The sideways deflecting force $\mathbf{F}_m$ (or the magnetic force $\mathbf{F}_m$) is related to $\mathbf{B}$ as

$$\mathbf{F}_m = q\, (\mathbf{v} \times \mathbf{B}). \qquad ...(1)$$

Obviously $\mathbf{F}_m$ is always perpendicular to $\mathbf{v}$ and also to $\mathbf{B}$ and thus is always a sideways force.

The magnitude of deflecting force, $\quad F_m = qv\, B \sin \theta$

where θ is the nagle between $\mathbf{v}$ and $\mathbf{B}$.

Therefore the magnitude of magnetic induction,

$$B = \frac{F_m}{qv \sin \theta}. \qquad ...(2)$$

If $\mathbf{v}$ is at right angles to $\mathbf{B}$, the force F_m is now a maximum and the magnitude of $\mathbf{B}$ may be defined in terms of the measured magnitude of this maximum force $(F_m)_{max.}$, as

$$B = \frac{(F_m)_{max.}}{qv}. \qquad ...(3)$$

The complete definition $\mathbf{B}$ is given by equations (1), (2) and (3). From these equations the following facts are obvious.

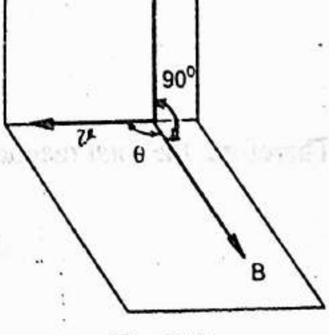
(i) The plane of magnetic field vector $\mathbf{B}$ is perpendicular to the plane of $\mathbf{F}_m$.

(ii) The magnitude of the magnetic force $\mathbf{F}_m$ is $qv\, B \sin \theta$.

(iii) The magnetic force $\mathbf{F}_m$ acting on the moveing charge q vanishes in the following cases :

(a) If $v = 0$, (b) $B = 0$, (c) If $\mathbf{v}$ either is parallel or antiparallel to the direction $\mathbf{B}$.

(iv) The magnetic force has maximum value $(F_m)_{max.}$, if $\mathbf{v}$ and $\mathbf{B}$ are perpendicular to each other *i.e.* $\quad (F_m)_{max} = qv\, B.$ ...(4)



(Fig. 6.6)

Fig. 6·6. represents the relations among the vectors $\mathbf{F}_m$, $\mathbf{v}$ and $\mathbf{B}$.

Summarising the above result the *magnetic induction* $\mathbf{B}$ *at any point in a magnetic field may be defined as a vector quantity whose magnitude is equal to*

$$\frac{F_m}{qv \sin \theta} \text{ or } \frac{(F_m)_{max.}}{qv}$$

and whose direction is always perpendicular to $\mathbf{F}_m$, *given by right handed screw rule as applied in vector product.*

If $\mathbf{v}$ is perpendicular to $\mathbf{B}$, the magnetic force $\mathbf{F}_m$ $(= q\mathbf{v} \times \mathbf{B})$ is constant in magnitude, but always acts at right angles to the velocity $\mathbf{v}$ of the charged particle. Hence the path traversed by the charged particle must be a circle, described with constant speed v, the necessary centripetal force being provided by the magnetic force,

i.e. centripetal force $\dfrac{mv^2}{r} = qvB.$

6.6. Some Applications of Biot-Savart Law

1. Magnetic field due to the current flowing in a striaght wire. Let CD be a straight wire carrying a current I ampere and P the point at a distance R from the wire at which the magnetic field is to be determined. Now consider a small element ab of length δl of the wire at a distance l from M, the magnetic field induction $\delta\mathbf{B}$ due to this element is

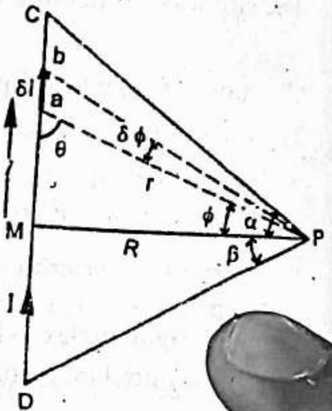
$$\delta\mathbf{B} = \frac{\mu I}{4\pi} \frac{\delta \mathbf{l} \times \mathbf{r}}{r^3} \qquad ...(1)$$

Here $\mathbf{r}$ is the position vector of point P relative to this element ab.

The direction of $\delta\mathbf{B}$ at point P is perpendicular to the plane of the paper and is directed into the paper.

The magnitude of $\delta\mathbf{B}$ is

$$\delta B = \frac{\mu I}{4\pi} \frac{\delta l \sin \theta}{r^2} \qquad ...(2)$$



(Fig. 6.7)

From fig. 6·7

$$l = R \tan \phi.$$

$\therefore$
$$\delta l = R \sec^2 \phi \, \delta\phi.$$

Also
$$r = R \sec \phi.$$

and
$$\sin \theta = \sin (90 - \phi) = \cos \phi.$$

Substituting these values in equation (2), we get

$$\delta B = \frac{\mu I}{4\pi} \frac{(R \sec^2 \phi \, \delta\phi) \cos \phi}{(R \sec \phi^2)}$$

$$= \frac{\mu I}{4\pi R} \cos \phi \, \delta\phi.$$

Therefore, the total magnetic induction B at point P due to the whole write is given by

$$B = \frac{\mu I}{4\pi R} \int_{-\beta}^{\alpha} \cos \phi \, d\phi = \frac{\mu I}{4\pi R} \Big[\sin \phi \Big]_{-\beta}^{\alpha}$$

$$= \frac{\mu I}{4\pi R} [\sin \alpha + \sin \beta]. \qquad ...(3)$$

If the wire is infinity long, then

$$\alpha = \frac{\pi}{2} \quad \text{and} \quad \beta = \frac{\pi}{2}.$$

Therefore magnetic induction

$$B = \frac{\mu I}{4\pi R}\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right]$$

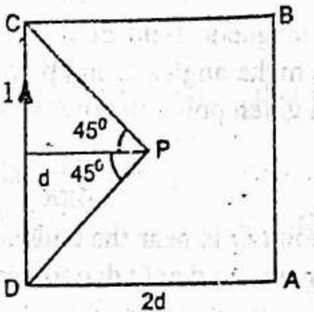
$$= \frac{\mu I}{4\pi R}\ \text{weber/m}^2. \qquad \text{...(4)}$$

Ex. 1. *Use Biot Savart law to obtain an expression for the magnetic field at the centre of a coil in the form of a square of side 2d carrying a current I.* **[Rohilkhand 1998; Agra 1991; Bombay 96]**

Solution. Let P be the centre of a square $ABCD$ of side $2d$ meters carrying a current I amp. (fig. 6·8). According to equation (3), of section 6·6, the magnetic field at the centre P due to the side CD.

$$= \frac{\mu I}{4\pi d}(\sin 45^\circ + \sin 45^\circ)$$

$$= \frac{\sqrt{2}\mu I}{4\pi d}\ (\text{since here } \alpha = \beta = 45^\circ \text{ and } R = d).$$



(Fig. 6.8)

By symmetry the magnetic field at P due to each side will be

$$\frac{\sqrt{2}\mu I}{4\pi d}\ \text{weber/m}^2.$$

Therefore, the magnetic field B at the centre of the square is given by

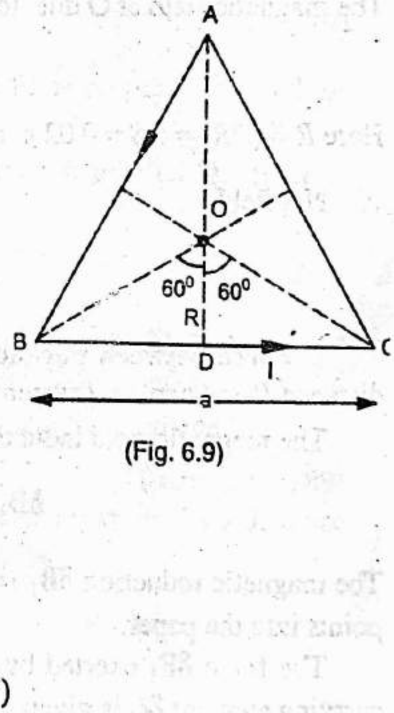
$$B = 4 \times \frac{\sqrt{2}\mu I}{4\pi d} = \frac{\sqrt{2}\mu I}{\pi d}\ \text{weber/m}^2.$$

The direction of the magnetic field is perpendicular to the plane of the paper.

Ex. 2. *A current of 1A is flowing in the sides of an equilateral triangle of side $4{\cdot}5 \times 10^{-2}$ m. Find the magnetic field at the centroid of the triangle.*

(Roorkee 91)

Solution. Let I be the current flowing in the sides of triangle. The magnetic field due to one side (say BC) of triangle at centroid O



(Fig. 6.9)

$$B_1 = \frac{\mu_0 I}{4\pi R}(\sin\alpha + \sin\beta)$$

Here $I = 1\,A$, $\frac{\mu_0}{4\pi} = 10^{-7}$ wb/A-m

$$\alpha = \beta = 60^\circ,\ R = \text{OD} = \frac{\text{BD}}{\tan 60^\circ}$$

$$= \frac{a/2}{\sqrt{3}} = \frac{a}{2\sqrt{3}} = \frac{4{\cdot}5 \times 10^{-2}}{2\sqrt{3}}\ m$$

$$\therefore \quad B_1 = \frac{10^{-7} \times 1}{(4{\cdot}5 \times 10^{-2}/2\sqrt{3})}(\sin 60^\circ + \sin 60^\circ)$$

$$= \frac{10^{-7} \times 2\sqrt{3}}{4{\cdot}5 \times 10^{-2}} \times \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{6}{4{\cdot}5} \times 10^{-5}\ \text{wb/m}^2$$

$\therefore$ The magnetic field due to whole triangle at centroid O is,

$$B = 3B_1 = \frac{3 \times 6}{4{\cdot}5} \times 10^{-5}$$

$$= \mathbf{4 \times 10^{-5}\ weber/m^2.}$$

Ex. 3. *A pair of stationary and infinitely long bent wires are placed in the XY plane as shown in fig. The wires carry currents of i = 10 amperes each as shown. The segments L and M are along the X-axis. The segments P and Q are parallel to the Y-axis such that OS = OR = 0·02 m. Find the magnitude and direction of the magnetic induction at the origin O.* **(IIT 89)**

Solution. Magnetic field due to wire LR

$$= \Sigma \frac{\mu_0}{4\pi} \cdot \frac{i\,dl \sin 0}{r^2} = 0$$

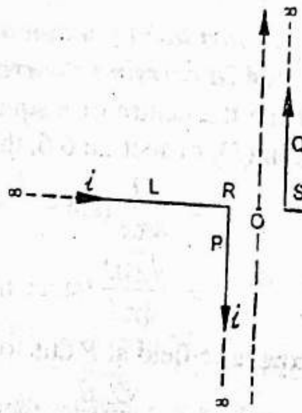
Magnetic field due to wire SM

$$= \Sigma \frac{\mu_0}{4\pi} \cdot \frac{i\,dl \sin 180^\circ}{2} = 0$$

The magnetic field due to a wire of finite length; whose ends make angles α and β with the perpendicular dropped from given point on wire is

$$\beta = \frac{\mu_0 i}{4\pi R}(\sin\alpha + \sin\beta)$$

As point O is near the ends of wires RP and QS, therefore magnetic field at O due to wire RP is



(Fig. 6.10)

$$B_1 = \frac{\mu_0 i}{4\pi R}\left(\sin 0 + \sin\frac{\pi}{2}\right) = \frac{\mu_0 i}{4\pi R} \text{ (upward)}$$

The magnetic field at O due to wire QS is

$$B_2 = \frac{\mu_0 i}{4\pi R}\left(\sin\frac{\pi}{2} + \sin 0\right) = \frac{\mu_0 i}{4\pi R} \text{ (upward)}$$

Here $R = (OR) = OS = 0{\cdot}02\ m$ and $i = 10$ A

∴ Net field

$$B = B_1 + B_2 = \frac{\mu_0 i}{4\pi R} + \frac{\mu_0 i}{4\pi R} = \frac{\mu_0 i}{2\pi R}$$

$$= \frac{4\pi \times 10^{-7} \times 10}{2\pi \times 0{\cdot}02} = \mathbf{10^{-4}\ weber/m^2}.$$

2. Force between parallel wires carrying currents. Let *a* and *b* be two parallel wires separated by a distance *R* and carrying currents I_1 and I_2 amp. respectively.

The magnetic field induction produced at element δl, due to elemenδl, is

$$\delta\mathbf{B}_1 = \frac{\mu I_1}{4\pi} \frac{\delta l_1 \times \mathbf{r}_{12}}{r_{12}^3} \quad \text{...(5)}$$

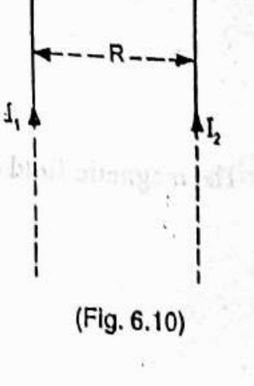
The magnetic induction $\delta\mathbf{B}_1$ is perpendicular to the plane containing δl_1 and $\mathbf{r}_1$, and points into the paper.

The force $\delta\mathbf{F}_1$ exerted by current carrying element δl_1 upon the second current carrying element δl_2 is given by

$$\delta\mathbf{F}_{12} = I_2\, \delta l_2 \times \delta\mathbf{B}_1$$

$$= I_2\, \delta l_2 \times \left\{\frac{\mu I_1}{4\pi} \frac{\delta l_1 \times \mathbf{r}_{12}}{r_{12}^3}\right\}$$

$$= \frac{\mu I_1 I_2}{4\pi r_{12}^3} \delta l_2 \times (\delta l_1 \times \mathbf{r}_{12}). \quad \text{...(6)}$$



(Fig. 6.10)

From the vector product it may readily be seen that the direction of $\mathbf{F}_{12}$ is as shown in fig. 6·9. Likewise the force $\delta\mathbf{F}_{21}$ exerted by current carrying element δl_2 on current carrying element δl_1 has the direction shown in figure.

Thus we may conclude : *if the current in parallel wires are in the same direction, the force between them is* **attractive;** *but if the current in the parallel wires are in opposite directions, the force between them is* **repulsive.**

If the wire a is long, the magnetic induction **B**, produced by it at a distance **R** from its axis has the magnitude given by

$$B_1 = \frac{\mu I_1}{2\pi R} \qquad \text{...(7)}$$

The direction of $\mathbf{B}_1$ is perpendicular to the paper and points into the paper.

We can now consider that the force on wire b due to the current flowing in wire a is the same as that in wire b located in a field of induction $\mathbf{B}_1$.

Thus the force experienced by current carrying element δl_2 is given by

$$\delta\mathbf{F} = I_2\,\delta l_2 \times \mathbf{B}_1 \qquad \text{...(8)}$$

or

$$\delta\mathrm{F} = I_2\,\delta l_2 \left\{\frac{\mu I_1}{4\pi R}\right\}$$

Thus, the magnitude of force extered per unit length on second wire b is

$$F = \frac{\mu_0 I_1 I_2}{2\pi R}\ \text{Newton/metre.} \qquad \text{...(9)}$$

Similarly, we can compute the field of induction $\mathbf{B}_2$ at wire a and then find the force on per unit length of wire a to be the same as given by equation (9).

Thus if the currents are parallel, the force is *attractive* and if the currents are anti-parallel, the force is repulsive; the magnitude of force per unit length on either wire being given by

$$F = \frac{\mu_0\, I_1 I_2}{2\pi R}\ \text{Newton/metre.} \qquad \text{...(10)}$$

Ex. 4. *A long horizontal rigidly supported wire carries a current i_a of 100 amp. Directly above it and parallel to it is a fine wire that carries a current i_b of 2θ amp and weighs* 0·03 *Newton/metre. How far above the lower wire should the second wire be kept if we wish to support it by magnetic repulsion* ?

Given permeability constant $\mu_0 = 4\pi \times 10^{-7}$ *weber/metre.* **(Meerut 1993, 89)**

Solution. If i_a and i_b are the anti-parallel currents flowing in two parallel wires separated by a distance R, then the repulsive force experienced by per metre length of either wire is given by

$$F = \frac{\mu_0 i_a i_b}{2\pi R}.$$

Weight per metre length of either wire = 0·073 Newton/m (given).

Therefore according to problem the weight of lower wire is suppored by upward magnetic repulsion; therefore

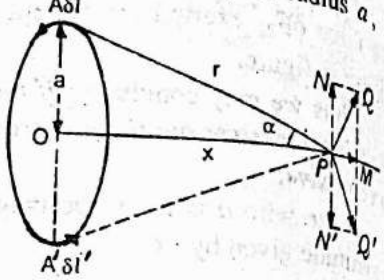
$$\frac{\mu_0 i_a i_b}{2\pi R} = 0{\cdot}073$$

or

$$R = \frac{\mu_0 i_a i_b}{2\pi \times 0{\cdot}073} = \frac{4\pi \times 10^{-7} \times 100 \times 20}{2\pi \times 0{\cdot}073}\ \text{metres}$$

$$= 5{\cdot}48 \times 10^{-3}\ \text{meters} = \mathbf{5{\cdot}48\ mm.}$$

3. Magnetic field induction on the axis of circular loop. Consider a circular loop of radius a, carrying current I amp. with its plane perpendicular to the plane of the paper. Let P be a point on the axis of this circular loop at a distance x from its centre O. We have to calculate the magnetic induction at P. Consider a small element of the coil length $\delta \mathbf{l}$ and point A. The magnetic induction $\delta \mathbf{B}$ at point P due to this element is given by



(Fig. 6.11)

$$\delta \mathbf{B} = \frac{\mu I}{4\pi} \frac{\delta \mathbf{l} \times \mathbf{r}}{r^3}. \qquad ...(11)$$

The direction of $\delta \mathbf{B}$ is perpendicular to the plane containing δl and $\mathbf{r}$ and is given by right handed screw rule. As the angle between δl and $\mathbf{r}$ is 90°, the magnitude of the magnetic induction δB is given by

$$\delta B = \frac{\mu I}{4\pi} \frac{\delta l \sin 90^\circ}{r^2} = \frac{\mu I \, \delta l}{4\pi r^2}. \qquad ...(12)$$

In fig. (6·11) $\delta \mathbf{B}$ has been represented by $\overrightarrow{PQ}$. The vector $\delta \mathbf{B}$ or $(\overrightarrow{PQ})$ can be resolved into the components namely PM and PN along and perpendicular to the axis respectively. Now consider another small elements of length $\delta l'$ at A'. The magnetic induction $\delta \mathbf{B}$ due to this element has been represented by $\overrightarrow{PQ'}$ whose magnitude is $\frac{\mu I \, \delta l'}{4\pi r^2}$ and which can also be resolved into two components; PM and PN' along the axis and perpendicular to the axis respectively. Thus if we consider the magnetic induction produced by whole of the circular coil, then by symmetry the components of magnetic induction perpendicular to the axis will be cancelled out, while those parallel to the axis will be added up. Thus the resultant magnetic induction $\mathbf{B}$ at axial point P is along the axis and may be evaluated as follows :

The components of $\delta \mathbf{B}$ along the axis,

$$\delta B_x = \frac{\mu I \, \delta l}{4\pi r^2} \sin \alpha. \qquad ...(13)$$

But $\sin \alpha = \frac{a}{r}$ and $r = (a^2 + x^2)^{1/2}$

$$\delta B_x = \frac{\mu I \, \delta l}{4\pi r^2} \cdot \frac{a}{r} = \frac{\mu I a}{4\pi r^3} \delta l = \frac{\mu I a}{4\pi (a^2 + x^2)^{3/2}} \delta l. \qquad ...(14)$$

Therefore the magnitude of resultant magnetic induction at axis point P due to whole circular coil is given by

$$B = \oint \frac{\mu I a}{4\pi (a^2 + x^2)^{3/2}} dl = \frac{\mu I a}{4\pi (a^2 + x^2)^{3/2}} \oint dl.$$

But $\oint dl$ = length on the loop $= 2\pi a$. ...(15)

Therefore,

$$B = \frac{\mu I a}{4\pi (a^2 + x^2)^{3/2}} (2\pi a)$$

$$= \frac{\mu I a}{2 (a^2 + x^2)^{3/2}} \text{ weber/metre}^2.$$

If the coil contains n turns, then

$$B = \frac{\mu n I a^2}{2 (a^2 + x^2)^{3/2}} \text{ weber/m}^2. \qquad ...(16)$$

Different cases :

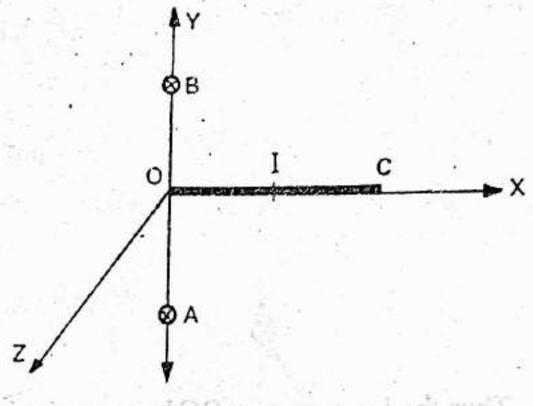
(i) The magnetic field induction at the centre of the circular coil. At the centre of the circular coil, $x = 0$ hence

$$B = \frac{\mu n I a^2}{2a^2} \text{ webers/m}^2. \qquad ...(17)$$

(ii) The magnetic field due to a small circular coil. If the coil is small, then $x \gg a$, hence

$$B = \frac{\mu n I}{2x} \text{ weber/m}^2. \qquad ...(18)$$

Ex. 5. *A straight segment OC (of length L metre) of a circuit carrying a current I amp is placed along the X-axis (fig.). Two infinitely long straight wires A and B, each extending from $z = -\infty$ to $+\infty$ are fixed at $y = -a$ metre and $y = +a$ metre respectively, as shown in the figure. If the wires A and B each carry a current I amp into the plane of the paper, obtain the expression for the force acting on the segment OC. What will be the force on OC if the current in the wire B is reversed ?* **(IIT 1992)**



(Fig. 6.12)

Solution. Consider an element at P of length dx at a distance x from origin O_y the magnetic field at P due to current in wire A

$$\vec{B_1} = \frac{\mu_0 I}{2\pi\sqrt{x^2 + a^2}} \text{ along } \vec{PQ}$$

The magnetic field at P due to current in wire B

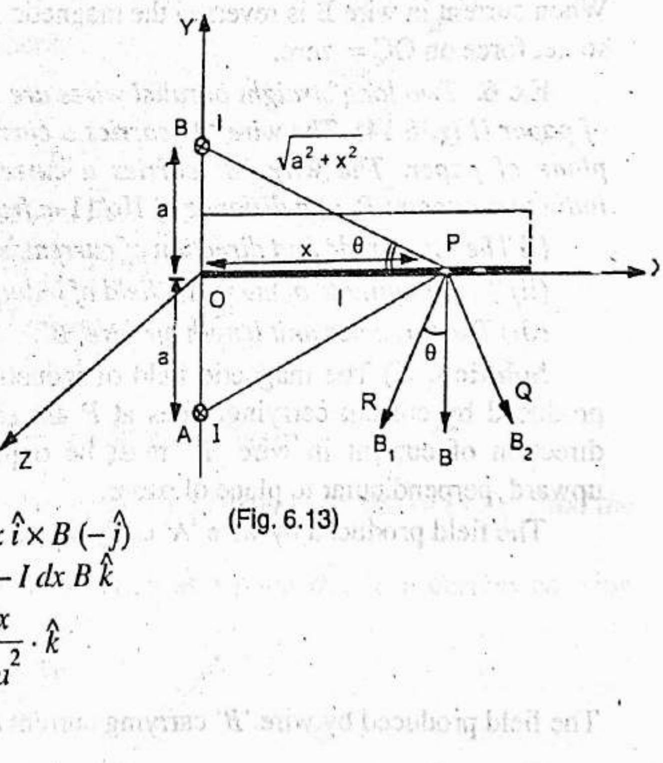
$$\vec{B_2} = \frac{\mu_0 I}{2\pi\sqrt{x^2 + a^2}} \text{ along } \vec{PR}$$

$\therefore$ Taking components of B_1 and B_2 along X and Y axes we note that the components along X-axis cancel out while those along (–) Y axis add up.

$\therefore$ Net magnetic field at P

$$B = \frac{\mu_0 i}{2\pi\sqrt{x^2 + a^2}}(\cos\theta + \cos\theta) = \frac{\mu_0 I}{\pi\sqrt{x^2 + a^2}}\cos\theta$$

$$= \frac{\mu_0 I}{\pi\sqrt{x^2 + a^2}} \cdot \frac{x}{\sqrt{x^2 + a^2}} = \frac{\mu_0 I x}{x(x^2 + a^2)} \text{ along (–) } Y\text{-axis} \therefore$$



(Fig. 6.13)

The magnetic force on element of length dx

$$dF = I\,\vec{dx} \times \mathbf{B} = I dx\,\hat{i} \times B\,(-\hat{j})$$
$$= -I\,dx\,B\,\hat{k} = -I\,dx\,B\,\hat{k}$$
$$= -I\,dx \cdot \frac{\mu_0 I x}{x^2 + a^2} \cdot \hat{k}$$

$\therefore$ Net force on wire OC

$$= -\frac{\mu_0 I^2}{x^2 + a^2} \cdot x\,dx\,\hat{k}$$

$$F = -\mu_0 I^2 \int_0^L \frac{x\,dx}{x^2+a^2} \cdot \hat{k}$$

Substituting $x^2 + a^2 = t$,

$$2x\,dx = dt \qquad \text{or} \qquad x\,dx = \frac{dt}{2}$$

We get

$$F = -\mu_0 I^2 \int_{a^2}^{a^2+L^2} \frac{dt/2}{t} \cdot \hat{k}$$

$$= -\frac{\mu_0 I^2}{2} [\log_e t]_{a}^{a_2^2+L^2} \cdot \hat{k}$$

$$= -\frac{\mu_0 I^2}{2} [\log_e (a^2+L^2) - \log_e a^2] \cdot \hat{k}$$

$$= -\frac{\mu_0 I^2}{2} \log_e \left(\frac{a^2+L^2}{a^2} \right) \cdot \hat{k}$$

$$= -\frac{\mu_0 I^2}{2} \log_e \left(1 + \frac{L^2}{a^2} \right) \cdot \hat{k}$$

Thus the force an wire OC has magnitude

$$F = \frac{\mu_0 I^2}{2} \log_e \left(1 + \frac{L^2}{a^2} \right) \text{ and is directed along negative Z-axis}$$

When current in wire B is reversed the magnetic field produced will be along the direction of current and so net force on OC = zero.

Ex. 6. *Two long straight parallel wires are 2m apart, perpendicular to plane of paper (Fig. 6·14). The wire 'A' carries a current of 9·6 amp, directed into the plane of paper. The wire 'B' carries a current such that the magnetic field induction at point P, at a distance of 10/11 m from the wire 'B', is zero. Find*

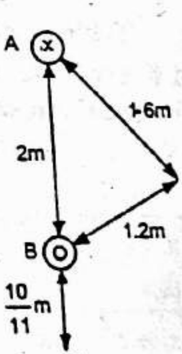
(i) The magnitude and direction of current in wire 'B'.

(ii) The magnitude of magnetic field of induction at point S.

(iii) The force per unit length on wire 'B'.

Solution. (i) The magnetic field of induction at point P is zero if the field produced by current carrying wires at P are equal and opposite. Therefore the direction of current in wire 'B' must be opposite to that of 'A' *i.e.* directed upward, perpendicular to plane of paper.

The field produced by wire 'A' carrying current i_1 (= 9·6 amp),



(Fig. 6.14)

$$B_1 = \frac{\mu_0 i_1}{2\pi r_1}$$

The field produced by wire 'B' carrying current i_2 (say) is

$$B_2 = \frac{\mu_0 i_2}{2\pi r_2}$$

For field at P to be zero, $\vec{B}_P = \vec{B}_1 + \vec{B}_2 = 0$

or

$$\overrightarrow{B}_1 = -\overrightarrow{B}_2$$

In magnitudes $B_1 = B_2$

i.e.

$$\frac{\mu_0 i_1}{2\pi r_1} = \frac{\mu_0 i_2}{2\pi r_2}$$

or

$$i_2 = \frac{r_2}{r_1} i_1$$

Here $r_1 = AP = \left(2 + \frac{2}{11}\right) m = \frac{32}{11}$ m,

$$r_2 = BP = \frac{10}{11} m, \; i_1 = 9{\cdot}6 \; A$$

$\therefore$

$$i_2 = \frac{\left(\frac{10}{11}\right)}{\left(\frac{32}{11}\right)} \times 9{\cdot}6 = \frac{10}{32} \times 9{\cdot}6$$

$$= \mathbf{3 \text{ amp.}}$$

That is in wire B, current is 3 amps directed upward, perpendicular to the plane of paper.

The magnetic field produced by wire 'A' at S

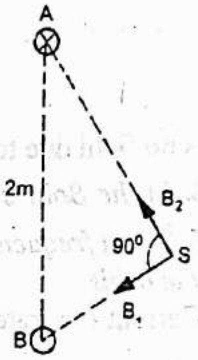
$$B_1 = \frac{\mu_0}{2\pi} \cdot \frac{i_1}{r_1}, \text{ along } \overrightarrow{SB} = \frac{2 \times 10^{-7} \times 9{\cdot}6}{1{\cdot}6} = 12 \times 10^{-7} \text{ weber/m}^2$$

The magnetic field produced by wire B at S

$$B_2 = \frac{\mu_0}{2\pi} \frac{i_2}{r_2}, \text{ along } \overrightarrow{SA} = \frac{2 \times 10^{-7} \times 3}{1{\cdot}2} = 5 \times 10^{-7} \text{ weber/m}^2$$

As $SA = 1{\cdot}6 \; m$, $SB = 1{\cdot}2 \; m$ and $AB = 2m$. Obviously

$$AB^2 = (SA)^2 + (SB)^2.$$



(Fig. 6.15)

That is angle between $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$ is $\angle ASB = 90°$, the resultant field at S

$$B_S = \sqrt{(B_1^2 + B_2^2)} = \sqrt{[(12 \times 10^{-7})^2 + (5 \times 10^{-7})^2]}$$

$$= 13 \times 10^{-7} = \mathbf{1{\cdot}3 \times 10^{-6} \text{ weber/m}^2}$$

(iii) Force per unit length on wire B is

$$f = \frac{\mu_0}{2\pi} \cdot \frac{i_1 i_2}{r_{12}} = \frac{2 \times 10^{-7} \times 9{\cdot}6 \times 3}{2} = \mathbf{2{\cdot}88 \times 10^{-6} \text{ N/m}}$$

Ex. 7. *A long wire having a semi-circular loop of radius r carries a current i as shown in fig. Find the magnetic induction at the centre O due to entire wire.*

Solution. According to Biot-Savart law the magnetic induction at a point due to a current carrying element $i \, \delta \overrightarrow{l}$ is given by

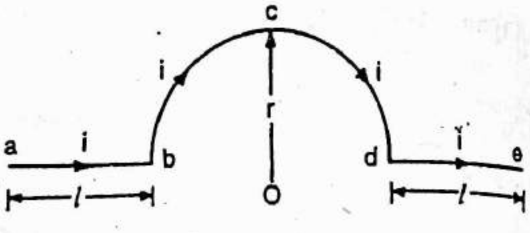
$$\delta \overrightarrow{B} = \frac{\mu_0 i}{4\pi} \frac{i \, \delta \mathbf{l} \times \mathbf{r}}{r^3}.$$

or $B = \frac{\mu_0}{4\pi} \frac{i \cdot \delta l \sin\theta}{r^2}$, directed normal to plane containing $\overrightarrow{\delta l}$ and $\overrightarrow{\mathbf{r}}$; θ being angle between $\delta \mathbf{l}$ and $\overrightarrow{\mathbf{r}}$.

Magnetic field due to linear portions of wire. If we consider any element of length $\overrightarrow{\delta l}$ of either linear portion *ab* or *de*, the angle between $\overrightarrow{\delta l}$ and $\vec{r}$ is O or π, therefore $\sin\theta = 0$.

Hence magnetic induction due to both linear portions of wire is

$$\frac{\mu_0}{4\pi}\frac{i\,\delta l \sin\theta}{r^2} = 0.$$



(Fig. 6.16)

Field due to semi-circular arc :

Now angle between a current element $\overrightarrow{\delta l}$ of semi-circular arc and the radius vector of the element to point C is $\pi/2$. Therefore the magnitude of magnetic induction $\vec{B}$ at O due to this element is

$$\delta B = \frac{\mu_0}{4\pi}\frac{i\,\delta l \sin\pi/2}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,\delta l}{r^2}$$

Hence magnetic induction due to whole semi-circular loop is

$$B = \Sigma\,\delta B = \Sigma\frac{\mu_0}{4\pi}\frac{i\,\delta l}{r^2} = \frac{\mu_0}{4\pi}\frac{i}{r^2}\Sigma\,\delta l$$

$$= \frac{\mu_0}{4\pi}\frac{i}{r^2}(\pi r) = \frac{\mu_0 i}{4r}.$$

As there is no field due to straight portions, the *field due to entire wire* = **(μ_0 i/4r).**

Ex. 8. *In the Bohr's model of hydrogen atom, the electron circulates around the nucleus on a path of radius 0·5 Å at a frequency of* 5×10^{15} *revolution/sec. Calculate the magnitude of the magnetic field at the centre of the orbit.* **(Rohilkhand 1999, 97 Agra 1993)**

Sol. Current I = rate of flow of charge

$$= \frac{dq}{dt} = \frac{e}{1/n} = ne$$

$$= (5 \times 10^{15}\ \text{rev/sec}) \times (1{\cdot}6 \times 10^{-19}\ \text{coul})$$

$$= 8 \times 10^{-4}\ \text{amp.}$$

The magnetic field at the centre of the orbit is given by

$$B = \left[\frac{\mu_0 I a^2}{2(a^2+x^2)^{3/2}}\right]_{x=0} = \frac{\mu_0 I}{2a}\ \text{(since at the centre } x = 0).$$

Here $\mu_0 = 4\pi \times 10^{-7}$ weber/amp-m, $I = 8 \times 10^{-4}$ amp.

$$a = 0{\cdot}5\ \text{Å} = 0{\cdot}5 \times 10^{-10}\ \text{m.}$$

$$\therefore \qquad B = \frac{(4\pi \times 10^{-7}) \times (8 \times 10^{-4})}{2 \times 0{\cdot}05 \times 10^{-10}} = 3{\cdot}2\,\pi\ \text{weber/m}^2$$

Ex. 9. *(i) A charge of* 1 *coul is placed at one end of a non-conducting rod of length* 0·6 *m. The rod is rotated in a vertical plane about a horizontal axis passing through the other end of the rod with angular frequency* $10^4\,\pi$ *radians/sec. Find the magnetic field at a point on the axis of rotation at a distance of* 0·8 *m from the centre of the path.*

(ii) Now half the charge is removed from one end and placed on the other end. The rod is rotated in a vertical plane about horizontal axis passing through the mid point of the rod with the same angular frequency. Calculate the magnetic field at a point on the axis at a distance of 0·4 m from the centre of rod.

Solution. (i) The charge q rotating at a frequency f is equivalent to a current (qf). Therefore the magnetic field at a point on axis, distant x from centre is

$$B = \frac{\mu_0 \, i \, a^2}{2\,(a^2 + x^2)^{3/2}} = \frac{\mu_0 \,(fq)\, a^2}{2\,(a^2 + x^2)^{3/2}}$$

Here $f = \frac{\omega}{2\pi} = \frac{10^4 \pi}{2\pi} = 5 \times 10^3$ rev/sec, $q = 1$ coul, $a = 0{\cdot}6$ m, $x = 0{\cdot}8$ m

$$\therefore \quad B = \frac{4\pi \times 10^{-7} \times (5 \times 10^3 \times 1) \times (0{\cdot}6)^2}{2\,[(0{\cdot}6)^2 + (0{\cdot}8)^2]^{3/2}}$$

$$= \frac{4\pi \times 10^{-7} \times 5 \times 10^3 \times (0{\cdot}6)^2}{2 \times (1{\cdot}0)^2}$$

$$= 11{\cdot}3 \times 10^{-4} \text{ weber/m}^2 = \mathbf{11{\cdot}3 \text{ gauss.}}$$

(ii) In this case current $i = f\left(\frac{q}{2}\right) + f\left(\frac{q}{2}\right) = fq$

Now $a = \frac{0{\cdot}6}{2} = 0{\cdot}3 \; m, \; x = 0{\cdot}4 \; m, \; f = 5 \times 10^3$ rev/sec

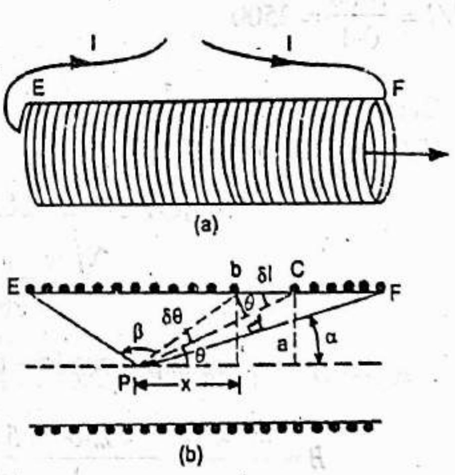
$$\therefore \quad B = \frac{\mu_0 \, i \, a^2}{2\,(a^2 + x^2)^{3/2}} = \frac{\mu_0 \,(fq)\, a^2}{2\,(a^2 + x^2)^{3/2}}$$

$$= \frac{4\pi \times 10^{-7} \times 5 \times 10^3 \times 1 \times (0{\cdot}3)^2}{2\,[(0{\cdot}3)^2 + (0{\cdot}4)^2]^{3/2}}$$

$$= \frac{4\pi \times 10^{-7} \times 5 \times 10^3 \times (0{\cdot}3)^2}{2 \times (0{\cdot}5)^3} = 22{\cdot}6 \times 10^{-4} \text{ weber/m}^2$$

$$= \mathbf{22{\cdot}6 \text{ gauss.}}$$

4. Magnetic field due to solenoid. A solenoid is a long wire wound in a close-packed helix and carrying an electric current. Let EF be a solenoid having n turns per unit length, each turn having radius a (fig. 6·17). Let I amp be the current flowing through the solenoid and P be the point on the axis of the solenoid at which the magnetic field is required. Consider a small portion of this solenoid of length δl at axial distance x from P. The number of turns in the length δl of the solenoid is $(n\,\delta l)$. Therefore the magnitude of the magnetic field at axial point P due to the element bc of length δl of the solenoid is



(Fig. 6.17)

$$\delta B = \frac{\mu\,(n\,\delta l)\, I a^2}{2\,(a^2 + x^2)^{3/2}} \qquad \ldots(19)$$

Now $\delta l = bc = bd \operatorname{cosec} \theta = r\,\delta\theta \operatorname{cosec} \theta$;
where bd is perpendicular drawn from b on PC.

Also $\quad \sqrt{[(a^2 + x^2)]} = r$ and $\sin \theta = a/r$.

$$\therefore \quad \delta B = \frac{\mu\,(nr\,\delta\theta\,\text{cosec}\,\theta)\,Ia^2}{2r^2} = \frac{\mu\,nI}{2}\frac{a^2}{r^2}\,\text{cosec}\,\theta\,\delta\theta$$

$$= \frac{\mu\,nI}{2}\sin^2\theta\,\text{cosec}\,\theta\,\delta\theta = \frac{\mu\,nI}{2}\sin\theta\,\delta\theta\,. \qquad ...(20)$$

If α and β are the angles subtended by the ends F and E of the solenoid at point P with axis, then total magnetic field at point P is given by

$$B = \frac{\mu\,nI}{2}\int_{\alpha}^{\beta}\sin\theta\,d\theta = \frac{\mu\,nI}{2}(\cos\alpha - \cos\beta). \qquad ...(21)$$

Different cases :

(i) Far an infinitely long and thin solenoid when point P is well within the solenoid. $\alpha \to 0$ and $\beta \to \pi$, so that the magnetic field inside a thin long solenoid as

$$B = \frac{\mu\,nI}{2}(\cos 0 - \cos\pi) = \mu\,uI. \qquad ...(22)$$

Thus the magnetic field inside a thin long solenoid is constant throughout the central region.

(ii) If point P is at an end (say F) of the long solenoid, then

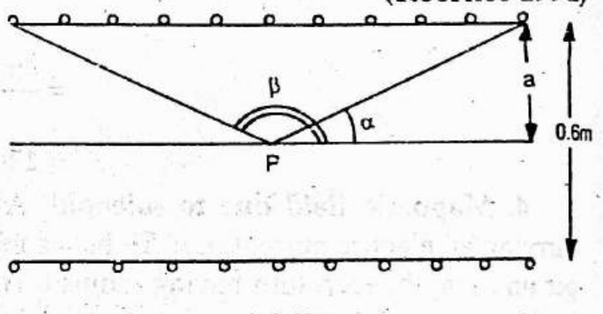
$$\alpha = \pi/2 \text{ and } \beta = \pi.$$

so the magnetic field at an end of long solenoid, from equation (21) is given as

$$B = \frac{\mu\,nI}{2}\left(\cos\frac{\pi}{2} - \cos\pi\right) = \frac{\mu\,nI}{2} \qquad ...(23)$$

Thus *for a long solenoid the magnetic field at the ends is half of that in the middle of the solenoid.*

Ex. 10. *A solenoid of length* 0·4 *m and diameter* 0·6 *m consists of a single layer of* 1000 *turns of fine wire carrying a current of* $5{\cdot}0 \times 10^{-3}$ *ampere. Calculate the magnetic field intensity on the axis at the middle and at the ends of the solenoid.* **(Roorkee 1992)**



(Fig. 6.18)

Solution. The magnetic field at the axis of the solenoid is given by

$$B = \frac{\mu_0\,nI}{2}(\cos\alpha - \cos\beta) \qquad ...(1)$$

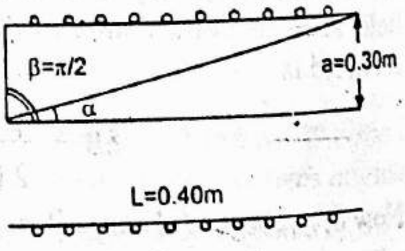
There n is number of turns per metre length $= N/l = \dfrac{1000}{0{\cdot}4} = 2500$

Current $I = 5{\cdot}0 \times 10^{-3}$ A

If P is middle point, then

$$\cos\alpha = \frac{l/2}{\sqrt{a^2 + \left(\frac{l}{2}\right)^2}} = \frac{0{\cdot}2}{\sqrt{(0{\cdot}3)^2 + (0{\cdot}2)^2}} = \frac{0{\cdot}2}{\sqrt{0{\cdot}13}}$$

$$\cos\beta = \cos(\pi - \alpha) = -\cos\alpha = \frac{0{\cdot}2}{\sqrt{0{\cdot}13}}$$



(Fig. 6.19)

$$\therefore \quad B = \frac{4\pi \times 10^{-7} \times 2500 \times 5{\cdot}0 \times 10^{-3}}{2}\left[\frac{0{\cdot}2}{\sqrt{0{\cdot}13}} + \frac{0{\cdot}2}{\sqrt{0{\cdot}13}}\right]$$

$$= \frac{4 \times 3{\cdot}14 \times 2500 \times 5{\cdot}0}{2} \times 10^{-3} \times 2 \times \frac{0{\cdot}2}{\sqrt{0{\cdot}13}}$$

$$= \frac{3{\cdot}14 \times 10^{-6}}{0{\cdot}3605} = \mathbf{8{\cdot}71 \times 10^{-3}\ weber/m^2}$$

Let Q be a point at the end of solenoid.

For Q,
$$\cos\alpha = \frac{l}{\sqrt{a^2 + l^2}} = \frac{0{\cdot}4}{(0{\cdot}3)^2 + (0{\cdot}4)^2} = \frac{4}{5}$$

$$\cos\beta = \cos\frac{\pi}{2} = 0$$

$$\therefore \qquad B = \frac{4\pi \times 10^{-7} \times 2500 \times 5{\cdot}0 \times 10^{-3}}{2}\left[\frac{4}{5} - 0\right]$$

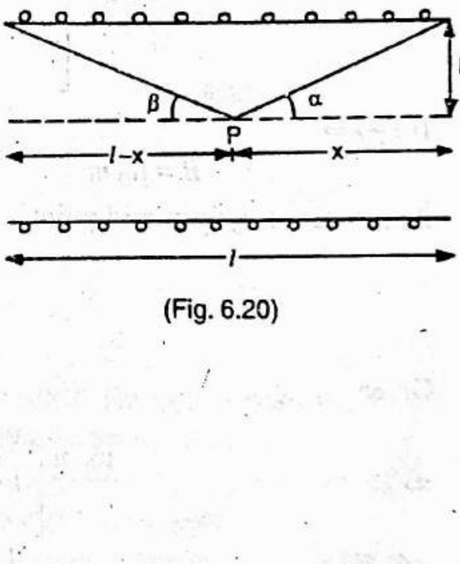
$$= \frac{3{\cdot}14 \times 10^{-7} \times 10^4 \times 5 \times 10^{-3} \times 4}{2 \times 5} = \mathbf{6{\cdot}28 \times 10^{-6}\ Tesla}$$

Ex. 11. *A very long straight solenoid has a cross-sectional radius R and number of turn per unit length equal to n. A direct current I flows through the solenoid. If x is the distance from the end of solenoid measured along its axis. Find*

(a) The magnetic induction B at the axis as a function of x. Draw an approximate plot of magnetic induction B versus the ratio $\frac{x}{R}$.

(b) The distance x_0 *to the point on the axis at which the value of B differs by* $\eta = 1\%$ *from that in the middle section of the solenoid.*

Solution. The magnetic field at any point in the middle of solenoid



(Fig. 6.20)

$$B = \frac{\mu_0\, ni}{2}(\cos\alpha + \cos\beta)$$

From fig.

$$\cos\alpha = \frac{x}{\sqrt{x^2 + R^2}}\ ,\ \cos\beta = \frac{l - x}{\sqrt{(l-x)^2 + R^2}}$$

$$\therefore \qquad \beta = \frac{\mu_0\, ni}{2}\left[\frac{x}{\sqrt{x^2+R^2}} + \frac{l-x}{\sqrt{(l-x)^2+R^2}}\right]$$

$$= \frac{\mu_0\, ni}{2}\left[\frac{x}{\sqrt{x^2+R^2}} + \frac{1}{\sqrt{1 + \left(\frac{R}{l-x}\right)^2}}\right]$$

If $l \to \infty$ (very long solenoid) *i.e.* $\left(\frac{R}{l-x}\right) \to 0$

$$B\,(x) = \frac{\mu_0\, ni}{2}\left[1 + \frac{x}{\sqrt{x^2+R^2}}\right]$$

This is an expression for magnetic field induction at a distance x from one end within the solenoid.

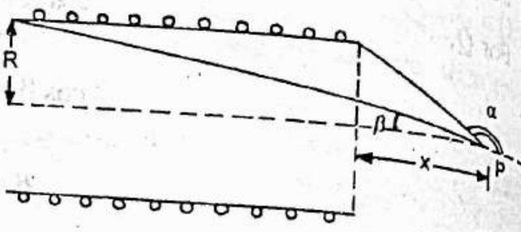
At a point outside the solenoid

$$B = \frac{\mu_0\, ni}{2}\,[\cos\alpha + \cos\beta]$$

$$\Rightarrow \qquad \cos\beta = +\frac{x+l}{\sqrt{(x+l)^2+R^2}}, \cos(\pi-\alpha) = \frac{x}{\sqrt{x^2+R^2}} \Rightarrow \cos\alpha = \frac{x}{\sqrt{x^2+R^2}}$$

$$\therefore \qquad B(x) = \frac{\mu_0 ni}{2}\left[-\frac{x}{\sqrt{x^2+R^2}} + \frac{x+l}{\sqrt{(x+l)^2+R^2}}\right]$$

$$= \frac{\mu_0 n i}{2}\left[\frac{1}{\sqrt{1+\frac{R^2}{(x+l)^2}}} - \frac{x}{\sqrt{x^2+R^2}}\right]$$



(Fig. 6.21)

If $l \to \infty$

$$B(x) = \frac{\mu_0 ni}{2}\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right]$$

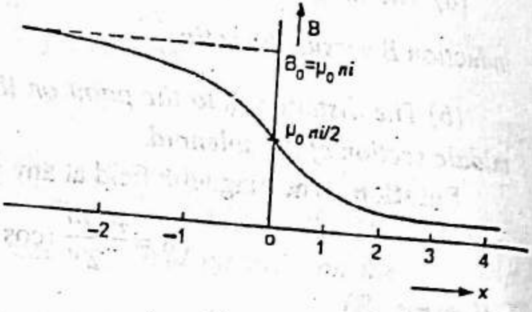
This is general expression

If $x < 0$, point is inside and if $x > 0$ point is outside

(b) For B in the middle of solenoid $x = l/2$

$$\therefore \qquad B_1 = \frac{\mu_0 ni}{2}\left[1 + \frac{l/2}{\sqrt{(l/2)^2+x^2}}\right]$$

$$= \frac{\mu_0 ni}{2}\left[1 + \frac{1}{\sqrt{\left\{1+\frac{R}{(l/2)^2}\right\}}}\right]$$



(Fig. 6.22)

If $l \to \infty$

$$B = \mu_0 ni$$

At a distance x_0 from mid point

$$B_2 = \frac{\mu_0 ni}{2}\left[1 - \frac{x_0}{\sqrt{x_0^2+R^2}}\right]$$

Given
$$\frac{B_1 - B_2}{B_1} = \eta \Rightarrow B_2 = B_1(1-\eta)$$

$$\Rightarrow \qquad \frac{\mu_0 ni}{2}\left[1 - \frac{x_0}{\sqrt{x_0^2+R^2}}\right] = \mu_0 ni(1-\eta) \Rightarrow x_0 = \frac{(1-2\eta)R}{2\sqrt{\eta(1-\eta)}}$$

Putting
$$\eta = \frac{1}{100} = 0{\cdot}01, \text{ we get } x_0 = \frac{(1-2\times 0{\cdot}01)R}{2\sqrt{0{\cdot}01\times(1-0{\cdot}01)}} \approx 5R$$

6.7. Magnetic Induction B and Magnetic Intensity H.

We have seen that the *magnetic induction* (or *magnetic flux density*) **B** is the property of the magnetic field that determines the force that is exerted upon a current or a moving charge in the field. Let us define another vector quantity in the magnetic field symbolised by **H** and known as *magnetic intensity* or the *magnetic field strength* the relation

$$\mathbf{B} = \mu\mathbf{H} \qquad ...(1)$$

(μ being permeability of the medium).

Equation (1) corresponds to electrical analogue

$$\mathbf{D} = \varepsilon\mathbf{E}. \qquad ...(2)$$

The correspondence between electrostatics and magnetostatics may result in misleading of concepts. Because in a dielectric medium the electrical force exerted upon a free charge is proportional to **E** and not to, **D**, but in a magnetic material the magnetic force that is exerted upon a moving charge is proportional to **B** and not to **H**. Thus in the sense that **E** is the electrical field for free charges **B** is the magnetic field for moving charges. Unfortunately **H** bears title that describes **B** in function.

6.8. Ampere's Law in Circuital Form

In studying electrostatics, we noted that

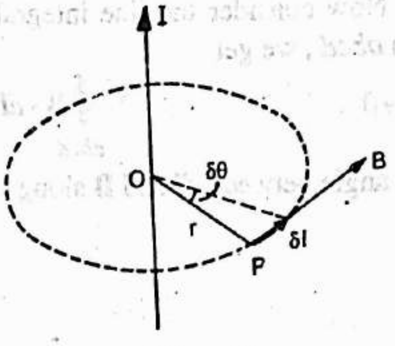
$$\oint \mathbf{E} \cdot dl = 0$$

Since electrostatic potential is single valued function of position. We shall now think for an analogous magnetic integral, namely, that of $\mathbf{B} \cdot dl$ around a closed path.

Let us assume for simplicity that the magnetic field induction **B** arises due to a long wire carrying a current I. Now consider aircular path of radius r centred on this carrying wire.

The magnitude of magnetic induction at any point P on the circular path is given by

$$B = \frac{\mu I}{2\pi r}. \quad ...(1)$$



(Fig. 6.23)

For all points on the circular path, the magnetic induction **B** has the same magnitude given by equation (1) and it is parallel to the tangent to the circular path. So the line integral of the magnetic induction **B** around the circular path centred on the current carrying wire is given by

$$\oint \mathbf{B} \cdot dl = \oint B \cdot dl = \oint \frac{\mu I}{2\pi} \cdot \frac{r d\theta}{r}$$

$$= \frac{\mu I}{2\pi} \oint d\theta = \frac{\mu I}{2\pi} \cdot 2\pi = \mu I.$$

Thus we have

$$\oint \mathbf{B} \cdot dl = \mu I. \quad ...(2)$$

The sign of integral depends upon the direction in which the current is encircled. The sign is positive, if the path followed for line integral is parallel to B and negative, if the path followed is anti-parallel.

Although equation (2) is derived by the use of certain simplified assumptions, namely the current is infinitely long and the path enclosing the current carrying wire is circular centred at the wire.

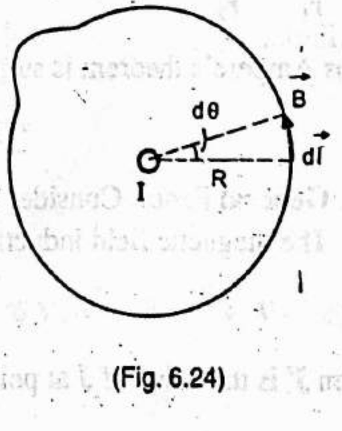
If the path enclosing the current is not circular but in irregular of any shape, then we may divide the path into large number of small elements.

Consider a small element dl of path. For this small element **B** and dl are directed along the same line *i.e.*

$$\mathbf{B} \cdot dl = Bdl = \frac{\mu_0 I}{2\pi R} d\theta$$

$$= \frac{\mu_0 I}{2\pi} d\theta. \quad ...(3)$$

This is independent of distnace R of element from the current. Along a radial path, **B** and dl are mutually perpendicular. Therefore $\mathbf{B} \cdot dl = 0$, thus if we resolve each path element along tangential and radial components, then the



(Fig. 6.24)

line integral of **B** along radial components vanishes, while that along tangential components is added.

Line integral of **B** along the whole path is given by

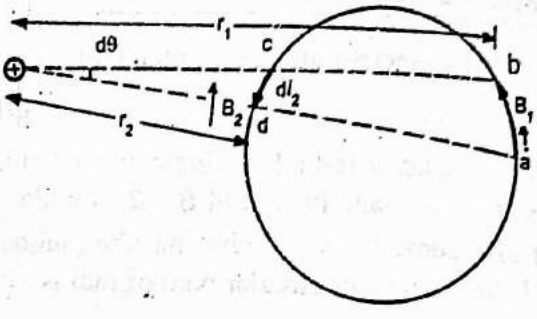
$$\oint \mathbf{B} \cdot dl = \frac{\mu_0 I}{2\pi} \oint d\theta = \frac{\mu_0 I}{2\pi} \times 2\pi = \mu_0 I. \qquad \text{...(4)}$$

This shows, that Ampere's law holds for closed path of any shape.

When the path does not enclose the current. Consider a current I and a neighbouring closed path which does not enclose any current. The path is again divided into large number of small elements.

Let us draw two lines Oa and Ob (from current I at O) making angle $d\theta$ between them and cutting the arcs ab and cd of the path.

Now consider the line integral, of **B** along the path $abcd$, we get



(Fig. 6.25)

$$\oint_{abcd} \mathbf{B} \cdot dl = \int_{ab} \mathbf{B} \cdot dl + \int_{bc} \mathbf{B} \cdot dl + \int_{cd} \mathbf{B} \cdot dl + \int_{da} \mathbf{B} \cdot dl.$$

The angle between dl and **B** along path bc and da is 90°,

i.e.
$$\int_{bc} \mathbf{B} \cdot dl = \int_{da} \mathbf{B} \cdot dl = 0.$$

$$\therefore \qquad \int_{abcd} \mathbf{B} \cdot dl = \int_{ab} \mathbf{B} \cdot dl + \int_{cd} \mathbf{B} \cdot dl.$$

Resolving path $\mathbf{ab} = dl_1$ and $\mathbf{cd} = dl_2$ along and perpendicular to radial and tangential directions, we note that $\mathbf{B} \cdot dl$ along radial directions is zero and the contribution is only due to tangential components.

$$\therefore \qquad \int_{abcd} \mathbf{B} \cdot dl = \int_{ab} B_1 dl_1 \cos\theta + \int_{cd} B_2\, dl_2 \cos\pi.$$

But $B_1 = \dfrac{\mu_0 I}{2\pi r_1}$ and $B_2 = \dfrac{\mu_0 I}{2\pi r_2}$.

$$\therefore \qquad \int_{abcd} \mathbf{B} \cdot dl = \int_{ab} \frac{\mu_0 I}{2\pi r_1} dl_1 - \int_{cd} \frac{\mu_0 I}{2\pi r_2} dl_2$$

$$= \frac{\mu_0 I}{2\pi} \left\{ \frac{ab}{r_1} - \frac{cd}{r_2} \right\}.$$

But $\dfrac{ab}{r_1} = \dfrac{cd}{r_2} = d\theta$

$$\therefore \int_{abcd} \mathbf{B} \cdot dl = 0.$$

Thus Ampere's theorem is summarised as

$$\oint \mathbf{B} \cdot dl = \begin{cases} \mu_0 I & \text{if path encloses the current } I \\ 0 & \text{if path does not enclose any current} \end{cases}$$

General Proof. Consider a current element $dI = \mathbf{J} \cdot d\mathbf{S}$ and a closed path C. (fig. 6·26).

The magnetic field induction at point P ($\mathbf{r}$) due to current at O ($\mathbf{r}'$) by Biot Savart law is given by

$$d\mathbf{B} = \frac{\mu}{4\pi} \frac{\mathbf{J}' \times \mathbf{R}}{R^3} \qquad \text{...(i)}$$

when $\mathbf{J}'$ is the value of $\mathbf{J}$ at point (x', y', z')

i.e.

$$\mathbf{J}' = \mathbf{J}(x, y, z)$$

and

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'$$

so that

$$R = |\mathbf{R}| = |\mathbf{r}' - \mathbf{r}| = \sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}$$

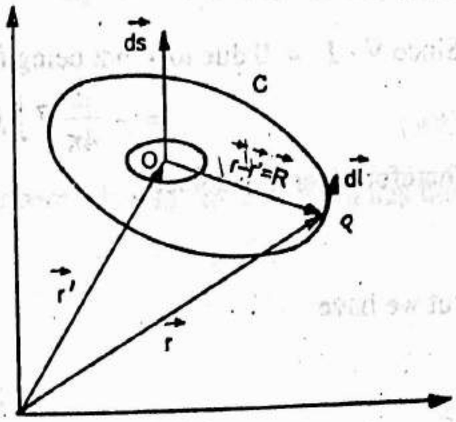
so that net magnetic induction

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu}{4\pi}\int \frac{\mathbf{J}' \times \mathbf{R}}{R^3}\, d\tau' \qquad \text{...(ii)}$$

when integration extends over whole volume element under consideration

i.e.

$$\mathbf{B} = \frac{\mu}{4\pi}\int \mathbf{J} \times \nabla\left\{-\frac{1}{R}\right\} d\tau' \;\text{...(iii)} \left[\because\ \nabla\left\{\frac{1}{R}\right\} = -\frac{\mathbf{R}}{R^3}\right]$$



(Fig. 6.26)

Using vector indentity

$$\text{curl}\,(\phi\,\mathbf{V}) = \phi\,\text{curl}\,\mathbf{V} - \mathbf{V} \times \text{grad}\,\phi,$$

we get

$$\text{curl}\left\{\frac{\mathbf{J}'}{R}\right\} = \frac{1}{R}\,\text{curl}\,\mathbf{J}' - \mathbf{J}' \times \text{grad}\left\{\frac{1}{R}\right\} \qquad \text{...(iv)}$$

As $\mathbf{J}'$ is not the function of coordinates (x, y, z),

where curl $\mathbf{J}' = \nabla \times \mathbf{J}' = 0$

so that equation (iv) gives

$$\text{curl}\left\{\frac{\mathbf{J}'}{R}\right\} = -\mathbf{J}' \times \text{grad}\left\{\frac{1}{R}\right\}$$

or

$$\mathbf{J}' \times \nabla\left\{-\frac{1}{R}\right\} = \nabla \times \left\{\frac{\mathbf{J}'}{R}\right\} \qquad \text{...(v)}$$

using (ii), equation (iii) may be written as

$$\mathbf{B} = \frac{\mu}{4\pi}\int \nabla \times \left\{\frac{\mathbf{J}'}{R}\right\} d\tau' \qquad \text{...(vi)}$$

$$\therefore \qquad \nabla \times \mathbf{B} = \frac{\mu}{4\pi}\nabla \times \int \nabla \times \left\{\frac{\mathbf{J}'}{R}\right\} d\tau' \qquad \text{...(vii)}$$

Now using the vector identity

$$\text{curl curl}\,\mathbf{V} = \text{grad div}\,\mathbf{V} - \mathbf{V}^2\,\mathbf{V}$$

i.e.,

$$\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla . \mathbf{V}) - \nabla^2 V$$

equation (viii) takes the form

$$\nabla \times \mathbf{B} = \frac{\mu}{4\pi}\int\left[\nabla\left\{\nabla \cdot \left(\frac{\mathbf{J}'}{R}\right)\right\} - \nabla^2\left(\frac{\mathbf{J}'}{R}\right)\right] d\tau'$$

$$= \frac{\mu}{4\pi}\int \nabla\left\{\nabla \cdot \left(\frac{\mathbf{J}'}{R}\right)\right\} d\tau' - \frac{\mu}{4\pi}\int \nabla^2\left(\frac{\mathbf{J}'}{R}\right) d\tau' \qquad \text{...(viii)}$$

Now considering the first term

$$\frac{\mu}{4\pi}\int \nabla\left\{\nabla \cdot \left(\frac{\mathbf{J}'}{R}\right)\right\} d\tau' = \frac{\mu}{4\pi}\nabla\int \nabla \cdot \left(\frac{\mathbf{J}'}{R}\right) d\tau'$$

$$= \frac{\mu}{4\pi}\nabla\int\left[\frac{1}{R}\nabla \cdot \mathbf{J}' + \mathbf{J}' \cdot \nabla\left(\frac{1}{R}\right)\right] d\tau'$$

(Since $\nabla \cdot (\phi\,\mathbf{V}) = \phi\,\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla\phi$)

$$= \frac{\mu}{4\pi}\nabla\int \mathbf{J}'\cdot\nabla\left(\frac{1}{R}\right)$$

[Since $\nabla\cdot\mathbf{J}' = 0$ due to $\mathbf{J}'$ not being function of coordinates x, y, z]

$$= -\frac{\mu}{4\pi}\nabla\int \mathbf{J}'\cdot\nabla'\left[\frac{1}{R}\right]d\tau'\left[\text{and } \nabla\left\{\frac{1}{R}\right\} = -\nabla'\left\{\frac{1}{R}\right\}\right]$$

Therefore I term becomes

$$= -\frac{\mu}{4\pi}\nabla\int \nabla'\cdot\left\{\frac{\mathbf{J}'}{R}\right\}$$

But we have

$$\nabla\cdot(\phi\,\mathbf{V}) = \phi\,\nabla\cdot\mathbf{V} + \mathbf{V}\cdot\nabla\phi$$

i.e.

$$\nabla'\cdot\left\{\frac{\mathbf{J}'}{R}\right\} = \frac{1}{R}\nabla'\cdot\mathbf{J}' + \mathbf{J}'\cdot\nabla'\left\{\frac{1}{R}\right\}$$

In steady state

$$\nabla'\cdot\mathbf{J}' = 0$$

$$\therefore \quad \nabla'\cdot\left\{\frac{\mathbf{J}'}{R}\right\} = \mathbf{J}'\cdot\nabla'\left\{\frac{1}{R}\right\}$$

Now in steady state

$$\int \nabla'\cdot\left\{\frac{\mathbf{J}'}{R}\right\}d\tau' = 0$$

(Since for steady state $\mathbf{J}' = 0$ or tangential at the boundary)

This implies that the I term

$$\frac{\mu}{4\pi}\nabla\int \nabla\cdot\left[\frac{\mathbf{J}'}{R}\right]d\tau' = 0 \qquad \text{...(ix)}$$

Further the second term in (viii) *i.e.*

$$= \frac{\mu}{4\pi}\int \nabla^2\left[\frac{\mathbf{J}'}{R}\right]d\tau' = -\frac{\mu}{4\pi}\int \mathbf{J}'\cdot\nabla^2\left[\frac{1}{R}\right]d\tau'$$

$$= -\frac{\mu}{4\pi}\int \mathbf{J}'\{-4\pi\,\delta(R-R')\}\,d\tau'$$

as $\nabla^2\left[\frac{1}{R}\right] = -4\pi\,\delta(R-R')$

$$= \mu\,\mathbf{J}' \qquad [\text{as } \delta(R-R')\,d\tau' = 1] \qquad \text{...(x)}$$

Using (ix) and (x), equation (viii) gives

$$\nabla\times\mathbf{B} = \mu\,\mathbf{J} \quad \text{(dropping the suffix)} \qquad \text{...(xi)}$$

Integrating (xi) over the surface S bounded by the loop C, we get

$$\int_S(\nabla\times\mathbf{B})\cdot d\mathbf{S} = \mu\int_S \mathbf{J}\cdot d\mathbf{S}$$

Using Stoke's theorem on L.H.S. to change surface integral into the integral, we get

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu\int_S \mathbf{J}\cdot d\mathbf{S}$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu\,I \qquad \text{...(xii)}$$

This is the required Ampere's law in circuital form.

In words equation (2) states that the *line integral of the magnetic induction around any closed path is numerically equal to permeability multiplied by the current enclosed by the path.* This result is known as *Ampere's law in circuital form.*

The Ampere's law in this form is powerful method of computing magnetic induction arising from certain symmetric configurations.

Equation (2) can also be written in the form

$$\oint \mathbf{H} \cdot dl = I. \qquad \text{...(3)}$$

Since **H** is defined as $\frac{\mathbf{B}}{\mu}$ [equation (1) of section 6·7]. This form of Ampere's law in circuital form has the advantage that it is independent of the nature of the medium.

6.9. Applications of Ampere's Law

(i) Magnetic Filed due to a long straight current carrying Wire :

Conside a long staright wire carrying a current *I*. From the symmetry of wire it is clear that the magnetic lines of force are concentric circles centred on wire. let *P* be a point at a distance *r* from the wire the magnetic field at point *P* is required. Consider a circular path of radius *r*, passing through *P*.

By symmetery the value of magnetic field is same at each point on the circular path.Moreover **B** and *dl* are always directed along the same direction therefore line integral of **B** along the boundary *C* of circular path

$$\oint_C \mathbf{B} \cdot dl = \int B\, dl \cos 0 = B \int dl = B\, 2\pi r$$

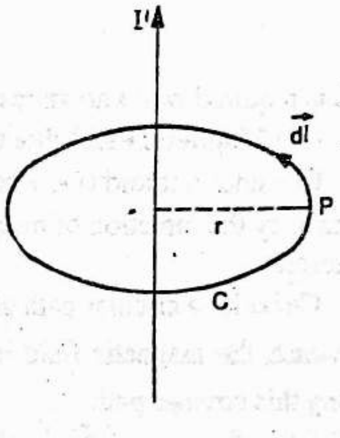
From Ampere's theorem

$$\oint \mathbf{B} \cdot dl = \mu_0 \times \text{current enclosed by path}$$

$$B\, 2\pi r = \mu_0 I$$

$$\therefore \qquad B = \frac{\mu_0 I}{2\pi r}$$

Thus is required well known result.



(Fig. 6.27)

(ii) Magnetic Field Inside a Long Solenoid, Consider a symmetrical long solenoid, having number of turns per unit length equal to *n*. Let *I* be the current flowing in the solenoid.

Field outside the solenoid. Consider a closed path *abcd*. Applying Ampere's law to this path

$$\oint \mathbf{B} \cdot dl = \mu \times 0$$

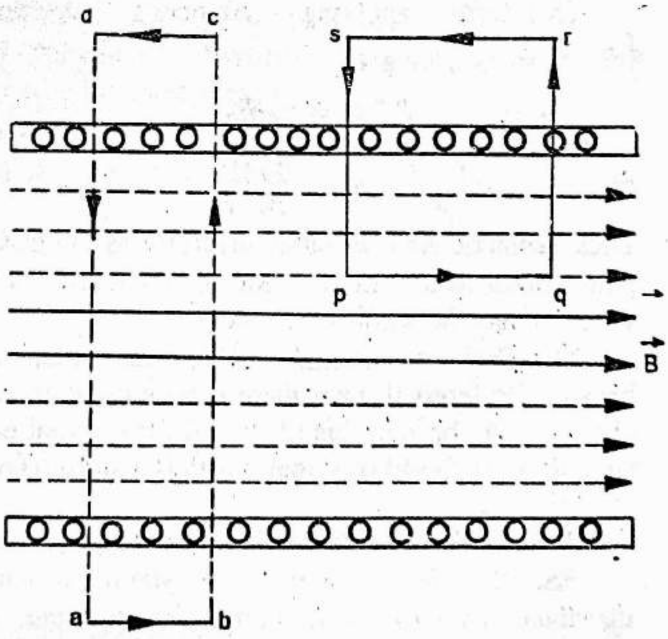
[Since net current enclosed by path is zero, the path encloses equal and opposite current in each turn if whole turn is enclosed in the path]

As $\qquad dl \neq 0 \qquad \therefore \mathbf{B} = 0$

This means that the *magnetic field outside the solenoid is zero.*

Field Inside the Solenoid. Consider a closed path *pqrs* the line integral of magnetic field **B** along path *pqrs* is

$$\oint_{pqrs} \mathbf{B} \cdot dl = \int_{pq} \mathbf{B} \cdot dl \int_{qr} \mathbf{B} \cdot dl + \int_{rs} \mathbf{B} \cdot dl + \int_{sp} \mathbf{B} \cdot dl \qquad \text{...(1)}$$



(Fig. 6.28)

For path pq, **B** and dl are along the same direction

$$\int_{pq} \mathbf{B}\cdot dl = \int \mathbf{B}\cdot dl = Bl \qquad [\because pq = l \text{ (say)}]$$

For paths qr and sp, **B** and dl are mutually perpendicular

$$\int_{qr} \mathbf{B}\cdot dl = \int_{sp} \mathbf{B}\,.\,\mathbf{d}l = \int B\,dl\cos 90^\circ = 0$$

For path rs, $B = 0$ (since field is zero outside a solenoid)

$\therefore$
$$\int_{rs} \mathbf{B}\cdot dl = 0$$

In view of these, equation (1) gives

$$\oint_{pqrs} \mathbf{B}\cdot dl = \int_{pq} \mathbf{B}\cdot dl = Bl \qquad \text{...(2)}$$

From Ampere's law $\oint \mathbf{B}\cdot dl = \mu \times$ current enclosed by path

$\therefore$
$$Bl = \mu\,(nl\,I)$$

$\therefore$
$$B = \mu nI$$

this is required well known result.

(iii) Magnetic Field due to a Toroid. [Endless solenoid]

Consider a toroid (*i.e.* a solenoid in the form of a circle with the ends Joined) carrying a current i_0. By symmetry the direction of magnetic field is tangential at any point, on the circular path centroid at the centre of toroid.

Consider a circular path of radius r, passing through p at which the magnetic field is required. We have $\oint B\cdot dl$ along this circular path

$$\oint \mathbf{B}\cdot dl = B\,2\pi r.$$

If N is the number of turns in the toroid, then the current enclosed $I = Ni_0$.

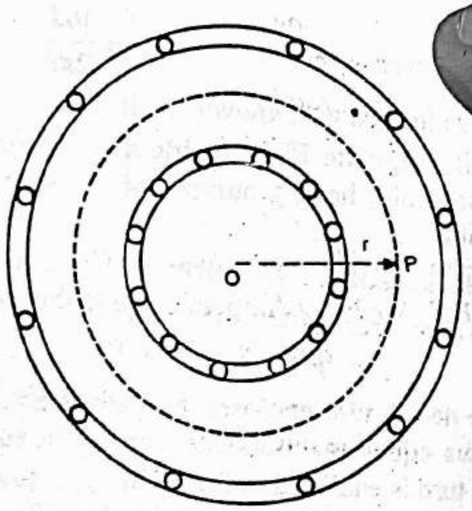
Therefore applying Ampere's theorem is $\oint \mathbf{B}\cdot dl = \mu_0 I$, we get

$$B\,2\pi r = \mu_0 N i_0$$

or
$$B = \frac{\mu_0}{2\pi}\frac{Ni_0}{r}$$

Thus magnetic field **B** varies inversely as the distance of point from centre in contrast to solenoid when B is constant over the whole cross-section.

The fields at internal and external point are zero because for internal point there is no current enclosed by circular path through this point and for external point the net current enclosed by circular path is also zero because each turn passes twice carrying equal and opposite currents.

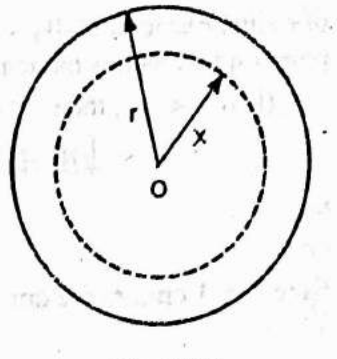


(Fig. 6.29)

Thus the field of a toroid is zero at all points except within the core.

Ex. 12. *A long straight solid cylindrical metal wire of radius R carries a total current I uniformly distributed over its circular cross-section. Find the magnetic field intensity inside as well as outside the wire.*

(Kanpur 1998; Rohilkhand 1997; Agra 1992)



(Fig. 6.30)

Solution. Figure 6·30 represents the section of a long straight solid metal wire perpendicular to its length. Now consider a circular path of radius x around the axis of the wire. By symmetry the magnetic field produced due to the current flowing in the wire at any point over this path is tangential to it and equal in magnitude everywhere. Therefore the line integral of **B** along this path is

$$\oint \mathbf{B} \cdot dl = B \, . \, 2\pi x \, .$$

(i) Inside the wire $x < r$, so that

$$\oint \mathbf{B} \cdot dl = \mu \times \text{ the current enclosed by the path in amp.}$$

i.e. $$B \, . \, 2\pi x = \mu \left(\frac{I}{\pi r} \right) \cdot \pi x^2$$

i.e. $$b = \frac{\mu \, Ix}{2\pi r^2}$$

(ii) Outside the wire $x > r$, so that

$$\oint \mathbf{B} \cdot dl = \mu \times \text{ current enclosed by the path}$$

i.e. $$B \, . \, 2\pi x = \mu I \, .$$

$\therefore$ $$B = \frac{\mu I}{2\pi x}$$

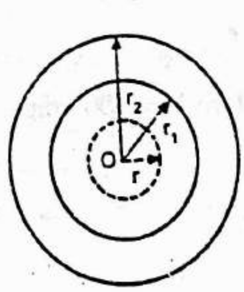
Note. On the surface of the wire $x = r$, so that

$$B = \frac{\mu I}{2\pi r}$$

Ex. 13. *(a) Show that the magnetic field of hollow circular cylindrical conductor carrying current vanishes everywhere inside the hollow region.*

(b) A long straight copper tube having an inside radius of 1 *cm. and outside radius of* 2 *cm. carries a current of* 200 *amperes. Compute the magnetic field at distance* 0·5 *cm,* 1·5 *cm and* 4 *cm from the axis.*

(Agra 1972)



(Fig. 6.31)

Solution. (a) Figure 6·31, represents the section of a hollow circular cylindrical conductor perpendicular to its length. The axis of the cylindrical conductor passes through the point O. Let the internal nad external radii of the cylindrical conductor be r_1 and r_2 respectivley. Consider a circular path of radius r around O inside the hollow. By symmetry the magnetic field produced due to the current flowing in the conductor at any point of this path will be tangential and equal in magnitude everywhere. Therefore the line integral of **B** along this path is

$$\oint \mathbf{B} \cdot dl = B \, . \, 2\pi r \, .$$

But $\oint \mathbf{B} \cdot dl = \mu \times$ (current enclosed by the circular path of radius r)

i.e. $$B = \mu \times . \, 0$$

i.e. $$B = 0.$$

Hence, the magnetic field of a hollow circular cylindrical conductor carrying current vanishes everywhere inside the hollow region.

(b) Let r_1 and r_2 be the inside and outside radii of long straight copper tube carrying a current of I amp. uniformly distrubuted over its circular cross section. Figure 6·31 represents the section of this cylindrical tube perpendicular to its length. The axis of this tube passes through point O. Now consider a circular path of radius r around O. By symmetry the magnetic field produced due to the current flowing in the tube at any point on this path is tangential and equal in magnitude everywhere.

(i) If $r < r_1$, then

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu \times \text{(current enclosed by the circular path of radius } r < r_1)$$

$$B \,.\, 2\pi r = \mu \times 0$$

i.e.
$$B = 0.$$

i.e.

Here $r_1 = 1$ cm., $r_2 = 2$ cm. and $r = 0{\cdot}5$ cm. That is $r < r_1$,

$\therefore$
$$B = 0.$$

(ii) If $r_1 < r < r_2$, then

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu \times \text{(current enclosed by the circular path of radius } r_1 < r < r_2)$$

i.e.
$$B \,.\, 2\pi r = \mu \times \frac{I}{\pi (r_2^2 - r_1^2)} \times \pi (r^2 - r_1^2)$$

i.e.
$$B = \frac{\mu I (r^2 - r_1^2)}{2\pi (r_2^2 - r_1^2)}.$$

Here $r_1 = 1$ cm. $= 0{\cdot}01$ m., $r_2 = 2$ cm. $= 0{\cdot}02$ m.,

$r = 1{\cdot}5$ cm. $= 0{\cdot}015$ m.,

$I = 200$ amp, $\mu = \mu_0 = 4\pi \times 10^{-7}$ Weber/amp-m.

$$B = \frac{(4\pi \times 10^{-7} \text{ weber/amp-m}) \times 200 \text{ amp} \times [(0{\cdot}015 \text{ m})^2 - (0{\cdot}01 \text{ m})^2]}{2\pi \times 0{\cdot}015 \{(0{\cdot}02 \text{ m})^2 - (0{\cdot}01 \text{ m})^2\}}$$

$$= 1{\cdot}1 \times 10^{-3} \text{ weber/m}^2.$$

(iii) If $r > r_2$, then

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu \times \text{(current closed by the circular path of radius } r > r_2)$$

i.e.
$$B \,.\, 2\pi r = \mu I$$

i.e.
$$B = \frac{\mu I}{2\pi r}$$

Here $I = 200$ amp., $\mu = \mu_0 = 4\pi \times 10^{-7}$ weber/amp-m,

$r = 4$ cm. $= 0{\cdot}04$ m.

$\therefore$
$$B = \frac{(4\pi \times 10^{-7} \text{ weber/amp-m}) \times (200 \text{ amp.})}{2\pi \times (0{\cdot}04 \text{ m})}$$

$$= 10^{-3} \text{ weber/m}^2.$$

6.10 The Curl of Magnetic Induction B.

From Ampere's law in circuital form, we have

$$\oint_C \mathbf{B} \cdot dl = \mu \times \text{(sum of currents enclosed by path).} \qquad \text{...(1)}$$

Now consider any surface S bounded by the path of integration C. Then the sum of current enclosed by the path is the total current passing through S. In terms of *current density* $\mathbf{J}$ (the current per unit area placed normal to the direction of current flow), the total current I through S is given by

$$I = \iint_S \mathbf{J} \cdot d\mathbf{a}\text{; where } d\mathbf{a} \text{ is small element of surface.}$$

Hence Ampere's law in circuital form (1) takes the form

$$\oint_C \mathbf{B} \cdot dl = \mu \iint_S \mathbf{J} \cdot da \qquad ...(2)$$

From Stoke's theorem of vector analysis for the vector function **B**, we have

$$\oint_C \mathbf{B} \cdot dl = \mu \iint_S \text{curl } \mathbf{B} \cdot da$$

comparing equa. (2) and (3), we obtain

$$\iint_S \text{curl B} \,.\, da = \mu \iint_S \mathbf{J} \,.\, da$$

or

$$\iint_S (\text{curl } \mathbf{B} - \mu\mathbf{J}) \cdot da = 0 \qquad ...(3)$$

Since surface S arbitrary, we must have

$$\text{curl } \mathbf{B} - \mu\, \mathbf{J} = 0$$

i.e.

$$\text{curl } \mathbf{B} = \mu\mathbf{J}. \qquad ...[4(a)]$$

In free space $\mu = \mu_0$; hence equation [4(a)] in free space takes the form

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J}. \qquad ...[4(b)]$$

Thus the curl of **B** *vanishes only where there is no current density.*

If the current density **J** is due to the motion of a charge density ρ with a mean velocity **v** (here all quantities are assumed to refer to the immediate neighbourhood of the point, where curl **B** is being evaluated), then

$$\mathbf{J} = \rho\mathbf{v}, \qquad ...(5)$$

hence equation [4(a)] for a *steady flow charge*, takes the form

$$\text{curl } \mathbf{B} = \mu\rho\mathbf{v}. \qquad ...(6)$$

6.11. The Divergence of Magnetic Induction B

We have adopted the approach that all magnetic fields can be attributed to the motion of electric charges or current elements. So we may consider **B** to be the sum or integral of fields such as those given by Biot-Savart law due to current element dl carrying current I, namely

$$d\mathbf{B} = \frac{\mu}{4\pi} \frac{I\, dl \times \mathbf{r}}{r^3} \qquad ...(1)$$

where **r** is the position vector of the point of observation relative to source (current) element *dl*. This equation tells us that the lines representing the direction of field *d***B** are circles about the axis of current element. Such lines, then, form closed curves. These lines do not start at (diverge from) any point not do they stop at (converge toward) any point. Thus the field *d***B** is *source free* or solenoidal, that is, its divergence is zero. Furthermore the sum of any number of such fields will be solenoidal *i.e.* **B** = 0. Let us now prove this fact mathematically.

The *magnetic induction* **B** due to the entire current loop is

$$B = \frac{\mu}{4\pi} \oint \frac{I\, d\, l \times \mathbf{r}}{r^3} \qquad ...(2)$$

$$\therefore \qquad \text{div } \mathbf{B} = \nabla \cdot \mathbf{B} = \frac{\mu I}{4\pi} \nabla \cdot \oint \frac{dl \times \mathbf{r}}{r^3} \qquad ...(3)$$

Since the differentiation and integration operators are interchangeable, therefore equation (3) may be written as

$$\text{div } \mathbf{B} = \frac{\mu I}{4\pi} \int \nabla \cdot \left(\frac{dl \times \mathbf{r}}{r^3} \right) \qquad ...(4)$$

Further from vector identity

$$\text{div}\,(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\text{curl}\,\mathbf{a} - \mathbf{a}\cdot\text{curl}\,\mathbf{b}$$

we have
$$\nabla\cdot\left[dl \times \left\{\frac{\mathbf{r}}{r^3}\right\}\right] = \left\{\frac{\mathbf{r}}{r^3}\right\}\cdot\text{curl}\,dl - dl\cdot\text{curl}\left\{\frac{\mathbf{r}}{r^3}\right\}$$
$$= \left\{\frac{\mathbf{r}}{r^3}\right\}\cdot(\nabla\times dl) - dl\cdot\left\{\nabla\times\left(\frac{\mathbf{r}}{r^3}\right)\right\}$$

Using this substitution, equation (4) may be written as

$$\text{div}\,\mathbf{B} = \frac{\mu I}{4\pi}\oint\left[\left\{\frac{\mathbf{r}}{r^3}\right\}\cdot(\nabla\times dl) - dl\cdot\left\{\nabla\times\frac{\mathbf{r}}{r^3}\right\}\right] \quad ...(5)$$

But $\nabla\times dl = 0$ since dl is not a function of coordinates (x, y, z) of the field point where we wish to find div **B**.

Also $\nabla\times(\mathbf{r}/r^3) = 0$, since this is of the same type of vector function as **E**, the electric field due to a point charge $\left\{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}}{r^3}\right\}$ and we have seen that $\nabla\times\mathbf{E} = 0$.

Thus the R.H.S. of equation (5) is zero, we therefore have

$$\text{div}\,\mathbf{B} = \nabla\cdot\mathbf{B} = 0. \quad ...(6)$$

That is *the divergence of the magnetic induction* **B** *is always zero.*

6.12. The Magnetic Scalar Potential

In a region such as that inside a current carrying conductor curl $\mathbf{B} = \nabla\times\mathbf{B} \neq 0$. Hence in such cases we can not hope to introduce a scalar potential function as we did in electrostatics. How ever if we have a current carrying conductor surrounded by empty space, then *outside of the conductor* $\nabla\times\mathbf{B} = 0$ and we can write

$$\mathbf{B} = -\,\text{grad}\,\phi_m = -\nabla\phi_m$$

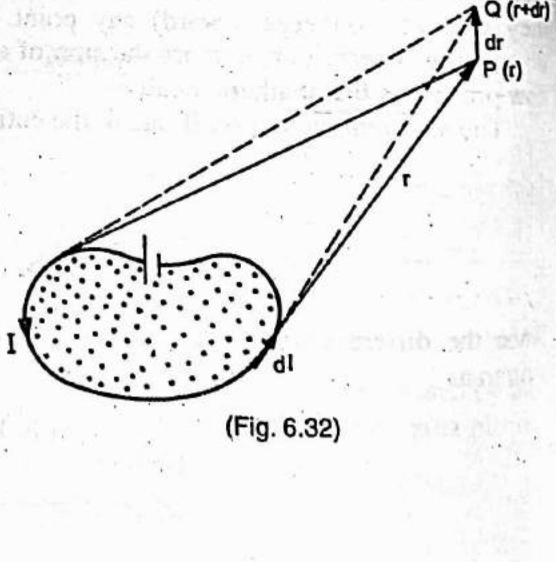
where ϕ_m is a magnetic scalar potential. ...(1)

For finding an expression for the magnetic scalar potential ϕ_m, let us consider a current loop carrying current I as shown in fig. 6·32. From biot-Savart Law that magnetic induction **B** at a field point P (**r**) having position vector **r** relative to current element $d\mathbf{l}$ is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\oint\frac{I\,dl\times\mathbf{r}}{r^3} = \frac{\mu_0 I}{4\pi}\oint\frac{dl\times\mathbf{r}}{r^3} \quad ...(2)$$

If the point of observation $P(r)$ is moved through an infinitesimal distance $d\mathbf{r}$ (say from P (**r**) to Q (**r** + d**r**), then we may write

$$\mathbf{B}\cdot d\mathbf{r} = \frac{\mu_0 I}{4\pi}\oint\frac{(dl\times\mathbf{r})\cdot d\mathbf{r}}{r^3}$$
$$= \frac{\mu_0 I}{4\pi}\oint\frac{d\mathbf{r}\cdot(dl\times\mathbf{r})}{r^3}$$
$$= \frac{\mu_0 I}{4\pi}\oint\frac{(d\mathbf{r}\times dl)\cdot\mathbf{r}}{r^3} \quad ...(3)$$



(Fig. 6.32)

Since the scalar triple product dot and cross are interchangeable if cyclic order of vectors is maintained.

or
$$\mathbf{B}\cdot d\mathbf{r} = -\frac{\mu_0 I}{4\pi}\oint\frac{\{d\mathbf{r}\times(-d\mathbf{l})\}\cdot\mathbf{r}}{r^3}$$

Obviously when the point of observation P is shifted to Q, the solid angle subtended by the loop at the point under consideration will change. Let this change be $d\omega$. For convenience we may calculate this change by assuming that the point of observation were held staitonary and the loop were moved by an amount $(-d\mathbf{r})$. Then the vector product

$$dl \times (-d\mathbf{r}) \qquad ...(4)$$

represents an element of area traced out by the current element dl during the displacements $(-d\mathbf{r})$.

Using (4), equations (3) takes the numerical form

$$\mathbf{B}\cdot d\mathbf{r} = -\frac{\mu_0 I}{4\pi}\oint\frac{d\mathbf{a}\cdot\mathbf{r}}{r^3} \qquad ...(5)$$

$\oint\frac{d\mathbf{a}\cdot\mathbf{r}}{r^3} = d\omega$, the *change* in solid angle subtended by current loop when point P is displaced to point Q.

Hence equation (5) may be expressed as

$$\mathbf{B}\cdot d\mathbf{r} = -\frac{\mu_0 I}{4\pi}d\omega \qquad ...(6)$$

As solid angle subtended by the circuit depends upon the point of observation and hence is a function of (x, y, z), the coordinates of the field point. So we may write

$$d\omega = \frac{\partial\omega}{\partial x}dx + \frac{\partial\omega}{\partial y}dy + \frac{\partial\omega}{\partial z}dz = \nabla\omega\cdot d\mathbf{r} \qquad ...(7)$$

Using equation (7), (6) takes the form

$$\mathbf{B}\cdot d\mathbf{r} = -\frac{\mu_0 I}{4\pi}\nabla\omega\cdot d\mathbf{r}$$

This gives

$$\mathbf{B} = -\frac{\mu_0 I}{4\pi}\nabla\omega = -\nabla\left\{\frac{\mu_0 I\omega}{4\pi}\right\} \qquad ...(8)$$

Comparing equation (8) with $\mathbf{B} - \nabla\phi_m$ we get

$$\textit{scalar magnetic potential}\ \phi_m = \frac{\mu_0 I}{4\pi}\omega. \qquad ...(9)$$

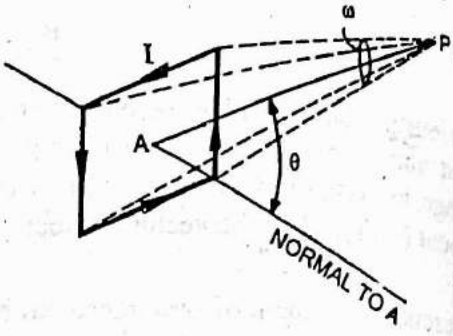
It may be pointed out that the concept of magnetic scalar potential is of practical utility only to derive magnetic field in the absence of continuous current distriutions. The scalar potential ϕ_m can be used if line integrals encircling any currents are considered or if the fields within current carrying media are desired.

6.13. Equivalence of a Small Current Loop and a Magnetic Dipole.

The most important application of the magnetic scalar potential is that it leads to an equivalence between a small current loop and a magnetic dipole.

Let us consider a small current loop of area A carrying current I. Let P be any field point having position vector $\mathbf{r}$ relative to centre of loop such that $\mathbf{r}$ makes an angle θ with the normal to the loop (fig. 6.22). Let us assume that r is large compared with the dimensions of the current loop A. Then the magnetic scalar potential at P, according to section 6·12 is

$$\phi_m = \frac{\mu_0 I}{4\pi}\omega. \qquad ...(1)$$



(Fig. 6.33)

Here ω is the solid angle subtended by current loop at point P, given by

$$\omega = \frac{\mathbf{A}\cdot\mathbf{r}}{r^3} = \frac{A\cos\theta}{r^2} \qquad ...(2)$$

Therefore $$\phi_m = \frac{\mu_0 I}{4\pi}\left(\frac{A\cos\theta}{r^2}\right) \qquad ...(3)$$

From $\mathbf{B} = -\nabla\phi_m$, the radial and transverse components of magnetic induction are given by

$$B_r = -\frac{\partial\phi_m}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{\mu_0 I A\cos\theta}{4\pi r^2}\right) = \frac{2\mu_0 I A\cos\theta}{4\pi r^3} \qquad ...(4a)$$

and $$B_\theta = -\frac{1}{r}\frac{\partial\phi_m}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu I A\cos\theta}{4\pi r^2}\right) = \frac{\mu_0 I}{4\pi}\frac{A\sin\theta}{r^3} \qquad ...(4b)$$

We know in the case of electric dipole (refer equations (10) and (11) of section 2·6).

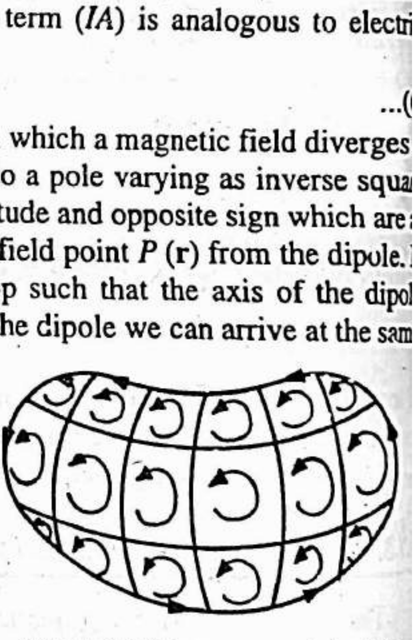
$$E_r = \frac{2p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^3} \qquad ...(5a$$

$$E_0 = \frac{p}{4\pi\varepsilon_0}\frac{\sin\theta}{r^3}. \qquad ...(5b$$

Comparing (4) and (5) we see that, as regard dependence on *r and* θ, *the magnetic field due to the sma current element is similar to the electric field an electric dipole.* The term (IA) is analogous to electri dipole moment ; hence it is known as magnetic dipole moment **m** *i.e.*

$$\mathbf{m} = IA. \qquad ...($$

Now let us imagine a *fictitious quantity;* known as a *magnetic pole,* from which a magnetic field diverges the pole is positive and converges if the pole is negative, the field due to a pole varying as inverse squa law. Then a *magnetic dipole* may be defined as two poles of equal magnitude and opposite sign which are a finite distance apart that is small compared with r, the distance of the field point P (**r**) from the dipole. we think of such a dipole situated at the location of small current loop such that the axis of the dipol coincides with the normal to loop area, then by adjusting the strength of the dipole we can arrive at the sam field with the dipole as we actually have due to the current loop.



(Fig. 6.34) A large current circuit is equivalent to large number of elementary loops.

A large current circuit may be thought of being formed of a large number of small elementary loops, as shown in fig. 6·34. If the same current I circulates in each small loop in the same direction, then the current on all interior boundaries will cancel out each other and hence will be equivalent to current I in the large circuit. Thus the magnetic field of the large current circuit must be the same as that due to all the elementary loops. Since the field of each elementary loop is same as that of a magnetic dipole at its centre, with its axis normal to the surface, the field of a large current circuit is same as that of a uniform surface layer of a large current dipoles, the surface being bounded by the circuit. Such a layer is termed as *double layer,* since it amounts to a layer of positive poles adjacent to a parallel layer of negative poles.

Although isolated magnetic dipoles do not exist, the magnetic dipole has reality in that a small ba magnet behaves like one. Although we shall retain concept that the field of such magnet is due to spinning electron within the metal, it may be more convenient for some purpose to adopt the dipole concept. Such a physical model is justified if it serves a purpose.

6.14. Magnetic Vector Potential

We have seen that the magnetic scalar potential can not be used if line integrals encircling any currents are considered or if the fields within current carrying media are desired; since in such regions the magnetic scalar potential can not be single valued function of position. Because of this shortcoming it is not *generally* possible to derive **B** from scalar magnetic potential ϕ_m, however a magnetic vector potential exists from which **B** can be derived.

From Biot-Savart law the magnetic induction due to acurrent element δl carrying current I at a field point having position vector **r** relative current element is given by

$$\delta \mathbf{B} = \frac{\mu}{4\pi} \frac{\delta l \times \mathbf{r}}{r^3}. \qquad \text{...(1)}$$

We have

$$\nabla\left(\frac{1}{r}\right) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left\{\frac{1}{(x^2+y^2+z^2)^{1/2}}\right\}$$

$$= -\frac{1}{2}\frac{(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{r^3} = -\frac{\mathbf{r}}{\mathbf{r}^3}.$$

Thus

$$\frac{\mathbf{r}}{r^3} = -\nabla\left(\frac{1}{r}\right) \qquad \text{...(2)}$$

using this, equation (1) may be written as

$$\delta \mathbf{B} = \frac{\mu I}{4\pi}\delta l \times \left\{-\nabla\left(\frac{1}{r}\right)\right\}$$

$$= \frac{\mu I}{4\pi}\left\{\nabla\left[\frac{1}{r}\right] \times \delta l\right\}. \qquad \text{...(3)}$$

From vector algebra we also have

$$\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$$

or

$$(\nabla \phi) \times \mathbf{A} = \nabla \times (\phi \mathbf{A}) - \phi(\nabla \times \mathbf{A}) \qquad \text{...(4)}$$

Substituting $\delta l / r$ in place of (ϕ **A**)
(*i.e.* δl for **A** and $1/r$ for ϕ) in above equation, we get

$$\nabla\left(\frac{1}{r}\right) \times \delta l = \nabla \times \left(\frac{\delta l}{r}\right) - \frac{1}{r}(\nabla \times \delta l). \qquad \text{...(5)}$$

Using equation (5) Biot-Savart law may be transformed from the form of equation (3) to

$$\delta \mathbf{B} = \frac{\mu I}{4\pi}\left[\nabla \times \left(\frac{\delta l}{r}\right) - \frac{1}{r}(\nabla \times \delta l)\right] \qquad \text{...(6)}$$

As δl is not a function of co-ordiantes (x, y, z), of the point at which the magnetic induction is required and the operation ∇ represents the differentiation with respect to these co-ordinates, we must have

$$\nabla \times \delta l = 0. \qquad \text{...(7)}$$

Substituting this in equation (6), Biot-Savart law can be written as

$$\delta \mathbf{B} = \frac{\mu I}{4\pi}\nabla \times \left(\frac{\delta l}{r}\right) \qquad \text{...(8)}$$

Therefore the total magnetic induction at the given point by a closed current loop is given by

$$\mathbf{B} = \frac{\mu I}{4\pi} \oint \nabla \times \left(\frac{dl}{r} \right) \qquad ...(9)$$

Further as the operation $\nabla \times$ is independent of the integration of $d\mathbf{l}/r$ around the closed loop, we may write equation (9) as

$$\mathbf{B} = \nabla \times \left[\frac{\mu I}{4\pi} \oint \frac{dl}{r} \right] = \text{curl}\, \frac{\mu I}{4\pi} \oint \frac{d\mathbf{l}}{\mathrm{r}}. \qquad ...(10)$$

The quantity within the square bracket is obviosuly a vector. From equation (10) we conclude that a vector exists such that by taking its curl (*i.e.* $\nabla \times$), the magnetic induction produced at any point by a closed loop carrying current may be obtained. This vector is known as *magnetic vector potential* and is designated by **A**.

Thus
$$\mathbf{B} = \text{curl}\, \mathbf{A} = \nabla \times \mathbf{A} \qquad ...(11)$$

where *magnetic vector potential*

$$\mathbf{A} = \frac{\mu I}{4\pi} \oint \frac{dl}{r}. \qquad ...(12)$$

If the region under consideration is specified by current density vector **J**, the using $Id\mathbf{l} = \mathbf{J}dv$, the magnetic vector potential represented by equation (12) takes the form

$$\mathbf{A} = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r})\, dv}{r} \qquad ...(13)$$

where the volume integral is taken over all regions containing a current density. Since $\mathbf{J} = \rho \mathbf{v}$, we may also write

$$\mathbf{A} = \frac{\mu}{4\pi} \int_V \frac{\rho\, \mathbf{v}}{r}\, dv, \qquad ...(14)$$

while ρ and **v** may vary throughout space we must make the restriction that they are independent of time, so that we have a *steady field.* In the case of a single charge, the field that it produces at a field point *P* will vary as the charge moves along, such a field will not be steady. Our expression will give the instantaneous field at *P* to a good approximation of $v/c \ll 1$, where *c* is speed of light.

If we assume the current element (say $I\, d\mathbf{l}$) at position $\mathbf{r}'$ instead of origin and **r** is the position vector of field point, then the distance of field point from current element $r = |\mathbf{r} - \mathbf{r}'|$, so that equations (12) and (13) take the form

$$\mathbf{A} = \frac{\mu}{4\pi} \oint \frac{I dl'}{|\mathbf{r} - \mathbf{r}'|} \qquad ...(15)$$

and
$$\mathbf{A} = \frac{\mu}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')\, dv'}{|\mathbf{r} - \mathbf{r}'|}. \qquad ...(16)$$

If vector potential **A** is calculated by any of above equation from (12) to (16), then equation (11) may be used derive magnetic induction **B** at the sepcified point. This situation is analogous to the electrostatic case in which the electric field, **E** is easily derived from the electric potential ϕ by using the relation,

$$\mathbf{E} = -\nabla \phi.$$

Characteristics of magnetic vector potential :

(i) *It satisfies Poisson's equation.*

We have
$$\text{curl}\ \mathbf{B} = \mu_0 \mathbf{J}$$

As
$$\mathbf{B} = \text{curl}\ \mathbf{A}$$

$\therefore$
$$\text{curl curl}\ \mathbf{A} = \mu_0 \mathbf{J} \text{ or } \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

i.e.
$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

If we restrict to cases of *steady currents* for which

$$\nabla \cdot \mathbf{A} = 0, \text{ we get}$$
$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

This equation is similar to *Poisson's equation* and is called Poisson's equation for *vector potential.*

(ii) The line integral of magnetic vector A around a closed curve give the magnetic flux linked with the surface enclosed by curve *i.e.*

$$\phi_m = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

By definition $\phi_m = \oint_S \mathbf{B} \cdot d\mathbf{S}$

But
$$\mathbf{B} = \text{curl } \mathbf{A}$$

$\therefore$
$$\phi_m = \oint_S \text{curl } \mathbf{A} \cdot d\mathbf{S}$$

Using Stoke's theorem to change surface integral into line integral, we get

$$\phi_m = \oint \mathbf{A} \cdot d\mathbf{l}$$

The divergence of the vector potential ; The Lorentz condition :

Consider a volume current distribution with current density $\mathbf{J}(\mathbf{r}')$ at position $\mathbf{r}'$.

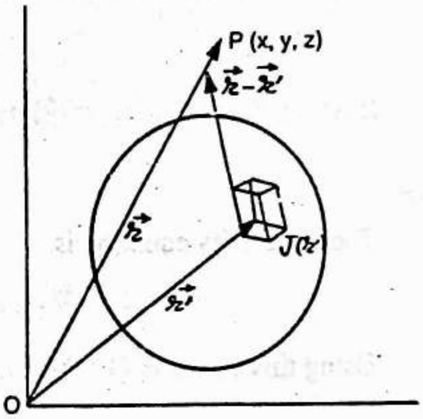
The vector potential A at point $P(\mathbf{r})$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')\, dv'}{|\mathbf{r} - \mathbf{r}'|}$$

The divergence of A

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \cdot \int_{V'} \frac{\mathbf{J}(\mathbf{r}')\, dv'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{...(1)}$$

The derivative is w.r. to coordinates (x, y, z) of field point and the integral is w.r. to (x', y', z') of current distribution, therefore these two operators are interchangeable.



(Fig. 6.35)

Therefore equation (1) can be written as

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \nabla \cdot \left[\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] dv' \qquad \text{...(2)}$$

According to vector identity ($S \rightarrow$ scalar function, $A \rightarrow$ vector function)

$$\nabla \cdot (S\,\mathbf{A}) = \mathbf{A} \cdot \nabla S + S\, \nabla \cdot \mathbf{A}, \qquad \text{...(3)}$$

setting $S = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$ and $\mathbf{A} = \mathbf{J}(\mathbf{r}')$, we get

$$\nabla \cdot \left[\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] = \mathbf{J}(\mathbf{r}') \cdot \nabla \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \cdot \mathbf{J}(\mathbf{r}')$$

As $\mathbf{J}(\mathbf{r}')$ is not the function of coordinates (x, y, z)

$\therefore$
$$\nabla \cdot \mathbf{J}(\mathbf{r}') = 0$$

$\therefore$
$$\nabla \cdot \left[\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] = \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dv'. \qquad \text{...(5)}$$

If the gradient in the integrand of (5) is evaluated at source point with respect to coordinates (x', y', z'), we get

$$\nabla\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] = -\nabla'\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]$$

Using this we get

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\int_{V'}\mathbf{J}(\mathbf{r}')\cdot\nabla'\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]dv' \quad ...(6)$$

Using vector identity (3) again, we get

$$\mathbf{J}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \nabla'\cdot\left[\frac{J(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{J}(\mathbf{r}') \quad ...(7)$$

Substituting this in (6), we get

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}[\nabla'\cdot\mathbf{J}(\mathbf{r}')]\,dv' - \frac{\mu_0}{4\pi}\int_{V'}\nabla'\cdot\left[\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right]dv' \quad ...(8)$$

Using Gauss divergence theorem in the second term on right hand side of equation (8), we get

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}[\nabla'\cdot\mathbf{J}(\mathbf{r}')]\,dv' - \frac{\mu_0}{4\pi}\int_{S'}\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot d\mathbf{S}' \quad ...(9)$$

where S' is the surface that encloses the volume V' within which the current density $\mathbf{J}(\mathbf{r}')$ is confined. Now, everywhere on the surface S', $\mathbf{J}(\mathbf{r}')$ either zero or tangential, so that the value of

$$\int\frac{\mathbf{J}(r')\cdot d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|} = 0$$

In view of this, equation (9) becomes

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}[\nabla'\cdot\mathbf{J}(\mathbf{r}')]\,dv' \quad ...(10)$$

The continuity equation is

$$\nabla\cdot\mathbf{J}(\mathbf{r}') = -\frac{\partial\rho(\mathbf{r}')}{\partial t}$$, where $\rho(\mathbf{r}')$ is charge at position $\mathbf{r}'$.

Using this equation (10) becomes

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\int_{V'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\rho(\mathbf{r}')}{\partial t}dv'$$

$$= -\mu_0\frac{\partial}{\partial t}\int_{V'}\frac{\rho(\mathbf{r}')\,dv'}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad ...(11)$$

If $\phi(\mathbf{r})$ is electric potential at point P due to total charge that can be associated with the current distribution at any instant, then

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_r\varepsilon_0}\int\frac{\rho(\mathbf{r}')\,dv'}{|\mathbf{r}-\mathbf{r}'|},\ \varepsilon_r$$ being dielectric constant or relative permittivity.

In view of this equation (11) implies that for a non-steady current distribution

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = -\mu_0\varepsilon_r\varepsilon_0\frac{\partial\phi(\mathbf{r})}{\partial t} \quad ...(12)$$

Equation (12) is called the Lorentz condition for the vector potential A associated with the magnetic field B due to a non-steady current.

In special case of *steady current* $\frac{\partial\rho(\mathbf{r}')}{\partial t} = 0$ and hence equation (12) gives

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = 0$$

This is the Lorentz condition for vector potential associated with steady current.

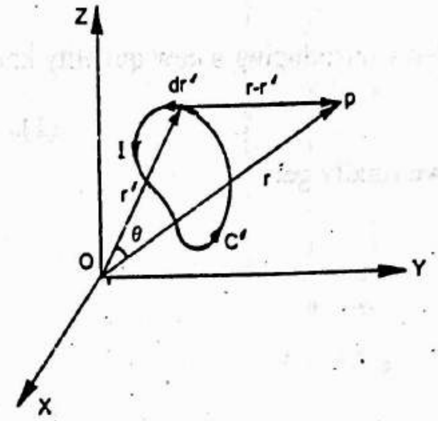
6.15. Applications of Vector Potential.

(i) The magnetic field of a distant current loop.

Figure 6·36 depicts a small current loop carrying current I. Let us consider a current element $I\,d\mathbf{r}'$ having position vector $\mathbf{r}'$ relative to origin. Then the vecor potential at distance field P having position vector $\mathbf{r}$ is given by equation (15), such that $r \gg \mathbf{r}'$ of section 6·14 (replacing $d\mathbf{l}'$ by $d\mathbf{r}'$ and μ by μ_0) viz.

$$A = \frac{\mu_0}{4\pi}\oint \frac{I\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \qquad ...(1)$$

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= (r^2 + r'^2 - 2rr'\cos\theta)^{1/2}\\ &= (r^2 + r'^2 - 2\mathbf{r}'.\mathbf{r}')^{1/2}\\ &= r\left(1 + \frac{\mathbf{r}'^2}{r^2} - \frac{2\mathbf{r}'\cdot\mathbf{r}'}{r^2}\right)^{1/2}\end{aligned}$$



(Fig. 6.36)

$$\begin{aligned}\therefore\quad \mathbf{A} &= \frac{\mu_0 I}{4\pi r}\oint \frac{d\mathbf{r}'}{\left(1 + \frac{r'^2}{r^2} - \frac{2\mathbf{r}'\cdot\mathbf{r}}{r^2}\right)^{1/2}}\\ &= \frac{\mu_0 I}{4\pi r}\oint d\mathbf{r}'\left(1 + \frac{r'^2}{r^2} - \frac{2\mathbf{r}'\cdot\mathbf{r}}{r^2}\right)^{-1/2}\end{aligned}$$

using binomial expansion and keeping only first order terms in r'/r since $r' \ll r$, we may write

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 I}{4\pi r}\oint d\mathbf{r}'\left[1 + \frac{\mathbf{r}'\cdot\mathbf{r}}{r^2} + \dots\right]\\ &= \frac{\mu_0 I}{4\pi r}\left[\oint_{C'} d\mathbf{r}' + \oint_{C'}\left(\frac{\mathbf{r}'\cdot\mathbf{r}}{r^2}\right)d\mathbf{r}'\right] \qquad ...(2)\end{aligned}$$

Obviously $\oint_{C'} d\mathbf{r}' = 0$ for an arbitrary closed loop C' (since initial and final positions are the same).

Therefore equation (2) takes the form

$$\mathbf{A} = \frac{\mu_0 I}{4\pi r^3}\oint_{C'}(\mathbf{r}'\cdot\mathbf{r})\,d\mathbf{r}'. \qquad ...(3)$$

Further from vector triple product, we have

$$\mathbf{r}\times(d\mathbf{r}'\times\mathbf{r}') = (\mathbf{r}\cdot\mathbf{r}')\,d\mathbf{r}' - (\mathbf{r}\cdot d\mathbf{r}')\,\mathbf{r}'. \qquad ...(4)$$

To eliminate the second term on R.H.S. of (3), let us write the differential of $\mathbf{r}'(\mathbf{r}\cdot\mathbf{r}')$ for a small change in $\mathbf{r}'$ *i.e.*,

$$d\,[\mathbf{r}'(\mathbf{r}\cdot\mathbf{r}')] = d\mathbf{r}'(\mathbf{r}\cdot\mathbf{r}') + \mathbf{r}'(\mathbf{r}\cdot d\mathbf{r}'). \qquad ...(5)$$

Adding equations (4) and (5), we get

$$\mathbf{r}\times(d\mathbf{r}'\times\mathbf{r}') + d\,[\mathbf{r}'(\mathbf{r}\cdot\mathbf{r}')] = 2\,(\mathbf{r}\cdot\mathbf{r}')\,d\mathbf{r}'.$$

Dividing throughout by 2 and rearranging, we get

$$(\mathbf{r}'\cdot\mathbf{r})\,d\mathbf{r}' = \tfrac{1}{2}\,\mathbf{r}\times(d\mathbf{r}'\times\mathbf{r}') + \tfrac{1}{2}\,d\,[\mathbf{r}'(\mathbf{r}\cdot\mathbf{r}')]. \qquad ...(6)$$

Making this substitution, equation (3) takes the form

$$\mathbf{A} = \frac{\mu_0 I}{4\pi r^3}\left\{\oint_{C'} \tfrac{1}{2}\, \mathbf{r} \times (d\mathbf{r}' \times \mathbf{r}') + \oint_{C'} \tfrac{1}{2}\, d\left[\mathbf{r}'\,(\mathbf{r} \cdot \mathbf{r}')\right]\right\} \quad ...(7)$$

Since the integrand in the second integral is a perfect differential, hence its integral is a perfect differential, hence its integral around a closed loop is always zero.

Hence equation (7) becomes

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 I}{4\pi r^3}\oint_{C'} \tfrac{1}{2}\, \mathbf{r} \times (d\mathbf{r}' \times \mathbf{r}') \\ &= \frac{\mu_0 I}{4\pi r^3}\oint_{C'} \tfrac{1}{2}\, (\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} \\ &= \frac{\mu_0}{4\pi}\left[\tfrac{1}{2}\oint_{C'} (\mathbf{r}' \times I\, d\mathbf{r}') \times \frac{\mathbf{r}}{r^3}\right] \quad ...(8)\end{aligned}$$

Now introducing a new quantity known as the *magnetic moment* **m** and defined by

$$\mathbf{m} = \tfrac{1}{2}\oint_{C'} (\mathbf{r}' \times I d\mathbf{r}')$$

we finally get

$$\mathbf{A}\,(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3} \quad ...(10)$$

$$= -\frac{\mu_0}{4\pi}\, \mathbf{m} \times \nabla\left\{\frac{1}{r}\right\} \quad ...(11)$$

$$\left[\text{since } \nabla\left\{\frac{1}{r}\right\} = -\frac{\mathbf{r}}{r^3}\right]$$

Now the magnetic induction **B** can be determined by taking the curl of equation (10)

$$\mathbf{B}\,(\mathbf{r}) = \text{curl}\, \mathbf{A}\,(\mathbf{r}) = \text{curl}\left\{\frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}\right\}$$

$$= \frac{\mu_0}{4\pi}\left[-(\mathbf{m} \cdot \nabla)\frac{\mathbf{r}}{r^3} + \mathbf{m}\nabla \cdot \left\{\frac{\mathbf{r}}{r^3}\right\}\right] \quad \text{for constant vector } \mathbf{m} \quad ...(12)$$

But

$$\text{div}\left\{\frac{\mathbf{r}}{r^3}\right\} = \nabla \cdot \left\{\frac{\mathbf{r}}{r^3}\right\} = 0$$

and

$$m_x \frac{\partial}{\partial x}\left\{\frac{\mathbf{r}}{r^3}\right\} - \frac{m_x \mathbf{i}}{r} - 3m_x x \frac{\mathbf{r}}{r^5}$$

so that

$$(\mathbf{m} \cdot \nabla)\frac{\mathbf{r}}{r^3} = \frac{\mathbf{m}}{r^3} - \frac{3\,(\mathbf{m} \cdot \mathbf{r})\,\mathbf{r}}{r^5}$$

using these substitutions equation (12) becomes

$$\mathbf{B}\,(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\,(\mathbf{m} \cdot \mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right] \quad ...(13)$$

This equation indicates that the *magnetic field of a distant circuit does not depend on its detailed geometry, but only on the magnetic moment of the circuit* **m**. The field is of the same form as the electric field due to an electric dipole. Hence the field may be called the *magnetic dipole field.*

Now we may also determine the scalar magnetic potential ϕ_m be the relation,

$$\mathbf{B} = -\nabla \phi_m \qquad \text{...(14)}$$

Equation (13) may be written as

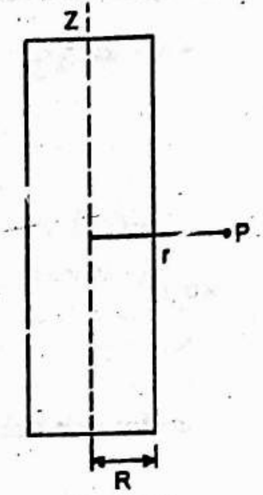
$$\mathbf{B} = -\nabla\left(\frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}\right). \qquad \text{...(15)}$$

Comparing (14) and (15) we note that

Scalar magnetic potential

$$\phi_m = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}\right)$$

Magnetic field induction due to an infinite current carrying wire : Consider a long wire carrying a current I whose axis coincides with z axis. The only component of current density and hence magnetic vector potential is z-component.



(Fig. 6.37)

Inside the wire :

In cylindrical coordinates (r, θ, z), the Poisson's equation is

$$\nabla^2 A_z = -\mu J_z \qquad \text{...(1)}$$

where μ is permeability inside the wire. If R is radius of wire,

then
$$J_z = \frac{I}{\pi R^2}$$

The value of $\nabla^2\phi$ in cylindrical coordinates is

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial^2\phi}{\partial z^2}.$$

As A_z is independent of θ and z, equation (1) may be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -\frac{\mu I}{\pi R^2} \qquad \text{...(1)}$$

Integrating, we get

$$r\frac{\partial A_z}{\partial r} = -\frac{\mu I r^2}{2\pi R^2} + C_1$$

where C_1 is constant of integration

At
$$r = 0,\ r\frac{\partial A_z}{\partial r} = 0, \quad \therefore C_1 = 0$$

$$\therefore\ r\frac{\partial A_z}{\partial r} = -\frac{\mu_0 I r^2}{2\pi R^2}$$

or
$$\frac{\partial A_z}{\partial r} = -\frac{\mu_0 I r}{2\pi R^2} \qquad \text{...(3)}$$

Integrating again

$$A_z = -\frac{\mu I r^2}{4\pi R^2} + C_2 \qquad \text{...(4)}$$

If we assume that $A_2 = 0$ at $r = R$, we get

$$C_2 = \frac{\mu I R^2}{4\pi R^2} = \frac{\mu I}{4\pi}$$

∴ Equation (4) becomes

$$A_z = \frac{\mu I}{4\pi}\left[1 - \frac{r^2}{R^2}\right] \qquad \text{...(5)}$$

As $\mathbf{B} = \text{curl } \mathbf{A} = \nabla \times \mathbf{A}$

the cylindrical components of curl **A** are

$$\left.\begin{aligned} (\text{curl } \mathbf{A})_r &= \frac{1}{r}\frac{\partial A_z}{r\theta} - \frac{\partial A_\theta}{\partial z} \\ (\text{curl } \mathbf{A})_\theta &= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ (\text{curl } \mathbf{A})_z &= \frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \end{aligned}\right\} \qquad \text{...(6)}$$

we get

$$B_\theta = \frac{\mu I r}{2\pi R^2} \qquad \text{...(7)}$$

Out side the wire : The Poisson's equation outside the wire takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = 0 \qquad \text{....(8)}$$

Integrating $r\dfrac{\partial A_z}{\partial r} = \text{constant}, C$

or

$$\frac{\partial A_z}{\partial r} = \frac{C}{r} \qquad \text{...(9)}$$

Integrating again

$$A_z = C \log_e r + C' \qquad \text{(another constant)} \qquad \text{...(10)}$$

As $r = R, \quad A_z = 0$

∴ $C' = -C \log_e R.$

Substituting this value in (10), we get

$$\begin{aligned} A_z &= C \log_e r - C \log_e R \\ &= C \log_e \left(\frac{r}{R}\right) \end{aligned} \qquad \text{...(11)}$$

Again magnetic field induction

$$\mathbf{B} = \text{curl } \mathbf{A} = \nabla \times \mathbf{A}$$

$$B_\theta = (\text{curl } \mathbf{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = -\frac{\partial A_z}{\partial r}$$

As B_θ must be continuous at $r = R$.

From (5)

$$\frac{\partial A_z}{\partial r} = \frac{\mu I}{4\pi}\left[0 - \frac{2r}{R}\right]_{r=R} = -\frac{\mu I}{2\pi R}$$

Also from (9),
$$\left(\frac{\partial A_z}{\partial r}\right)_{r=R} = \frac{C}{R}$$

∴
$$\frac{C}{R} = -\frac{\mu I}{2\pi R} \text{ or } C = -\frac{\mu I}{2\pi}$$

ubstituting this value in (11), we get
$$A_z = -\frac{\mu I}{2\pi}\log_e\left(\frac{r}{R}\right) \qquad ...(12)$$

nd hence
$$B_\theta = -\frac{\partial A_z}{\partial r} = \frac{\mu I}{2\pi R} \qquad ...(13)$$

Ex. 14. *Show that the magnetic vector potential for two long straight parallel wires carrying the same urrent I in opposite directions is given by*
$$\mathbf{A} = \frac{\mu_0 I}{2\pi}\log\left(\frac{r_2}{r_1}\right)\mathbf{n}$$
here r_1 and r_2 are the distances from the field point to the wires and **n** *is a unit vector parallel to the wire.*

(Rohilkhand 1995; Meerut 1972)

Solution. Consider a wrie of length L, carrying a current I. Let P be a point at a distance x from the ire. For convenience we consider the point P to be situated symmetrically. However the results obtained ould hold for any position of P. Let O be the origin of coordinate system, with Y axis along the current and -axis towards the field point P, the magnetic potential at P due to a small element dy at a distance y from is given by
$$d\mathbf{A} = \frac{\mu_0 I}{4\pi}\frac{dy}{r}$$

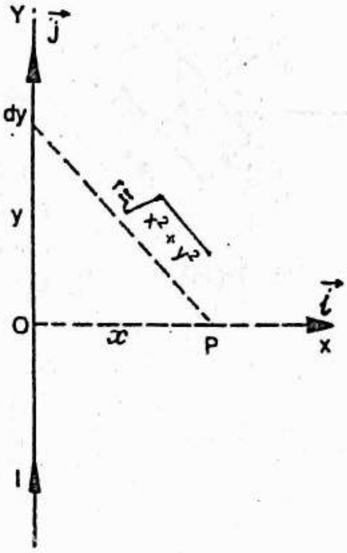
irected along the current

e.
$$d\mathbf{A} = \frac{\mu_0 I}{4\pi}\frac{dy}{\sqrt{(x^2+y^2)}}\mathbf{j},$$

being unit vector along Y axis.

The magnetic potential due to whole length of wire is
$$\mathbf{A} = \mathbf{j}\frac{\mu_0 I}{4\pi}\int_{-L/2}^{+L/2}\frac{dy}{\sqrt{(x^2+y^2)}}$$
$$= \mathbf{j}\frac{\mu_0 I}{4\pi}\left[\log\{y+\sqrt{y^2+x^2}\}\right]_{-L/2}^{+L/2}$$

nce from standard integral



(Fig. 8.38)

$$\int\frac{dx}{\sqrt{(x^2+a^2)}} = \sin^{-1}\frac{x}{a}$$
$$\log\{x+\sqrt{(x^2+a^2)}\}$$
$$= \mathbf{j}\frac{\mu_0 I}{4\pi}\left[\log\left\{\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}\right\} - \log\left\{-\frac{L}{2}+\sqrt{\left(\frac{L^2}{2}+x^2\right)}\right\}\right]$$

$$= \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}}{-\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}}\right]$$

$$= \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{\frac{L}{2}\left\{1+\left(1+\frac{4x^2}{L^2}\right)^{1/2}\right\}}{\frac{L}{2}\left\{-1+\left(1+\frac{4x^2}{L^2}\right)^{1/2}\right\}}\right]$$

As wire is infinitely long, then $L \to \infty$

$$\therefore \qquad \frac{x^2}{L^2} \ll 1\,;$$

hence on using bionomial theorem and neglecting higher order terms, we have

$$\mathbf{A} = \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{1+\left(1+\frac{4x^2}{2L^2}\right)}{-1+1+\frac{4x^2}{2L^2}}\right]$$

$$= \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{\left(2+\frac{2x^2}{L^2}\right)}{\left(\frac{2x^2}{L^2}\right)}\right]$$

$$= \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\frac{1+\frac{x^2}{L^2}}{\left(\frac{x^2}{L^2}\right)}$$

$$= \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\left(1+\frac{L^2}{x^2}\right)$$

$$= \mathbf{j}\,\frac{\mu_0 I}{4\pi}\log\left(\frac{L}{x}\right)^2 \qquad \left(\text{since } \frac{L^2}{x^2} \gg 1\right)$$

$$\mathbf{A} = \frac{\mu_0 I}{2\pi}\log\left(\frac{L}{x}\right)\mathbf{j} \qquad \text{...(1)}$$

This is required expression for *magnetic potential due to a long straight current carrying* wire.

Now instead of one wire, we are given two long parallel wires carrying current in oposite directions : hence using result (1) the magnetic vector potential A at field point P due to both the wires is given by

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$$

$$= \frac{\mu_0 I}{2\pi}\log\left(\frac{L}{x}\right)\mathbf{n}_1 + \frac{\mu_0 I}{2\pi}\log\left(\frac{L}{x}\right)\mathbf{n}_2 \qquad \text{...(2)}$$

where $x = r_1$ and $\mathbf{n} = \mathbf{n}_1$ for wire 1.

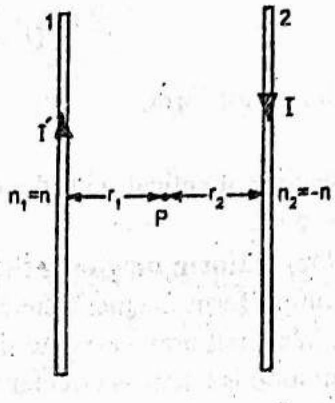
and $x = r_2$ and $\mathbf{n} = \mathbf{n}_2$ for wire 2.

But from figure 6·28, it is obvious that

$$\mathbf{n}_1 = \mathbf{n} \text{ and } \mathbf{n}_2 = -\mathbf{n}.$$

Therefore equation (2), gives

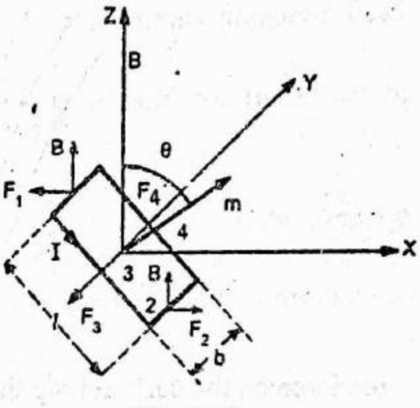
$$\begin{aligned} \mathbf{A} &= \frac{\mu_0 I}{2\pi}\log\left(\frac{L}{r_1}\right)\mathbf{n} - \frac{\mu_0 I}{2\pi}\log\left(\frac{L}{r^2}\right)\mathbf{n} \\ &= \frac{\mu_0 I}{2\pi}\log\left(\frac{L/r_1}{L/r_2}\right) \\ &= \frac{\mu_0 I}{2\pi}\log_e\left(\frac{r_2}{r_1}\right)\mathbf{m}. \end{aligned}$$



(Fig. 6.39)

6.16. A Magnetic Dipole in a Magnetic Field

Uniform magnetic field. Consider a loop of current, which is equivalent to a magnetic dipole, placed in an external uniform magnetic field as shown in fig. 6·40, where we have considered the loop as rectangular for simplicity and the uniform magnetic field **B** along the z-direction. The side of the loop are l and b and the plane of the loop is placed through the y-axis, making an angle θ with the x-y-plane as shown in figure. Then the magnetic moment of the loop which is normal to its plane-will make an angle θ with the magnetic field.



(Fig. 6.40)

Since the current are oposite on oposite sides of the loop, the forces are also opposite, so there is no net translatory force on the loop in a uniform magnetic field. The force on sides 3 and 4 of the loop are IlB and these do not form a couple. The force IbB on sides 1 and 2 of the loop form a couple which tends to rotate the loop about the y-axis. The perpendicular distance between the force $\mathbf{F}_1$ and $\mathbf{F}_2$ is $l \sin\theta$ and so the torque is

$$\tau = IBb\, l \sin\theta \qquad ...(1)$$

Since Ilb is the magnetic moment m of the loop, we can write

$$\tau = mB\sin\theta \qquad ...(2)$$

or writing vectorially, we have

$$\vec{\tau} = \mathbf{m} \times \mathbf{B}. \qquad ...(3)$$

Although we have derived this expression for torque for the special case of a rectangular loop, but this result is true for a small loop of any shape. Note that the torque is of the same kind as the torque on an electric dipole in a uniform field.

$$\vec{\tau} = \mathbf{p} \times \mathbf{E}.$$

Since a toruqe acts on a current loop or other magnetic dipole, when it is placed in an external magnetic field, it follows that work (positive or negative) must be done by an external agent to change the orientation of such a dipole. Thus a magnetic dipole has *potential energy* associated with its orientation is an external magnetic field. We may take this energy as zero for any arbitrary position of the loop. As for electric dipoles we take the magnetic potential energy U as zero, when **m** and **B** are at right angles to each other *i.e.*, when $\theta = 90°$. This choice of a zero energy configuration for U is arbitrary because we are generally interested in the change in energy that occur when the dipole is rotated, The ***magnetic potential energy*** in any position θ of the magnetic dipole simply represents the work done by an external agent to turn the dipole from its zero energy position (θ = 90°) to the given position θ. Thus

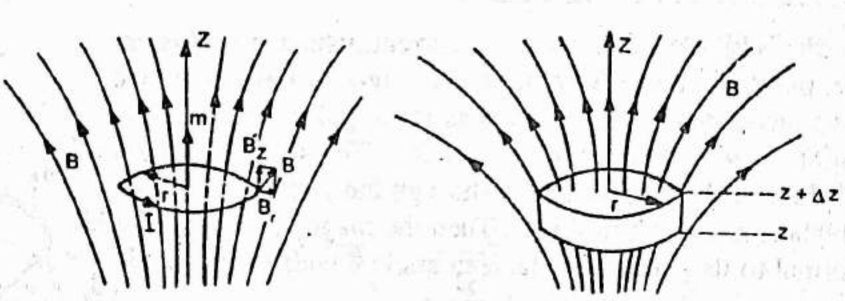
$$U = \int_{\pi/2}^{\theta} \tau \,.\, d\theta = mB \int_{\pi/2}^{\theta} \sin\theta \, d\theta = -mB\cos\theta \qquad ...(4)$$

or in the vector form,

$$U = -\mathbf{m} \cdot \mathbf{B} \qquad ...(5)$$

The result is identical with the expression for the energy of an electric dipole in an external electric field $U = -\mathbf{p} \cdot \mathbf{E}$.

Non-uniform magnetic field. When a current loop, which is equivalent to a magnetic dipole is placed in a non-uniform magnetic field, then in addition the the torque $\mathbf{m} \times \mathbf{B}$ the dipole experiences a translatory force. We shall now compute the magnitude of the translotory force in reference to fig. 6·41 in which the current loop is taken as circular and the non-uniform magnetic field **B** is generlaly along the z-direction. The field **B** goes on decreasing in the increasing z-direction. We take for simplicity, the field **B** to be symmetric about the z-axis. The magnetic field shown in the figure does not include the field of the current loop. The net force on the current loop due to its own field as zero, so we shall ignore its own field in this discussion.



(Fig. 6.41)

At every position of the current loop, the field **B** has components $\mathbf{B}_z$ along the z-direction and $\mathbf{B}_r$ in the plane of the loop directed radially outwards. It is clear from the circular symmetry of the loop about the z-axis that the component $\mathbf{B}_z$ produces no force on the loop. However, the component $\mathbf{B}_r$ exerts a force $Idl\mathbf{B}_r$ acting downwards (towards the negative z-direction) on energy element dl of the current loop. Thus the total downward force that the loop experiences is

$$\mathbf{F} = -\mathbf{k} \cdot 2\pi \, IB_r \,. \qquad ...(6)$$

Thus, value of B_r can easily by evaluated in terms of the gradient of B_z using Gauss's law. Consider a small cylindrical box type surface with its axis along the z-direction, the base area πr^2 and height Δz, as shown in fig. 6·33 (b). The outward flux from the curved side surface is $2\pi r \,.\, \Delta z \,.\, B_r$, and the net outward flux from the end surface is

$$\pi r^2 \, [B_z(z) - B_z(z + \Delta z)].$$

The quantity in square bracket approaches to $\partial B_z/\partial z \,.\, \Delta z$ in the limit when Δz is very small so the total flux coming out of the box is,

$$\oint \mathbf{B} \cdot d\mathbf{a} = 2\pi r \,.\, \Delta z B_r \,.\, \Delta z \,.\, B_r + \pi r^2 \cdot \frac{\partial B_z}{\partial z} \cdot \Delta z \qquad ...(7)$$

and according to Gauss's law the total magnetic flux coming out of a box is equal to zero since there exist no isolated poles in magnetism. Thus setting equation (7) equal to zero, we get

$$B_r = -\frac{r}{2} \cdot \frac{\partial B_z}{\partial z}. \qquad ...(8)$$

Since B_z decreases in the increasing z direction, $\partial B_z/\partial z$ is negative and B_r is positive as expected. Substituting the value of B_r from equation (8) into (6), we get the force on the dipole,

$$\mathbf{F} = \mathbf{k} \,.\, 2\pi l \cdot \frac{r}{2} \cdot \frac{\partial B}{\partial z} = \mathbf{k} l \pi r^2 \frac{\partial B}{\partial z}$$

$$= m \frac{\partial B}{\partial z} \mathbf{k} \qquad ...(9)$$

where $m = \pi r^2 I$ is the magnetic dipole moment of the current ring.

Although we have derived this result for the special case of a circular loop, but we can say that, in general the force on any small current loop of any shape depends only on is dipole moment m and not the details of its shape : provided the loop is small enough so that only the first order variation of the external field, over the span of the loop is effective.

The current ring considered in fig. 6·41 has a magnetic dipole moment **m** pointing upwards and the force on its acting downwards. If the direction of current in the ring is reversed, so will also be for **m** and the force on the ring will act upwards. We can thus say :

1. That when the magnetic dipole moment of a current loop is parallel to the external magnetic field, the forces on the loop act in the direction of increasings field strength ;

2. That when the magnetic dipole moment is anti-parallel to external magnetic field, the forces on the loop act in the direction of decreasing field strength; and

3. That in a uniform external magnetic field the net force on a current loop is zero.

We have derived equation (9) for force on a current loop only in a special case. In general, the moment **m** could be pointing at some odd angle with respect to the field **B**, and the different components of **B** could be varying, in diffeent ways. In such a case the general formula for force on a magnetic dipole of moment **m** in a non-uniform field **B** may be derived as follows

General Expression

The potential energy of magnetic dipole of moment **m** in a magnetic field **B** is

$$U = -\mathbf{m} \,.\, \mathbf{B}$$

The relation between force and potential energy is

$$\mathbf{F} = -\nabla U \qquad ...(10)$$

Substituting value of U, we get

$$\mathbf{F} = +\nabla(\mathbf{m} \,.\, \mathbf{B})$$

Using vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$$

we get

$$\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{m} + \mathbf{B} \times (\nabla \times \mathbf{m}s) + \mathbf{m} \times (\nabla \times \mathbf{B})$$

As magnetic moment **m** is not a function of space-coordinates, we have

$$(\mathbf{B} \cdot \nabla)\mathbf{m} = 0 \text{ and } \nabla \times \mathbf{m} = 0$$

$$\therefore \qquad \mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B} + \mathbf{m} \times (\nabla \times \mathbf{B})$$

This is the general expression for the force

In cases $\nabla \times \mathbf{B} = 0$ (for no current density region)

$$\therefore \qquad \mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B}$$

This expression is analogous to that of electric dipole in non-uniform electric field.

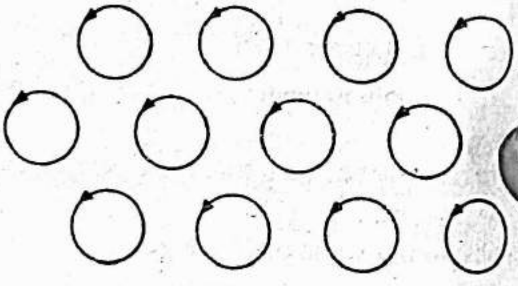
6.17. Magnetisation

We known that all matter consists ultimately of atom, and each atom consists of electron in motion. These electron circuits, each of which is confined to a single atom are known as *atomic currents.* We thus have two types of currents : (1) A true current which consists of charge transport, *i.e.* the current due to motion of free electrons or charged ions, and (2) Atomic currents, which are pure circulatory currents and give rise to no charge transport. However both kinds of current may produce magnetic field :

Each atomic current is a tiny closed circuit of atomic dimension and it may be approximately described as a magnetic dipole. Let the magnetic moment of ith atom be $\mathbf{m}_i$. We now define a *microscopic vector quantity* called the *magnetisation (or intensity of magnetisation)* **M** *as the magnetic dipole moment per unit volume in that limit when the volume considered is infinitely small i.e.*

$$\mathbf{M} = \underset{\Delta v \to 0}{\text{Limit}} \frac{\sum_i m_i}{\Delta v} \qquad ...(1)$$

In the limiting process Δv is made very small from the macroscopic point of view; but not so small that it does not contain a statistically large number of atoms. The magnetisation **m** is thus a vector point function. In an unmagnetised matter, the summation $\sum_i \mathbf{m}_i$ will sum to zero as a result of random orientation of $\mathbf{m}_i$; but in the presence of external exciting field the magnetisation **M** usually depends on the field. The magnetisation **m** provides us with a macroscopic description of the atomic currents inside matter. Specifically **M** *measures the number of atomic current circuits per unit volume times the average of effective magnetic moment of each circuit.* From the purely macroscopic point of view all magnetic effects due to matter can be described in terms of **m** or its derivatives. An important derivative is curl **M** which is equivalent to *transport current density* **J** also known as *magnetisation current density.* Before we derive an important relation between **J** and **M** ; let us consider a simplified model of magnetised matter in which the magnetised matter is assumed to consists of atomic loop currents circulating in the same direction side by side as shown in fig. 6·42.



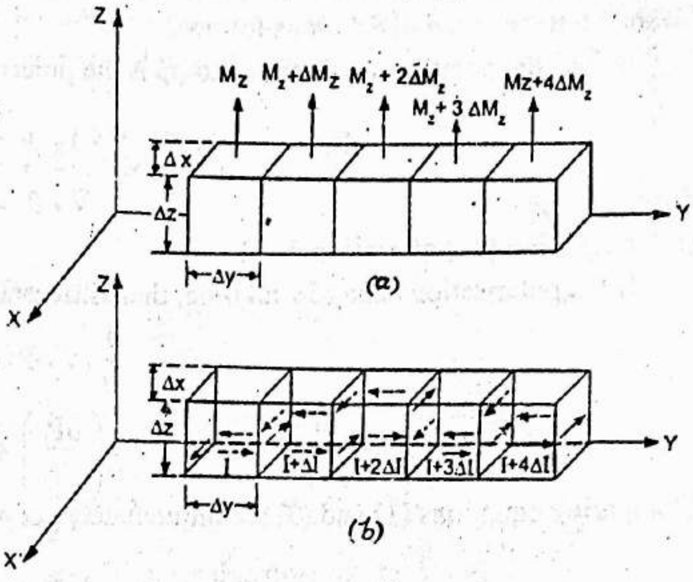
(Fig. 6.42)

If the magnetisation is uniform, then the currents in various loops cancel each other in such a way that there is no net effective current within the material. On the other hand *if the magnetisation is non-uniform, the cancellation of current is incomplete* and *hence there will be a net current even though there is no charge transport.* This current can produce a magnetic field and is known as the *magnetisation current* symbolised by **J**.

Now let us derive the important relationship between **J** and **M**.

Let us consider small volume elements in a piece of magnetic material, each element of volume $\Delta x \, \Delta y \, \Delta z$ and located next to each other in the direction of y-axis, as shown in fig. 6·43 (a). We, first suppose that the non-uniform magnetization in the magnetic material is along the Z-direction and that it becomes stronger as we proceed in the Y-direction. The volume elements are supposed to be so small that we consider the magnetisation as uniform within a single element. If the magnetization in the first volume element is M_z, then the magnetization in the second element is

$$M_z + \frac{\partial M_z}{\partial y} \cdot \Delta y$$



(Fig. 6.43)

and that in the third element is,

$$M_z + \frac{\partial M_z}{\partial y} 2\Delta y$$

and so on. Thus in every element the magnetization is large than the one just left to it by an amount.

$$\Delta M_z = \frac{\partial M_z}{\partial y} \Delta y$$

We can replace each magnetized volume element by its equivalent current ribbon. The first element equivalent current ribbon will carry a current of, $I = M_z \Delta z$. The second element equivalent current ribbon will carry a current of $I + \Delta I = (M_z + \Delta M_z) \cdot \Delta z$, and so on [fig. (6.32)]. Thus the current in each loop is greater than the current in the loop to the left by an amount.

$$\therefore \Delta I = (\Delta M_z)\,\Delta z$$

$$= \frac{\partial M_z}{\partial y} \Delta y\, \Delta z$$

A every interface in this row of elements there is a net current in the x-direction of magnitude ΔI. This x-current is flowing acrosss the cross-section of area $\Delta y\, \Delta z$, of every element. Thus the equivalent current density along the x-direction is

$$J_x = \frac{\Delta I}{\Delta y\, \Delta z} = \frac{\partial M_z}{\partial y}. \quad \text{...(1)}$$

If the non-uniform magnetization also has a component along the y-direction and that if it varies in the z-direction, it will contribute to the current density in the negative x-direction given by,

$$J_x = -\frac{\partial M_y}{\partial z}. \quad \text{...(2)}$$

Combining equation (1) and (2) we will get general formula for the x-component of current density **J** at any point of a magnetized material. Thus,

$$J_x = \left[\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right]$$

$$= (\text{curl } \mathbf{M})_x \quad \text{...(3)}$$

When the non-uniform magnetization **M** varies at every point (x, y, z), then we will get the total current density **J** as,

$$\mathbf{J} = \text{curl } \mathbf{M}. \quad \text{...(4)}$$

Thus a current density equal to that given by equation (4) can produce the same magnetic field as the field produced by the magnetization **M**.

6.18. Currents in Material Media

The currents in material media may be classified as :

1. True Currents J. These currents arises from the physical motion of true charges.

2. Polarisation Currents J. These currents arises from the change of polarisation with time and may be shown to be equal to $\partial \mathbf{P}/\partial t$ as follows :

If J_P is the polarisation current and ρ_P is the polarisation charge density, then by equation of continuity,

$$\nabla \cdot \mathbf{J}_P + \frac{\partial \rho_P}{\partial t} = 0. \qquad ...(1)$$

Also

$$\nabla \cdot \mathbf{P} = -\rho_P \qquad ...(2)$$

(refer equation (14) of section 2·13)

If the polarisation change with time, then differentiating (2) with respect to time t, we get

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_P}{\partial r}$$

or

$$\nabla \cdot \left(\frac{\partial \mathbf{P}}{\partial t} \right) + \frac{\partial \rho_P}{\partial t} = 0 \qquad ...(3)$$

Comparing equations (1) and (3) we immediately get *polarisation current.*

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \qquad ...(4)$$

3. Magnetisation Currents. These are stationary currents that flow within regions that are inaccessible to observation but which might give rise to net boundary or volume currents due to incomplete orbit cancellation on an atomic scale. In the last section magnetisation current $\mathbf{J}_m$ are shown to be

$$\mathbf{J}_m = \text{curl}\, \mathbf{M} = \nabla \times \mathbf{M} \qquad ...(5)$$

where **M** denotes magnetisation.

4. Convective Currents. If the material medium in motion contains charges of various types, additional currents, known as convective currents, are also present. These convective currents may be derived from the motion of both true and polarisation charges. The currents are absent if the medium is stationary.

6.19. Vacuum Displacement Current

If the medium **stationary,** then the total current is given by

$$\mathbf{J}_{total} = \mathbf{J}_{true} + \mathbf{J}_{polarisation} + \mathbf{J}_{magnetisation}$$

$$= \mathbf{J}_{true} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}, \qquad ...(1)$$

For the conservation of charge, total current must obey the equation of continuity, *i.e.*

$$\nabla \cdot \mathbf{J}_{total} + \left(\frac{\partial \rho}{\partial t} \right)_{total} = 0. \qquad ...(2)$$

From *Poisson's equation*

$$\nabla \cdot \mathbf{E} = \frac{\rho_{total}}{\varepsilon_0} \quad \textit{i.e.} \quad \rho_{total} = \varepsilon_0 \nabla \cdot \mathbf{E}. \qquad ...(3a)$$

This yields

$$\left(\frac{\partial \rho}{\partial t} \right)_{total} = \varepsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) \qquad ...(3b)$$

using above equation (2) gives

$$\nabla \cdot \mathbf{J}_{total} = -\varepsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) \qquad ...(4)$$

Obviously the divergence of total current ($\nabla \cdot \mathbf{J}_{total}$) is not zero. Hence the total current is not *solenoidal.* Now we want to find a quantity that should be added to the total current, so that the divergence of ($\mathbf{J}_{total}$ + quantity) may be equal to zero. For the purpose taking divergence of (1) and substituting in (4), we get

$$\nabla \cdot \left(\mathbf{J}_{true} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right) = -\varepsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}$$

This equation may be written as

$$\nabla \cdot \mathbf{J}_{true} + \nabla \cdot \frac{\partial}{\partial t}(\mathbf{P} + \varepsilon_0 \mathbf{E}) + \nabla \cdot (\nabla \times \mathbf{M}) = 0.$$

Using $\mathbf{D} = \mathbf{P} + \varepsilon_0 \mathbf{E}$, we obtain

$$\nabla \cdot \mathbf{J}_{true} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} + \nabla \cdot (\nabla \times \mathbf{M}) = 0$$

or

$$\nabla \cdot \left(\mathbf{J}_{true} + \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{M} \right) = 0. \qquad \text{...(5)}$$

The divergence of the quantity in bracket is zero. Let us express this quantity by **C** *i.e.*

$$\begin{aligned} \mathbf{C} &= \mathbf{J}_{true} + \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{M}. \\ &= \mathbf{J}_{true} + \frac{\partial (\mathbf{P} + \varepsilon_0 \mathbf{E})}{\partial t} + \nabla \times \mathbf{M} \\ &= \mathbf{J}_{true} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_{total} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \qquad \text{...(6)}$$

It is clear that $\mathbf{C} = \mathbf{J}_{total} + \varepsilon_0 (\partial \mathbf{E}/\partial t)$ is a *solenoidal current.*

The need for addition of the term $\varepsilon_0 (\partial \mathbf{E}/\partial t)$ in the total nonsolenoidal current to produce *solenoidal current vector* **C** was recognised by Maxwell. This additional term $\varepsilon_0 (\partial \mathbf{E}/\partial t)$ is known as *vacuum displacement current.* It does not have the significance of a current in the sense of being the motion of charges. It may be pointed out that the magnetic effects of currents can be formulated only in terms of solenoidal currents and hence the vacuum displacement current term must be introduced in order to be able to apply the formulas for the magnetic interaction to cases involving non-stationary currents.

6.20. Definition and Role of Magnetic Field Intensity.

We have seen in section 5·9 that for stationary cases.

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J} \text{ (for free space)}$$

where **J** represents *total current density* which is not solenoidal if polarisation **P** varies with time. To apply this relation to non-stationary cases, it is necessary for the current density to remain *solenoidal.* In the previous section, we have seen that current density $\mathbf{C} = \mathbf{J}_{total} + \varepsilon_0 (\partial \mathbf{E}/\partial t)$ is solenoidal. Therefore if **C** is used as the total current, then the defining equations for vector field **B** are

$$\text{div } \mathbf{B} = \nabla \cdot \mathbf{B} = 0 \qquad \text{...(1)}$$

and

$$\text{curl } \mathbf{B} = \nabla \times \mathbf{B} = \mu_0 \mathbf{C} = \mu_0 \left(\mathbf{J}_{true} + \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{M} \right) \qquad \text{...(2)}$$

It is obvious from the expression for curl **B** that the total magnetic field arises due to true currents, magnetisation currents and displacement currents. It is convenient to separate from the total field that part whose circulation density arise from atomic magnetisation currents. Therefore writting equation (2) in the

$$\nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) = \mu_0 \left(\mathbf{J}_{true} + \frac{\partial \mathbf{D}}{\partial t} \right) \qquad \text{...(3)}$$

Now let us introduce a new vector quantity **H** known as magnetic field intensity and defined by

$$\mathbf{H} = \frac{\mathbf{B}-\mu_0\mathbf{M}}{\mu_0} = \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) \quad ...(4)$$

then equation (3) takes the form

$$\nabla \times \mathbf{H} = \mathbf{J}_{true} + \frac{\partial \mathbf{D}}{\partial t} \quad ...(5)$$

This equation shows that the circulation density of A arises from the true current plus the total displacemnet current, the latter including both the polarisation current $\partial \mathbf{P}/\partial t$ and the vacuum displacement current $\varepsilon_0 (\partial \mathbf{E}/\partial t)$.

Under *stationary or quasi-static state* $\partial \mathbf{D}/\partial t$ is smali campared to $\mathbf{J}_{true}$, hence

$$\nabla \times \mathbf{H} = \mathbf{J}_{true} \quad ...(6a)$$

or in integral form

$$\oint \mathbf{H} \cdot d\mathbf{l} = \mathbf{J}_{true} \quad ...(6b)$$

If may be noted that **B** plays a role that corresponds to **E** while **H** plays a role that corresponds to **D** as may be seen by comparing above equations with analogous electrostatic equations

$$\left.\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{q_{total}}{\varepsilon} \\ \oint \mathbf{D} \cdot d\mathbf{a} &= q_{true} \end{aligned}\right\} \quad ...(7)$$

5.21. Magnetic Materials : Magnetic Susceptibility and Permeability.

All materials, with respect to their bulk magnetic properties, may be classified as :
(i) diamagnetic, (ii) paramagnetic, (iii) ferromagnetic.

For distinguishing above classes of magnetic materials, let us first define *magnetic permeability* and *susceptibility*. As already pointed out that the *magnetic polarisation* or *magnetisation* **M** is defined as magnetic moment per unit volume

i.e.

$$\mathbf{M} = \frac{d\mathbf{m}}{dv} = \frac{\mathbf{m}}{V} \quad ...(1)$$

and *permeability* μ is defined as

$$\mu = \frac{\mathbf{B}}{\mathbf{H}} \quad i.e. \quad \mathbf{B} = \mu \mathbf{H}. \quad ...(2)$$

Also the *magnetic field intensity* **H** is defined as

$$\mathbf{H} = \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) \quad ...(3a)$$

i.e.

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \quad ...(3b)$$

For simple isotropic materials **M** is found to be proportional to **H** and parallel to it *i.e.*

$$\mathbf{M} \propto \mathbf{H}$$

or

$$\mathbf{M} = \chi_m \mathbf{H}, \quad ...(4)$$

where χ_m is a dimensionless constant known as *magnetic susceptibility and is a characteristic of the medium.*

In most hand books on physical data χ_m is not listed directly but instead is given as the *mass susceptibility* $\chi_{m,\,mass}$ and molar susceptibility $\chi_{m,\,molar}$ defined by

mass susceptibility

$$\chi_{m,\,mass} = \frac{\chi_m}{\rho}, \quad ...(5)$$

where ρ is the mass density.

,molar susceptibility
$$\chi_{m,\,molar} = \frac{\chi_m \rho}{A}, \qquad ...(6)$$

where A is the molecular weight.

Since **M** and **m** both have the dimensions of magnetic moment per unit volume,, it is evident that $(\chi_{m,\,mass}\,\mathbf{H})$ and $(\chi_{m,\,molar}\,\mathbf{H})$ give magnetic moment per unit mass and magnetic moment per mole respectively.

Important relations. Let us now find important relations between various magnetic quantities.

Substituting values of **B** and **M** from equations (2) and (4) in equation (3), we get

$$\mathbf{H} = \left(\frac{\mu \mathbf{H}}{\mu_0} - \chi_m \mathbf{H}\right)$$

This equation yields

$$\mu = \mu_0 (1 + \chi_m) \qquad ...(7)$$

or
$$\frac{\mu}{\mu_0} = (1 + \chi_m). \qquad ...(8)$$

But
$$\frac{\mu}{\mu_0} = K_m = \text{relative permeability of medium.}$$

Hence equation (8) may be expressed as

$$K_m = 1 + \chi_m.$$

Now let us distinguish between diamagnetic, paramagnetic and ferromagnetic substances.

Diamagnetic Substances. Diamagnetic substances have negative values of magnetic susceptibility which is practically independent of temperature and consequently possess permeability [refer equation (7)] that is less than free space. When placed in a strong magnetic field they display magnetic moments that are antiparallel to the field. By the presence of such materials the magnetic induction **B** is weekened than the free space value. In a non-uniform field they experience a force towards the weaker portions of the field. Most of the simple insulators and approximately half the simple metals are diamagnetic. The examples are copper, silver, gold, bismuth, diamond, sodium, water, carbon dioxide (1 atm), hydrogen (1 atm) and nitrogen (1 atm) etc.

Paramagnetic Substances. Paramagnetic substances are characterised by small positive values of magnetic susceptibility. When placed in a strong magnetic field they display magnetic moments that are parallel to the applied field. The magnetic induction is increased by the presence of such materials. In a non-uiniform field they experience a force towards the stronger portion of the field. For paramagnetic substance χ_m usually increases with decrease of temperature and is about 10^{-3} at room temperature.

When the temperature is not very small, the paramagnetic substances obey **Curie law,** according to which the magnetic susceptibility of paramagnetic substances is inversely proportional to the absolute temperature *i.e.*

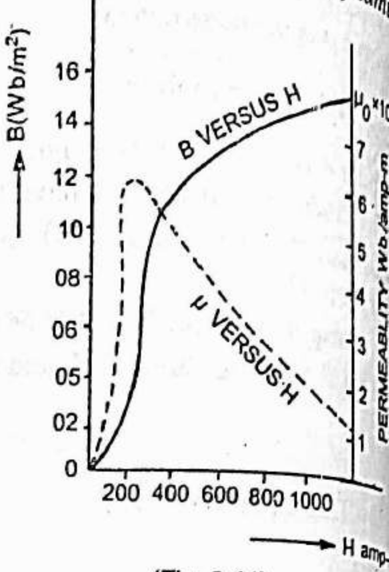
$$\chi_m \propto \frac{1}{T}$$

or
$$\chi_m = \frac{\lambda}{T}$$

where λ is a constant, called **Curie constant.**

For these substances permeability is greater than that of free space, thus for air it is $\mu_{air} = (1 + 0{\cdot}38 \times 10^{-6})\,\mu_0$ and for aluminium $\mu_{Al} = (1 + 0{\cdot}25 \times 10^{-4})\,\mu_0$. The examples of paramagnetic substances are aluminium, magnesium, titanium, tungsten, oxygen (1 atm) and air etc.

Ferromagnetic Substances. Ferromagnetic substances are characterised by large positive values of magnetic susceptibility. The relative permeability of these substances is very high of the order of hundered or even several thousands. In general the flux density B is not a linear function of magnetic intensity H or in other words the *permeability* μ *is not constant but depends upon the strength of the magnetic field and upon their past history.* Thus the equation $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{M} = \chi_m \mathbf{H}$ are not applicable to these substance. It is customary to use $\mathbf{B} = \mu\mathbf{H}$ as defining equation for μ, but this practice can lead to a difficulty to certain situations. If the μ of a ferromagnetic substance is defined by $\mathbf{B} = \mu\mathbf{H}$, then depending on the value of $\mathbf{H}$, μ goes through an entire range of values from infinity to zero and they may be either positive or negative. Iron, nickel, cobalt and godolinium are the only ferromagnetic elements at room temperatue, but a number of elements at low temperatures and Heusler alloys show these effects.



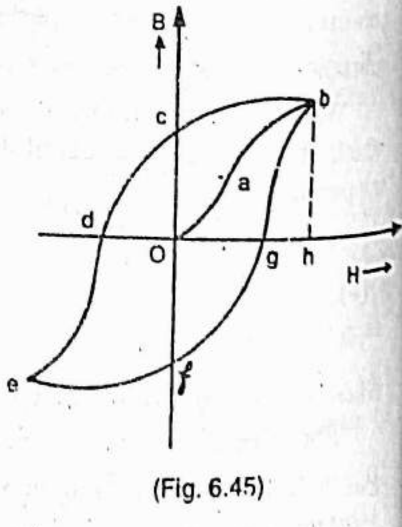
(Fig. 6.44)

The susceptibility of ferromagnetic substances decreases with rise of temperature. Above a certa temperature, called the Curie temperature, ferromagnetic substances change to paramagnetics. It advisable to consider each problem involving ferromagnetism separately and try to determine which regio of B-H curve is important for the particular problem and make approximations appropriate to this region.

Let us consider an unmagnetised sample of ferromagnetic material say *annealed iron.* If the magnet intensity initially zero is increased *monotonically,* then B—H curve obtained is shown in fig. 6·44. This called the magnetisation curve of the material. It is obvious that the value of μ taken fro magnetisation curve using the relation $\mu = B/H$ is not a constant, the dotted curve represents graph of μ as a function of H. The maximum permeability occurs at the *knee* of the curve, in som materials the maximum permeability is $10^5\,\mu_0$; in others it is much lower. The reason for the *knee* in th curve is that the magnetisation $\mathbf{M}$ reaches a maximum value in the material and $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ continue to increase at large values of $\mathbf{H}$ due to the term $\mu_0\,\mathbf{H}$ only. The maximum value of $\mathbf{M}$ is called the *saturatio magnetisation* of the material.

6.22. Hysteresis and Rowland Ring Method for B-H Curve

As already pointed out that for ferromagnetic substances the flux density B is not a linear function of magnetic intensity H: because in such cases the relative permeability μ_r or K_m is not constant, but is a function of H. In other words there is no unique value of K for a particular ferromagnetic substance. The relationship between flux density B and corresponding magnetic intensity H for such a material initially unmagnetised is represented by a typical curve shown in fig. 6·45. known as the *magnetisation curve or B-H curve.* Thus in fig. 6·45, if the unmagnetised sample of ferromagnetic substance is magnetised by steadily increasing the magnetic intensity from zero until the magnetic intensity corresponds to the abscissa Oh the flux density B follows the curve Oab. In a field of a few hundred amp/m, the value of B becomes practically constant. If the magnetic intensity H is now reduced to zero, the magnetic flux density B does not return on the path baO but follows the path bc and even at $H=0$, the flux

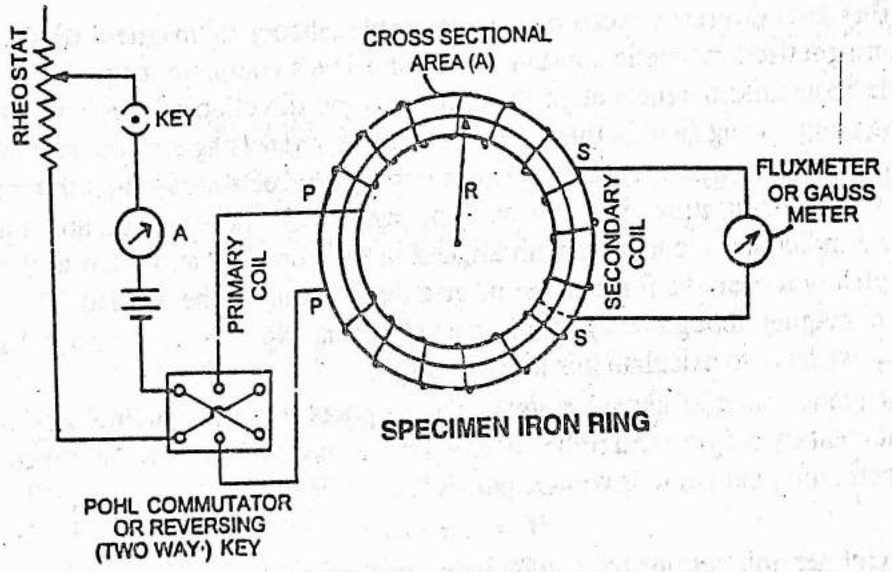


(Fig. 6.45)

density is Oc. The flux density in the sample is seen to depend upon not on the magnetic intensity alone but on the magnetic history of the sampel as well. At point c the specimen has become a *permanent magnet* since magnetisation is still present even though the magnetising field H has been cut off. The value of B *i.e.* Oc at $H = 0$ is known as *residual induction* (or residual density) adn the retention of magnetisation by the specimen is known as *retentivity* or *remanence.* When the magnetic field H is reversed, the magnetic flux density B falls and becomes zero for $H = Od$. This value of H is denoted by H_c and is called the *coercive force* or the *coercivity.* When the magnetic field is increased still further in the reverse direction, a reverse induction is set up in the specimen which quickly, attains the *saturation value.* If we reduce the reverse field gradually, the induction follows the curve ef and attains value Of for $H = 0$. When the field H is increased again in the positive direction, the induction traces out the curve fgb. The closed curve $bcdefgb$ is known as *hysteresis loop.* In the entire hysteresis loop *the magnetic induction (and magneitsation too) lags behind the applied magnetic field H.* This phenomenon is known as *hysteresis,* from the Greek world meaning "to lag".

Rowland ring method of obtaining B—H curve of a Ferromagnatic Substance.

To obtain B—H curve of a given ferromagnetic substance such as iron, a thin closed ring, known as *Rowland ring,* is cut from an iron sample and then a uniform primary winding is placed over the ring forming an iron cored toroid. The secondary coil placed over the ring is connected to a fluxmeter or a gaussmeter. The circuit diagram is shown in fig. 6·46. If the radius of the Rowland ring is large compared with dimensions of its crossection, a current carrying coil would uniformly round the ring, will produce a uniform field H everywhere within the ring. If N is the total number of turns in the primary coil carrying a current I and R the mean radius of toroid; then according to Ampere's circuital law along the path of radius R inside the toroid, we have



(Fig. 6.46)

$$\oint \mathbf{H} \cdot d\mathbf{l} = NI. \qquad ...(1)$$

Since by symmetry **H** is same along the circular path everywhere, then

$$H \cdot 2\pi R = NI$$

or

$$H = \frac{NI}{2\pi R} \text{ ampere turn/metre,} \qquad ...(2)$$

This value of H is called the *applied magnetising field* and it can be varied by changing the current I. The flux density **B** in the ring may be considered to due to be applied magnetising field H and is measured by the fluxmeter or gaussmeter connected to secondary coil. If ϕ_m is the *magnetic flux* read by fluxmeter, then flux density $\mathbf{B} = (\phi_m/A)$, where A is the cross-sectional area of the ring. If magnetic field is a read by gaussmeter, then $B =$ (reading of gaussmeter) gauss $=$ (reading of gaussmeter) $\times 10^{-4}$ weber/m^2, the relationship between B and H for an iron sample by using Rowland ring method is shown in fig. 6·34, provided the sample is initially unmagnetised and the magnetic intensity H is steadily increased from zero value by increasing the value of current I from is zero value.

To plot part ob of the curve. The unmagnetised sample is magnetised by the increasing current I in the circuit from its zero value in steps of 0·1 ampere and flux density B is measured by fluxmeter or Gaussmeter each time until the flux density B attains constant value. This gives part *ob* of the curve.

To plot part bc of the loop. The specimen is now demagnetised by decreasing current in steps of 0·1 amperes and corresponding flux density B is measured by fluxmeter. When current is zero, magneting field is zero, the corresponding flux density B is noted. This gives part *bc* of the curve.

To plot part cde of the curve. The direction of current I now reversed by Pohl commutator and current I is again increased in steps of 0·1 ampere until fluxmeter gives zero reading. This corresponds to point d of curve. The current is further increased in steps of 0·1 ampere. This time fluxmeter gives reading in reverse direction and fluxmeter reading is noted for each value of current until the reading of fluxmeter becomes maximum. This gives part *cde* of the loop.

To complete the loop. The part *efgh*, *i.e.*, remaining half loop may be plotted either by symmetry or by repeating experiment using *e* as refrence point.

Energy loss due to hysteresis. According to molecular theory of magnetisation, the molecule of a magenetised or unmagnetised magnetic substance are themsleves complete magnets. When a magnetising field is applied, the molecular magnets align themselves in the direction of the field. During this process work is done by the magnetising field in turning the molecular magnets against the mutual attraction forces. This energy required to magnetise a specimen is not completely recovered when the magnetising field is turned off, since the magnetisation does not become zero. The specimen retains some magnetisation because some of the molecular magnets remain aligned in the new formation due to the group forces. To tear them out completely a coercive force in the reverse direction has to be applied. Thus there is a loss of energy in taking a magnet though a cycle of magnetisation. Now we have to calculate this loss of magnetisation. Now we have to calculate this loss of energy.

Consider a magnetic material having n molecular magnets per unit volume. Let m be the magnetic moment of each elementary magnet and θ the angle which its axis makes with the direction of magnetising field H, then magnetic moment per unit volume parallel to the field is

$$M = \Sigma m \cos\theta . \quad ...(1)$$

The magnetic moment per unit volume perpendicular to the field is

$$\Sigma m \sin\theta ,$$

and this is equal to zero since there can be no magnetisation perpendicular to H.

Now, the torque due to the field acting on the dipole of moment m when it is inclined at an angle θ to field is

$$\tau = \mu_0 m H \sin\theta \quad ...(2)$$

and the work done when it moves through a small angle $+ d\theta$ is $- \mu_0 mH \sin\theta$. Here minus sign comes in because the work has to be done against the magnetic field in increasing θ by $d\theta$. Hence the work done per unit volume of the material is

$$dW = -\Sigma \mu_0 mH \sin d\theta \quad ...(3)$$

As θ increases by $d\theta$, the intensity of magnetisation M also increases by dM obtained from equation (1) as

$$dM = d(\Sigma m \cos\theta)$$

$$= -\Sigma m \sin\theta \, d\theta . \qquad ...(4)$$

Inserting equation (4) into (3), we get the work done per unit volume in turning the elementary magnets through a small angle θ is

$$dW = \mu_0 H dM . \qquad ...(5)$$

Thus work done by the magnetising field per unit volume of the material for completing a cycle is

$$W = \oint \mu_0 H dM = \mu_0 \oint H \, dM$$

$$= \mu_0 \times \text{area of } M - H \text{ loop.}$$

Since the magnetic induction B is related to magnetisation M by the relation,

$$B = \mu_0 (H + M)$$

or

$$dB = \mu_0 (dH + dM). \qquad ...(6)$$

This implies,

$$dM = \frac{dB}{\mu_0} - dH \qquad ...(7)$$

Thus equation (6) may be written as

$$W = \mu_0 \oint H \left(\frac{dB}{\mu_0} - dH \right)$$

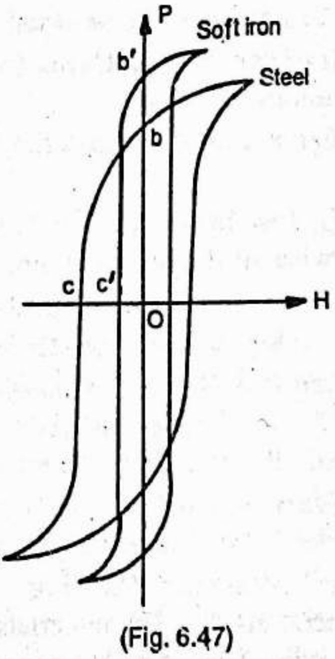
$$= \oint H \, dB - \mu_0 \oint H \, dH \qquad ...(8)$$

but $\oint H \, dH = 0$ because the plot of H against H is a straight line and the area enclosed by it is zero. Thus equation (8), gives

$$W = \oint H \, dB = \text{area of } B - H \text{ loop.}$$

Thus, the work done per unit volume of the material per cycle is equal to area of μ_0 times the area of $M - H$ loop or the area of $B - H$ loop. The work is measured in joule/m^3 per cycle and is dissipated in the form of heat.

6.23. Uses of Hysteresis Cycle :



(Fig. 6.47)

The hysteresis curves of a ferromagnetic can be used to study its magnetic properties. The B—H curves for soft-iron and steel are shown in Fig. 6·47. A study of these curves gives the following information regarding the magnetic properties of soft-iron and steel :

(i) Permeability : The value of the flux density B is greater for soft iron than that for steel for the same value of H. Therefore, *the permeability* ($\mu = B/H$) *is also greater for soft-iron than for steel.*

(ii) Retentivity : When the magnetising field H is reduced to zero, soft-iron retains flux density Ob' which is greater than the magnetic flux Ob retained by steel. Hence *the retentivity of soft-iron is greater than that of steel.*

(iii) Coercivity : To demagnetise the soft iron a coercive force Oc' is required which is smaller than the coercive force, Oc required for steel. Hence, *the coercivity of soft-iron is less than that of steel.*

(iv) Hysteresis Loss : The area of the B—H loop for soft-iron is smaller than that for steel. Hence *the hysteresis loss per unit volume per cycle is less for soft iron than for steel.*

Use of Hysteresis curve : *Choice of materials* (For (i) permanent magnets (ii) electromagnets (iii) Transformer cores (iv) Magnetic shielding). The choices of magnetic materials for making permanent magnets, electromagnets, cores of transformer and magnetic shielding can be decided from the hysteresis curve of the specimen.

(a) Permanent Magnets : The materials used for permanent magnet must have the following characteristics :

(i) high retentivity so that the magnet may cause strong magnetic field (even in the presence of mechanical disturbance).

(ii) high retentivity so that the magnetisation is not wiped out by strong external fields, mechanical ill-treatment and temperature changes. The loss due to hysteresis is immaterial because the magnet in this case is never put to cyclic changes.

According to these considerations *steel is better for permanent magnets than soft iron.* The fact that the retentivity of iron is a little greater than and steel is completely outweighed by its much smaller coercivity which makes it very easy to demagnetise. The materials used for permanent magnets are cobalt-steel, tungstensteel, chromium steel and a number of other alloys.

Recently various alloys such as Alnico (Ni 18%, Al 10%, Co 12%, Cu 6%, Fe 54%) and vicalloy (composed of iron, cobalt, vanadium) etc. have been developed which are most suitable for permanent magnets.

Alnico has a coercivity of 500 oersteds = 0·05 Tesla and retains 70% of its magnetism at even a high temperature of 600°C and more than 80% in alternating fields which will be sufficient to demagnetise cobalt-steel.

(b) Electromagnets : The material used for cores of electromagnets must have.

(i) maximum flux density with comparatively small magnetising field.

(ii) high initial permeability.

(iii) low hysteresis loss.

(iv) low coercivity.

(v) high retentivity.

considering these facts *soft iron* is an ideal material for electromagnets.

(c) Transformer Cores. (or Telephone Diaphragms, Armature of Dynamos and Motors and Cores of Choke).

In these cases the material is subjected to cycle changes of magnetisation. The material, therefore, must have

(i) low hysteresis loss [for dissipation of energy per cycle and hence a small heating of material otherwise insulation of windings may break]

(ii) high initial permeability (to get a large flux density at low fields)

(iii) high specific resistance (to reduce eddy current losses)

Soft iron is a good material for these purposes. The initial relative permeability of soft iron is 250 and an addition of 4% silicon produces *transformer steel* which is ideal for cores of transformers. Alloys of iron and nickel called *Permalloys* have high values of initial, permeabilities and are suitable for cores.

Mumetal (Ni 41%, Fe, 48·5% Gu 3%, Mn 0·5%) have very high values of initial permeabilities and low hysteresis loss and hence is very suitable for such purposes.

(d) Magnetic shielding : It is often required to shield the electronic instruments, watches etc. from magnetic effects. The materials used for magnetic shield must have high saturation induction and very low coercivity. The momental and permalloy are widely used for magnetic shielding.

6.24. Measurement of Susceptibility

(a) Quincke's Method :

Principle : When a paramagnetic substance is placed in a *non-uniform* magnetic field, it tends to move towards the region of greatest field intensity and so it is subjected to a net force. A measurement of this force enables us to determine the susceptibility of the substance.

If F is the force acting on the substance which moves through a distance δx, then

work done by the force $= F\,\delta x$ ...(1)

and is equal to the change in potential energy resulting from the displacement.

Let us consider that the substance is initially outside the field. Let us consider a point where the field intensity is H. If μ_0 be the permeability constant, then the energy per unit volume of the medium is $\frac{1}{2}\mathbf{B}\cdot\mathbf{H} = \frac{1}{2}\mu_0\,\mathbf{H}\cdot\mathbf{H} = \frac{1}{2}\mu_0 H^2$. If V is the volume of substance, then energy contained by volume V of field into which the substance is to be placed $\frac{1}{2}\mu_0 H^2 V$ joules. When the substance is in field on, the energy is $\frac{1}{2}\mu H^2 V$ joules, where μ is the permeability of the substance. Thus the energy resulting from the introduction of the substance into the field is $\frac{1}{2}(\mu - \mu_0)\,H^2 V$ joules.

The value of H^2 at displacement δx is

$$\left(H^2 + \frac{dH^2}{dx}\,\delta x \right)$$

When substance moves through a distance δx, the energy becomes $\frac{1}{2}(\mu - \mu_0)\left(H^2 + \frac{dH^2}{dx}\,\delta x \right) V$ joule.

Hence

$$\text{change in energy } -\tfrac{1}{2}(\mu - \mu_0)\left(\frac{dH^2}{dx}\,\delta x \right) V$$

As
$$\mu_r = 1 + \chi$$
or
$$\mu = \mu_0 + \mu_0 \chi.$$

Therefore, in terms of susceptibility, we have change in energy

$$= \tfrac{1}{2}\mu_0 \chi \left(\frac{dH^2}{dx} - \delta x \right) V \qquad ...(2)$$

Equation (1) and (2), we get

$$F = \tfrac{1}{2}\mu_0 \chi \frac{dH^2}{dx} \qquad ...(3)$$

Experimental Arrangement : The given specimen (paramagnetic solution) is placed in U-tube consisting of a wide and a narrow limb. The wide limb is placed outside the field, and the narrow limb inside the magnetic field, provided by an electromagnet with wedge-shaped pole-pieces (Fig. 6·48). The field varies rapidly along the vertical direction due to the wedging of the polepieces. Thus, the force on the specimen will be vertical.

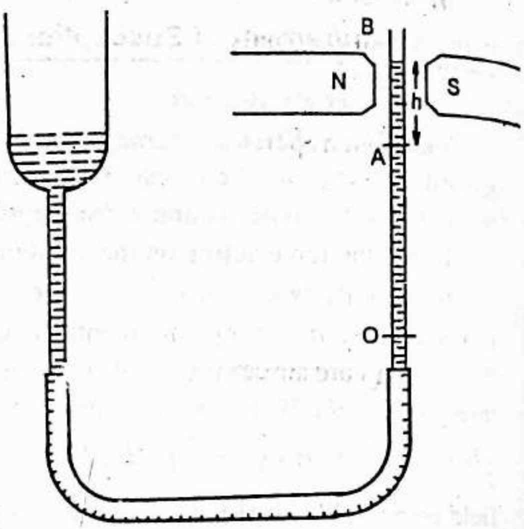
Let A be the position of the liquid level before the magnetic field is generated. This position is noted by means of a micrometer microscope. The electromagnet is now switched on. The level of the paramegnetic liquid in the narrow limb rises to B. This new position of the level is noted by the microscope. Thus the height AB is obtained. Let it be h. This is the height of the liquid column supported by the forces arising from the magnetic field. If a be the area of cross-section of the narrow limb ρ the density of the liquid, and g the acceleration due to gravity, then $h\rho ga$ is the vertical force on the liquid arising from the magnetic field.

Now let O be a section where the field is negligible. Let x be the vertical coordinate of O. The force on an element dx on the liquid at O is, from eq. (3),

$$dF = \frac{1}{2}\mu_0 x \frac{dH^2}{dx} V$$

$$= \frac{1}{2} F_0 x \frac{dH^2}{dx} (a\,dx).$$

The force on the entire liquid above the potent O is therefore,



(Fig. 6.48)

$$F = \frac{1}{2}\mu_0 \chi a \int_0^{H_1} \frac{dH^2}{dx}\, dx,$$

where H_1 is the field intensity at the upper level of the liquid *i.e.* at B. Thus

$$F = \frac{1}{2}\mu_0 \chi a H_1^2.$$

The force F on the liquid due to the magnetic field has been shown above as equal to $h\rho ga$. Thus

$$\frac{1}{2}\mu_0 \chi a H_1^2 = h\rho g a$$

or

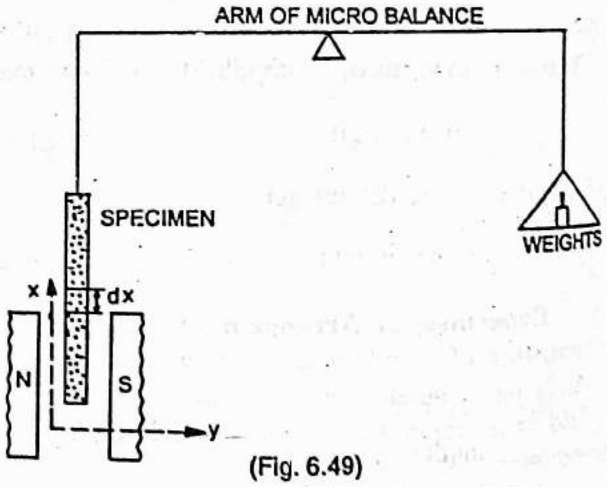
$$x = \frac{2h\rho g}{\mu_0 H_1^2}.$$

h has already been determined. H_1 can be measured by the fluxmeter. Hence knowing the density ρ of the liquid, susceptibility χ can be evaluated.

(b) Gouy's Method for a solid or a liquid.

Apparatus : It consits of an electromagnet having parallel and flat pole pieces with a gap between them. The electromagnet provides a strong horizontal magnetic field of nearly 1 weber/m^2.

The specimen is taken in the form of a cylindrical rod nearly 10 cm. long. If the magnetic substance is in liquid or powder form, it is taken in a glass or quartz cylindrical tube. The specimen in the form of cylinder is suspended vertically within the gap of pole pieces in such a way that its lower end is near the mid point of the magnetic field where the magnetic intensity H is large while its upper end is out side the field in the region of low intensity H_0. The cylindrical is then weighed by suspending it from a sensitive microbalance in the two cases :



(Fig. 6.49)

(i) when field is off (ii) when field is on.

Theory : Consider a magnetic field of intensity H in a region of permeability μ_1. This will usually be air but not free space. Let a small body of volume V have a permeability μ_2. When inserted in the field it will replace a volume V of medium of permeability μ_1. Hence the energy of magnetic flux in this region was

$\frac{1}{2}\mu_1 H^2 V$ before inserting the specimen and $\frac{1}{2}\mu_2 H^2 V$, after inserting the specimen. The difference of these energies will be equal to the work done in inserting the specimen.

$$\therefore \qquad \text{Work done} = \tfrac{1}{2}(\mu_2 - \mu_1) H^2 V$$

This is stored as potential energy of field *i.e.* change in potential energy of field,

$$U = \tfrac{1}{2}(\mu_2 - \mu_1) H^2 V \qquad ...(1)$$

As

$$\mu_0 = 1 + \chi_m \quad \text{or} \quad \frac{\mu}{\mu_0} = 1 + \chi_m$$

$$\mu = \mu_0 + \mu_0 \chi_m$$

If χ_1 and χ_2 are susceptibilities of air and specimen, we have

$$\therefore \qquad \mu_1 = \mu_0 + \mu_0 \chi_1 \quad \text{and} \quad \mu_2 = \mu_0 + \mu_0 \chi_2$$

$$\therefore \qquad \mu_1 - \mu_1 = \mu_0 (\chi_2 - \chi_1) \qquad ...(2)$$

Substituting this in (1), we get

$$U = \tfrac{1}{2}\mu_0 (\chi_2 - \chi_1) H^2 V$$

If E_x is the force acting on the specimen along X-axis then

$$F = -\frac{dU}{dx} = -\frac{d}{dx}\{\tfrac{1}{2}\mu_0(\chi_2 - \chi_1) H^2 V\}$$

$$= -\tfrac{1}{2}\mu_0 (\chi_2 - \chi_1) \frac{d}{dx}(H^2) V$$

If H_x, H_y and H_z are components of **H**, then

$$H^2 = H_x^2 + H_y^2 + H_z^2$$

$$\therefore \qquad F = -\tfrac{1}{2}\mu_0 (\chi_2 - \chi_1) \left\{ \frac{d}{dx}(H_x^2 + H_y^2 + H_z^2) \right\} V$$

$$= -\tfrac{1}{2}\mu_0 (\chi_2 - \chi_1) V \left[2H_x \frac{dH_x}{dx} + 2H_y \frac{dH_y}{dx} + 2H_z \frac{dH_z}{dx} \right]$$

$$\therefore \qquad F_x = +\mu_0 (\chi_2 - \chi_1) \left[H_x \frac{dH_x}{dx} + H_y \frac{dH_y}{dx} + H_z \frac{dH_z}{dx} \right] V \qquad ...(4)$$

Now considering X-axis to be vertical, the vertical force on a small element dx of rod of volume dV, at a distance x from origin, is

$$dF_x = -\mu_0 (\chi_2 - \chi_1) \left[H_x \frac{dH_x}{dx} + H_y \frac{dH_y}{dx} + H_z \frac{dH_z}{dx} \right] dV$$

If a is cross-sectional area of rod, then

$$dV = a\,dx$$

$$\therefore \qquad dF_x = -\mu_0 (\chi_2 - \chi_1) \left[H_x \frac{dH_x}{dx} + H_y \frac{dH_y}{dx} + H_z \frac{dH_z}{dx} \right] a\,dx \qquad ...(5)$$

In the narrow gap between pole pieces, the magnetic flux will consist of straight lines direct from one pole face to another, in y direction say. Thus only the component H_y will be of significant magnitude while H_z and H_x will be negligible. Hence equation (5) is expressible as

$$\therefore \qquad dF_x = -\mu_0 (\chi_2 - \chi_1) H_y \frac{dH_y}{dx} a\,dx \qquad ...(6)$$

To find the total vertical force on the rod due to whole field variation along the length under the limit $H_y = H$ to $H_y = 0$

i.e.
$$F_x = -\mu_0 (\chi_2 - \chi_1)\, a \int_H^{H_0} H_y \frac{dH_y}{dx}\, dx$$
$$= -\mu_0 (\chi_2 - \chi_1)\, a \int_H^{H_0} H_y\, dH_y$$
$$= -\mu_0 (\chi_2 - \chi_1)\, a \left[\frac{H_y^2}{2}\right]_H^{H_0}$$
$$= -\mu_0 (\chi_2 - \chi_1)\, a\, (H_y^2 - H^2)$$
$$= \mu_0 (\chi_2 - \chi_1)\, a\, (H^2 - H_a^2) \qquad ...(7)$$

If m_1 and m_2 (kg) are the weights to counterpoise the weight on rod with magnetic field off and on respectively, then

$$F_x = (m_2 - m_1)\, g. \qquad ...(8)$$

Equating (7) and (8), we get

$$(m_2 - m_1)\, g = \mu_0 (\chi_2 - \chi_1)\, a\, (H^2 - H_0^2)$$

i.e.
$$(\chi_2 - \chi_1) = \frac{(m_2 - m_1)\, g}{\mu_0\, a\, (H^2 - H_0^2)}$$

knowing the value of area a, H_0 and H by gauss-meter, $\chi_2 - \chi_1$ and hence χ_2 may be determined.

6.25. Langevin's Theory of Diamagnetism

Langevin, in 1905, provided a satisfactory explanation for the diamagnetism exhibited by all matter on the basis of electron theory. According to electron an atom is made up of a central positive nucleus and a number of electrons revolving around it in descrete circular orbits. We know that a revolving electron is equivalent to an electric current ring which produces the magnetic field prependicular to the plane of the orbit and has a magnetic dipole moment. Since an atom, in general, contains several such current rings, the resultant magnetic moment is the vector sum of magnetic moments of all the current rings. According to *Langevin an atom of a diamagnetic substance is such that the vector sum of all the magnetic moments of current rings is zero.* When an external magnetic field is applied to such an atom, the electronic orbit undergoes precessional motion about the direction of the field without any change in the form of the orbits, in much the same manner as a spinning top that is tilted with respect to vertical, precesses about the vertical. This precession is known as *Larmor precession* and represents criculation of charge (in effect, either an increase or decrease of the angular velocity, depending upon their direction of motion). This simply means that the motion of the system is the same as it would be in the absence of the field except that a uniform rotation around an axis parallel to the field is superimposed. Consequently, the perfect balance of the magnetic moment within the atom is disturbed. *The electronic orbits shift slightly relative to each oher but it such a manner so as to produce a negative magnetic moment of the atom as a whole* (*i.e.* the material is magnetised in the direction opposite to that of the field). Let us calculate this change in magnetic moment mathematically.

Let us consider a single electron circulating in a diamagnetic atom at angular freqeuency ω_0 in a circular orbit of radius r in the absence of external field. Each electron is moving in circular orbit with speed v_0 under the action of centripetal force F_e of electrostatic origin *i.e.*

$$F_e = \frac{mv_0^2}{r} = m\,\omega_0^2 r. \qquad ...(1)$$

Each rotating electron has an orbital magnetic moment; but for the atom as a whole the orbits are randomly oriented, so that the net magnetic moment is zero. What a uniform external field

$$\mathbf{B} = \mu_0 \mathbf{H},$$

is applied perpendicular to the plane of the orbit, an *additional magnetic force* given by

$$\mathbf{F}_m = -e\,(\mathbf{v} \times \mathbf{B})$$

acts on the electron. This force always acts at right angles to the direction of motion, its magnitude is

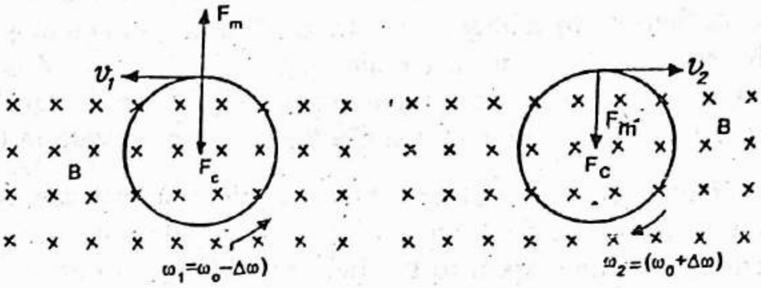
$$\mathbf{F}_m = ev\,B = e\,(\omega r)\,B. \qquad ...(2)$$

It may be noted that in figure [6·50 (a)] B_c and $\mathbf{F}_m$ point in opposite directions; while in figure [6·50 (b)[they point in the same directions.

Due to the action of F_m the angular velocity would change : thus ω and ω_0 are different.

This change in angular frequency with sign may also be calculated by the use of Faraday's law of induction as follows :

When an external magnetic field B is applied perpendicular to the plane of the electronic orbit; then it will induce an e.m.f. ε in the loop given by


F_m
v_1
v_2
B
F_c
F_m
B
F_c
$\omega_1 = \omega_0 - \Delta\omega$
$\omega_2 = (\omega_0 + \Delta\omega)$

(Fig. 6.50)

$$\varepsilon = -\frac{d\phi}{dt}.$$

This e.m.f. results in an electric field $\mathbf{E}$ which acts tangentially to the direction on electronic motion. This field given by $\mathbf{E} = \varepsilon/2\pi r$ will impart energy to the electron. By Newton's law

$$m\frac{dv}{dt} = eE$$

$$= e\frac{\varepsilon}{2\pi r} = -\frac{e}{2\pi r}\frac{d\phi}{dt}. \qquad ...(5)$$

If the field changes from O to B, the change in flux will be given by $\Delta\phi = \pi r^2 B$. Due to this change in flux the change in linear velocity v from relation (4) is given by

$$\Delta v = -\frac{e}{2\pi\,rm}\Delta\phi = -\frac{e}{2\pi\,rm}(\pi r^2 B) = -\frac{erB}{2m}.$$

∴ change in angular velocity $\quad \Delta\omega = \dfrac{\Delta v}{r} = -\dfrac{eB}{2m}. \qquad ...(5)$

The frequency given by (5) is called *Larmor frequency*. This change in angular frequency produces a change in magnetic moment. The magnetic moment of the circular current is iA where i is the current and A the area of the loop.

From definition $\quad i - ne = \dfrac{\omega}{2\pi}e = \dfrac{\omega e}{2\pi}$

∴ The magnetic moment of circulating electron $= iA = \dfrac{e\,\omega}{2\pi}\pi r^2$

$$= \frac{er\,\omega^2}{2}. \qquad ...(6)$$

When the angular velocity changes by $\Delta\omega$, the magnetic moment of the electron is changed by

$$\Delta M = \frac{er^2}{2}\Delta M = \frac{er^2}{2m}\left(-\frac{eB}{2m}\right) \quad \text{using (8)}$$

$$= -\frac{e^2r^2B}{4m} \quad \text{...(17)}$$

The negative sign indicates that the induced magnetic moment $\Delta\mathbf{M}$ is always opposite to the direction of applied $\mathbf{B}$. This brings out the *intrinsic nature* of the diamagnetic *i.e.* an induced effect is always opposite to the induction effect. Consider, for example, an atom consisting of two rings of electons of equal moments oppositely oriented so that it has zero magnetic moment in the absence of an external magnetic field. The effect of a magnetic field, parallel to the axis of the first ring is to decrease the moment of the ring and to increase that of oppositely directed ring. Thus a resultant magnetic moment opposite to the field is imparted to the atom. In an atom of atomic number $\mathbf{Z}$, there are $\mathbf{Z}$ electrons hence we must consider $\mathbf{Z}$ electronic orbits. The change of magnetic moment for the atom is equal to $\mathbf{Z}$ times that of a single electron given by equation (17). But since all the electrons may not rotate in the same circular orbit; hence in summing up their effects, we must replace r^2 by a suitable mean value, $\overline{r^2}$. Further the orbits may not all be perpendicular to the direction of field but inclined at different angles to it. Supposing that the orbits have all possible orientations with respect to the field and taking into account the spherical symmetry of the diamagnetic atom let us first calculate $\overline{r^2}$.

If $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of any point on an orbit of radius R, then

$$R^2 = \overline{x^2} + \overline{y^2} + \overline{z^2}.$$

For a spherically symmetric atom

$$\overline{x^2} = \overline{y^2} = \overline{z^2} = \frac{R^2}{3}.$$

If the field $\mathbf{B}$ is along Z-axis, then $\overline{r^2}$, which represents the average of the squares of the projections of the electron's radii of an atom on a plane perpendicular to field (*i.e.* on x-y plane) is evidently given by

$$\overline{r^2} = \overline{x^2} + \overline{y^2} = \frac{R^2}{3} + \frac{R^2}{3} = \frac{2R^2}{3} \quad \text{...(8)}$$

where R is the radius of the orbit.

Hence the *total magnetic moment of the atom* is given by

$$\mathbf{M} = -\frac{e^2\mathbf{B}}{4m}\Sigma\overline{r^2} = -\frac{Ze^2\mathbf{B}}{4m} \times \frac{2R^2}{3}$$

$$= -\frac{Ze^2R^2B}{6m} \quad \text{...(9)}$$

If the substance contains n electrons per unit volume, then the *magnetic moment per unit volume*

$$\mathbf{M} = -\frac{nZe^2R^2}{6m}\mathbf{B} = -\frac{nZe^2R^2}{6m}\mu_0\mathbf{H}. \quad \text{...(10)}$$

$\therefore$ The volume susceptibility of the diamagnetic substance is

$$\chi_m = \frac{\mathbf{M}}{\mathbf{H}} = -\frac{\mu_0 nZe^2R^2}{6m} \quad \text{...(11)}$$

From this equation it is clear that *the diamagnetic susceptibility χ_m is negative and does not depend upon the sign of electronic charge. Also diamagnetic succeptibility is independent of temperature and field strength.* Also diamagnetism is an induced opposing effect which disappears as soon as the field is removed (since $\mathbf{M} = 0$ when $\mathbf{B} = 0$).

6.26. Langevin's Theory of Paramagnetism

Paramagnetic occurs in those substance where the individual atoms, ions or molecules possess a permanent magnetic dipole moment. In the absence of an external magnetic field, the atomic or molecular dipoles point in random directions due to thermal agitation. Therefore the net magnetization of the substance as a whole in any direction is zero. When an external magnetic field is a applied the magnetic atoms or dipoles tend to orient themselves parallel to the field, because this is a state of lower energy than the antiparallel position. Then there is a net magnetization parallel to the field and the susceptibility is positive. The thermal agitation which tends to give a random orientation is less at low temperatures, a bigger proportion of the dipoles are able to align themselves parallel to the field and hence the magnetization is greater for a given field. Curie discovered that susceptibility of many substances, for ordinary fields and temperatures, follows the relation :

$$\chi = \frac{M}{H} = \frac{\lambda}{T}$$

where λ is a constant and T is the absolute temperature. This relation is known Curie's law. For large fields at low temperatures the magnetization produced is no longer proportional to the applied field, and approaches a constant value. This saturation effect is produced when all the atomic dipoles are aligned parallel to the field, and the magnetization takes a limiting maximum value.

The theoretical explanation for paramagnetism was given by Langevin using the classical statistics of Boltzmann. He assumed that each atom consists of a permanent magnetic moment m, and that the only force acting on the atom is that due to the external field $\mathbf{B}$. If a given atomic or molecular dipole is pointing in a direction making an angle θ with $\mathbf{B}$; its magnetic potential energy is, $W = -\mathbf{m} \cdot \mathbf{B} = -mB \cos \theta$. According to Boltzmann's classical statistics the number of atoms or molecules per unit volume having an energy in the range W and $W + dW$ is,

$$dn = Ce^{-W/kT}\, dW \qquad \text{...(1)}$$

where C is a constant to be determined, k is Boltzmann constant and T is the absolute temperature.

On differentiating the potential energy expression, we get

$$dW = mB \sin \theta\, d\theta. \qquad \text{...(2)}$$

Substituting the values of W and dW in equation (1), we get

$$dn = Ce^{mB \cos \theta/kT} \cdot mB \sin \theta\, d\theta \qquad \text{...(3)}$$

Now we shall integrate this expression over all possible orientations, *i.e.*, for θ from 0 to π, in order to find the total number of atoms or molecules n per unit volume of the substance. Thus

$$\int dn = n = \int_0^\pi Ce^{mB \cos \theta/kT} \cdot m\mathbf{B} \sin \theta\, d\theta.$$

Thus gives,

$$C = \frac{n}{\int_0^\pi e^{mB \cos \theta/kT} \cdot mB \cos \theta\, d\theta} \qquad \text{...(4)}$$

The component of each dipole moment parallel to B is $m\cos\theta$, and so the resultant moment due to all the dn atoms or molecules along the field is, $dn\, m\cos\theta$. The intensity of magnetization M which is, the magnetic moment per unit volume is thus given by,

$$M = \int_0^\pi m\cos\theta \,.\, dn$$

$$= C\int_0^\pi m^2 B e^{mB\cos\theta/kT}\sin\theta\cos\theta\, d\theta \qquad \text{[using eq. (3)]}$$

$$= nm^2 B\,\frac{\int_0^\pi e^{mB\cos\theta/kT}\sin\theta\cos\theta\, d\theta}{\int_0^\pi e^{mB\cos\theta/kT}\, mB\sin\theta\, d\theta} \qquad \text{[using equation (4)]}$$

$$= nm\,\frac{\int_0^\pi e^{mB\cos\theta/kT}\sin\theta\cos\theta\, d\theta}{\int_0^\pi e^{mB\cos\theta/kT}\sin\theta\, d\theta} \qquad \text{...(5)}$$

Let us write $\dfrac{mB}{kT} = x$, and $\cos\theta = y$, then equation (5) becomes,

$$M = nm\,\frac{\int_{-1}^{+1} e^{xy}\, y\, dy}{\int_{-1}^{+1} e^{xy}\, dy}$$

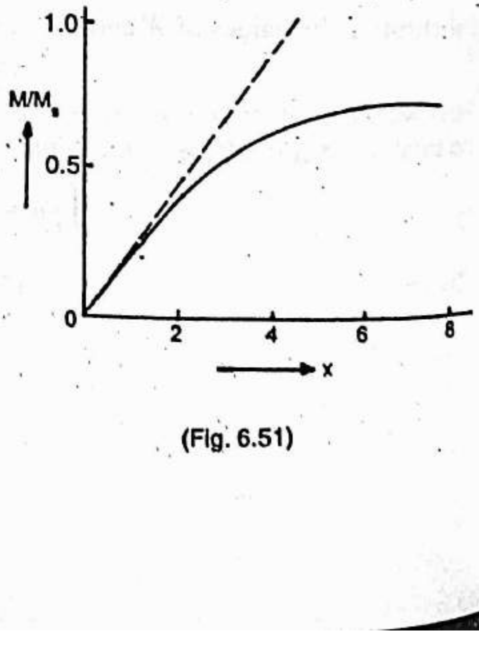
$$= nm\,\frac{\left[\dfrac{y\,e^{xy}}{x} - \dfrac{e^{xy}}{x^2}\right]_{-1}^{+1}}{\left[\dfrac{e^{xy}}{x}\right]_{-1}^{+1}}$$

$$= nm\left[\frac{(e^x + e^{-x})}{(e^x - e^{-x})} - \frac{1}{x}\right]$$

$$= M_S\left[\coth x - \frac{1}{x}\right] \qquad \text{...(6)}$$

where $M_S = nm$ represents the saturation value of M, *i.e.*, when all the atoms or molecules turn parallel to the field. A plot of the function $M/M_S = \coth x - 1/x$ function) is shown in fig. 6·40. For large values of x, the function tends to unity, saturation being reached when all the atomic or molecular dipoles are parallel to B. For *small values* of x the curve is linear and coincides with tangent to the curve at the origin. For ordinary magnetic field strengths x is small and we can expand $\coth x$ in equation (5) in power series of x retaining only the important terms. Equation (6) then becomes,

$$M = M_S\left[\left(\frac{1}{x} + \frac{x}{3}\right) - \frac{1}{x}\right]$$



(Fig. 6.51)

$$= M_S \frac{x}{3}$$

$$= \frac{n.m^2 B}{3kT} \qquad \left[\begin{array}{l}\text{Since } x = \dfrac{mB}{kT} \\ \text{and } M_S = nm\end{array}\right]$$

or
$$\mathbf{M} = \frac{n.m^2}{3kT}\mathbf{B} = \frac{mn^2 \mu_0 \mathbf{H}}{3kT} \qquad \text{...(7)}$$

The volume susceptibility for a paramagnetic substance is thus given by

$$\chi_m = \frac{\mathbf{M}}{\mathbf{H}} = \frac{\mu_0 nm^2}{3kT} = \frac{\mu_0 M_S^2}{3nkT}. \qquad \text{...(8)}$$

This is the same as Curie's law stated earlier where Curie constant λ is $\frac{\mu_0 nm^2}{3k} = \frac{\mu_0 \mathbf{M}_S^2}{3nk}$. The paramagnetic behaviour is exhibited only when the atoms or molecules of the system have permanent magnetic moments, and these tend to orient in the applied field. Except for temeprature near absolute zero and simultaneous large fields, the magnetization is far below the saturation value which would obtain when all the dipole moments are aligned. *Langevin's theory applies strictly only to gases, where the molecules are sufficiently far apart of their mutual interactions to be negligible. In liquids and solids and* such interactions may be large, and many substances obey the modified *Curie-Weiss law,*

$$\chi = \frac{\lambda}{T - \theta} \qquad \text{...(9)}$$

θ is called the *'Weiss constant'* and represents the characteristic of the substance. Equation (8) can be rationalised, as was first pointed out by Weiss (1907), by assuming that the actual field acting upon a dipole is the sum of the applied field H and that arising from the presence of neighbouring dipoles *i.e.* effective magnetising force may be expressed as

$$H_i = H + \alpha M \qquad \text{...(10)}$$

explicity assuming that the contributions of the neighbouring dipoles to the effective field is proportional to the magnetic polarisation. This form of dependence is not self evident but was chosen by Weiss because it leads to equation (9). The factor of proportionality α is known as the *molecular field coefficient.*

Substituting (10) in (7), we get

$$M = \frac{\mu_0 nm^2 (H + \alpha M)}{3kT} = \frac{\mu_0 M_S (H + \alpha M)}{3nkT} \qquad (\text{since } \mathbf{M}_S = um) \text{...(11)}$$

∴ volume susceptibility
$$\chi_m = \frac{\mu_0 nm^2 (H + \alpha M)}{3kTH}$$

i.e.
$$\chi_m = \frac{\mu_0 nm^2}{3kT} + \frac{\alpha\mu_0 nm^2}{3kT}\chi_m$$

or
$$\chi_m \left(1 - \frac{\alpha\mu_0 nm^2}{3kT}\right) = \frac{\mu_0 nm^2}{3kT}$$

∴
$$\chi_m = \frac{(\mu_0 nm^2/3kT)}{1 - \dfrac{\alpha\mu_0 nm^2}{3kT}} = \frac{(\mu_0 nm^2/3k)}{\left(T - \dfrac{\alpha\mu_0 nm^2}{3k}\right)} = \frac{\lambda}{T - \theta} \qquad \text{...(12)}$$

Equation (12) represents required *Curie Weiss law*. The constant $\theta = \alpha \lambda$ is known as *Curie temperature*. One might predict from Curie Weiss law that below Curie temperature, the susceptibility of an otherwise paramagnetic material would be negative *i.e.* a paramagnetic material would become diamagnetic below Curie temperature. This prediction is wrong because it overlooks the fact that the analysis leading upto equation (7) is predicted upon the existence of high temperatures.

Thus, Curie Weiss law holds only for temperatures $T > |\theta|$. For most paramagnetic material the Curie temperature is so low that even the Curie law [equation (8)] is approximately correct at room temperature.

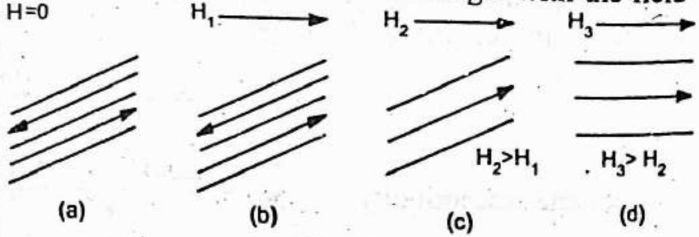
6.27. Wiess Theory of Ferromagnetism.

In order to explain the relation that exists between para and ferromagnetism as well as to account for the special properties, Weiss introduced the idea of *domains of molecules*. According to Weiss a ferromagnetic substance contains atoms with permanent magnetic moments, as in a paramagnetic substance but due to special form of interaction called *exchange coupling* occuring between adjacent atoms, there exist strong forces between neighbouring atomic dipoles which cause groups of them such that all atomic dipoles of a group point in the same direction : such groups are called the *domains* and their size is now known to vary from about 10^{6} cm^{3} to 10^{-2} cm^{3} and even greater in some crystals. Thus every domain is magnetically saturated and the direction of magnetisation in different domains is different. When a ferromagnetic substance is in unmagnetised state, the magnetisation vectors in different domains are oriented in different directions such that the resultant magnetisation is zero. When the temperature of a ferromagnetic substance is raised above the Curie point, exchange coupling suddenly disappears and the material becomes paramagnetic.

In passing from one domain to adjacent one, the atomic magnetisation vector gradually rotates from its original to its new direction over a thickness of about 1000 atomic dimensions, while the domains themselves have dimensions of the order of tenth of a miliimeter. The region between two domains is called *domain wall.*

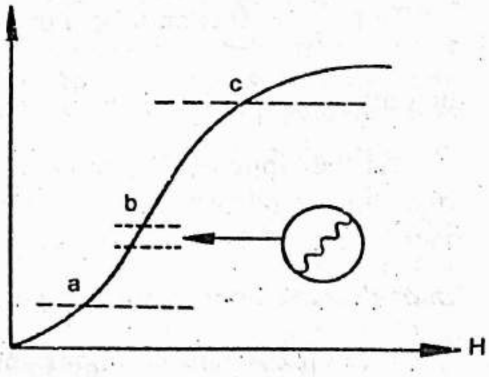
When a magnetic field is applied, the substance in magnetized and hence it is necessary that some of magnetisation vectors in domains must align parallel to the applied field.

When the magnetizing field is weak ; the magnetic polarisation of the sample changes as a result of *motion of domain walls*. In this process domains whose axes are parallel or at a small angle with the field grow at the expense of those which are unfavouably oriented, as shown in fig. 6·52 (b). This growth is reversible so long as the field stays very small : if we turn the field off, the magnetisation will return to zero. This part of magnetisation curve is marked *a* in fig. 6·53. [In stronger fields the magnetisation {part (b) in fig. 6·53} proceeds by irreversible wall motion]. In this region of field strength the magnetization is not a continuous process but takes place in a series of infinite small discrete steps because of the irregularities in crystal structure which might be due to strains, dislocation, impurities, dirt and imperfections. This can be shown by inserting the sample in a coil connected to an amplifier and loudspeaker. If the field surrounding the speaker is slowly increased, individual clicks are hard in a speaker which represent small discrete flux increments. This is known as *Barkhausen effect.* The step like nature of the magnetisation curve, if viewed on a macroscopic scale, would appear as shown in fig. 6·53.



(Fig. 6.52)

Near the knee of the curve in fig. 6.52, we have moved all domain walls and magnetised each crystal of the sample in its best direction [fig. 6·53 (c)]. The application of a very strong field then causes *domain rotation,* during which the magnetic polarisation assumes a direction parallel to the applied field as in fig. 6.52 (d), and this accounts for the curve *c* in fig. 6.52. Thus motion of the domain wall accounts for portion of the magnetisation curve below the knee of the curve, while domain rotation accounts for the portion above the knee. If the applied field is removed, there is little change in the domain structure so that the magnetisation remains quite high until reverse fields are applied, this accounts for the hysteresis described earlier.



(Fig. 6.53)

Introducting the idea of domains Weiss exactly made the same assumption as for paramagnetics given by relation

$$\mathbf{H}_i = \mathbf{H} + \alpha\mathbf{M}. \qquad ...(1)$$

When the molecular field coefficient α is positive, the possibility of *spontaneous magnetisation of these domains arises* even in the absence of external magnetic field. The value of such a spontaneous magnetisation due to internal molecular field may be found by putting the external field H to zero in the above relation. This relation can also determine the condition for spontaneous magnetisation.

Since the domains are assumed to obey the general theory of paramagnetism,

$$\frac{M}{M_S} = \coth x - \frac{1}{x} \qquad ...(2)$$

where

$$x = \frac{mB_i}{kT} = \frac{\mu_0 m H_i}{kT} = \frac{\mu_0 m\,(H + \alpha\mathrm{M})}{kT} \qquad ...(3)$$

Since the dipole aligning force $H_i = (H + \alpha M)$ depends partially upon the degree of alignment (magnetic polarisation) already prevailing, any increase in magnetic polarisation will result in an increase in the aligning force, which in turn, will result in an increase in the magnetic polarisation (or magnetisation) etc. In short a cumulative effect is to be expected. In this way magnetisation once induced, could occur without the application of an external field. Let us now investigate the conditions under which such *spontaneous magnetisation* could take place. When the applied field is $H = 0$ equation (3) reduces to

$$x = \frac{\mu_0 m \alpha M}{kT} = \alpha\frac{\mu_0 M M_S}{nkT} \qquad ...(4)$$

Since

$$M_S = n\mathrm{m}\,.$$

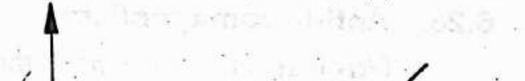
Equation (4) may be expressed as

$$\frac{M}{M_S} = \frac{nkT}{\alpha\,\mu_0 M_S^2}\,x\,, \qquad ...(5)$$

Equations (2) and (5) constitute a pair of equation which may be solved simultaneously for two variables $\frac{M}{M_S}$ and x. The solutions are most easily obtained by graphical method, Fig. 6·54 represents the curves corresponding to two equations (2) and (5); equation (2) gives the Langevin's curve 1 while equation (5) gives a straight line 2 passing through the origin whose slope equal to $\frac{nkT}{\alpha\,\mu_0 \mathrm{M}_S^2}$ increases with increase of T.

The curves (1) and (2) intersect at origin O and at another point A, so that the solutions are

$$\frac{M}{M_S} = 0 \quad \text{or} \quad AP$$



(Fig. 6.54)

But $\frac{M}{M_S} = 0$ cannot be a true solution, since there is spontaneous magnetisation by supposition. Hence the only correct solution is $\frac{M}{M_S} = AP$ *i.e.* a non-zero point of intersection of two curves.

As the slope of the straight line curve (2) increases with T. if this slope coincides or becomes greater than that of tangent at the origin to the Langevin curve, then there will not exist a non-zero point of intersection of two curves : hence the spontaneous magnetisation will not occur. On the other hand, we know that the slope of the tangent at the origin to the Langevin curve is $\frac{1}{3}$. Since $\frac{M}{M_S} = \frac{x}{3}$, when x is small.

From these considerations, the condition of *stable spontaneous magnetisation* may be expressed as

$$\frac{nkT}{\alpha\,\mu_0\,M_S^2} < \frac{1}{3}$$

i.e.

$$T < \frac{\alpha\,\mu_0\,M_S^2}{3nk}.$$

But $\alpha\,\mu_0\,M_S^2/3nk = \Theta$, the Curie point; hence $T < \Theta$.

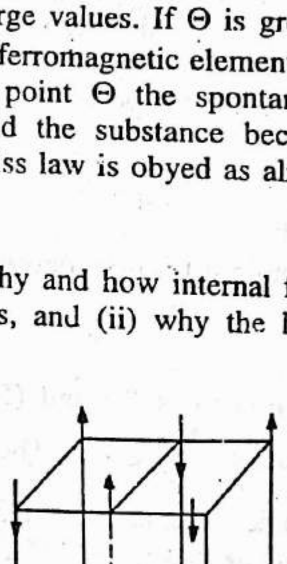
Hence below Curie point Θ a state of magnetisation is possible without an applied field. The degree of spontaneous magnetisation depends on the temperature approaching the saturation value as the temperature approaches absolute zero, since the nonzero point of intersection will take place at

$$M/M_S = 1 \text{ for } x = \mu_0\,mH_i/kT \to \text{infinite when } T = 0$$

It may be noted from the relation $\Theta = \alpha\mu_0 M_S^2/3nk$ that the absolute value of Θ is greater if the magnetic moment of domains (M_S) and molecular field coefficient α have large values. If Θ is greater, greater will be the transition temperature. Incidently the Curie temperature for ferromagnetic elements Fe, Co and Ni are 770°C, 111°C and 358°C respectively. Above the Curie point Θ the spontaneous magnetisation no longer occurs and ferromagnetic properties disappear and the substance becomes paramagnetic. At temperatures, not too near the transition point the Curie-Weiss law is obyed as already shown in preceding section $\chi_m = \frac{\lambda}{(T-\Theta)}$.

The chief drawbacks of Weiss' theory are that (i) it could not explain why and how internal fields between the molecules of ferromagnetic materials possess such large vlaues, and (ii) why the linear relationship expressed by Curie-Wiess law breaks near Curie point.

6.28. Anti-ferromagnetism :



(Fig. 6.55)

In ferromagnetic substances the atomic dipoles are aligned parallel to one another in the absence of the external magnetic field. This is due to the positive exchange interaction energy between neighbouring dipoles. But there are substances in which exchange interaction energy between neighbouring dipoles is negative, the atomic dipoles of such substances would be aligned in anti-parallel manner and there will be no spontaneous magnetisation. This phenomenon is called ***antiferromagnetism*** and the substances exhibiting this phenomenon are called ***antiferromagnetics.*** The antiferromagnetism was studied by Neel and Bitter and the theory was extended by Van-Veleck. The antiferromagnetic arrangement for a simple cubic lattice is shown is fig, the dipoles at adjacent corners of each cubic cell point in opposite directions. Another simple case is the body centred cubic lattice, where there is an atom at the centre of each cube as well as at the corners. The dipoles of corner atoms are parallel to each other but antiparallel to those at the centres. In each of these cases a given dipole is surrounded by a number of equidistant oppositely directed dipoles, while the next nearest neighbouring atomic dipoles point in the same direction again. Thus the system may be thought of as consisting two interlocking sub-lattices, one of which is spontaneously magnetised in one direction, while the other in opposite direction. This spontaneous magnetisation of sub-lattices sets in only below a certain transition temperature, called the **'Neel point'**. Above this temperature the dipoles are randomly oriented and the substance is paramagnetic obeying a Curie-Weiss law with a Weiss constant of opposite sign to that corresponding to ferromagnetism. The substance as a whole shows no spontaneous magnetisation in zero field, since the two sub-lattices are equally and oppositely polarised. When an external magnetic field is applied, a small magnetisation occurs giving a positive susceptibility. The examples of antiferromagnetics are Cr, MnF_2, MnO.

Molecular field theory of antiferromagnetism

Let us consider two interlocked sublattices, A and B of anti-ferromagnetic. Let all nearest neighbours of an atom of A are atoms of B and vice-versa. Let a dipole in lattice A be subjected to an external magnetic field H and an internal field proportional to magnetisation of sub-lattice B *i.e.* I_B the effective field on dipole A is

$$H_A = H - \alpha I_B \quad \text{...1 (a)}$$

Similarly effective field on dipole B is, $H_B = H - \alpha I_A$ ...1(b)

Case (1) At high temperatures $T > T_N$: When the temperature is above **Neel temperature,** the dipoles are randomly oriented and the magnetisation of each sub-lattice obeys Curie law if we take the effective magnetic field instead of external field *i.e.*

$$\left.\begin{aligned} I_A &= \frac{1}{2}\lambda\frac{H_A}{T} \\ I_B &= \frac{1}{2}\lambda\frac{H_B}{T}\end{aligned}\right\} \quad \text{...(2)}$$

where λ is Curie constant and the factor $1/2$ appears because only one-half of the dipoles are in given sub-lattice.

The net magnetisation $I = I_A + I_B$

$$= \frac{1}{2}\frac{\lambda}{T}(H_A + H_B)$$

Using (1)

$$I = \frac{1}{2}\frac{\lambda}{T}[2H - \alpha(I_A + I_B)]$$

$$= \frac{\lambda}{T}\left[H - \frac{\alpha I}{2}\right]$$

or

$$I + \frac{\lambda}{T}\cdot\frac{\alpha I}{2} = \frac{\lambda}{T}H$$

or

$$I\left[1 + \frac{\alpha\lambda}{2T}\right] = \frac{\lambda}{T}H$$

$\therefore$

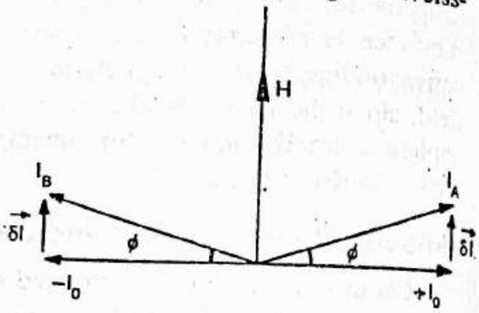
$$I = \frac{\lambda}{T\left[1 + \frac{\alpha\lambda}{2T}\right]}H = \frac{\lambda}{T + \frac{\alpha\lambda}{2}}H$$

$$\therefore \text{ Magnetic susceptibility } \chi = \frac{I}{H} = \frac{\lambda}{T + \frac{\alpha\lambda}{2}}$$

$$= \frac{\lambda}{T + \Theta} \text{ where } \Theta = \frac{\alpha\lambda}{2}$$

This equation is similar to that found for ferro-magnetism except that we have reversed sign of Weiss-constant Θ.

Case (II) At low temperatures $T < T_B$:



(Fig. 6.56)

When an external magnetic field is applied at a temperature below the Neel point, a positive magnetisation results. The effect of the external field is to change the magnetisation of each sub-lattice slightly, so that we can write

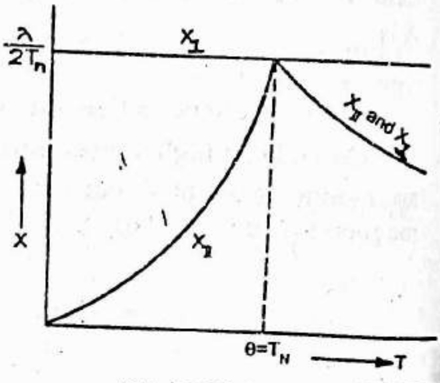
$$\vec{I_A} = \vec{I_o} + \delta\vec{I}$$

$$\vec{I_B} = -\vec{I_o} + \delta\vec{I}$$

These are vector relations because external field may be applied at arbitrary angle to the direction of spontaneous magnetisation $\vec{I_o}$. If $\vec{H}$ is parallel to $\vec{I_o}$ and hence $\delta\vec{I}$, then the effective field becomes

$$\vec{H_A} = \vec{H} - \alpha(\vec{I_o} + \delta\vec{I}) = \alpha\vec{I_o} + (\vec{H} - \alpha\,\delta\vec{I})$$

$$\vec{H_B} = \vec{H} - \alpha(\vec{I_o} + \delta\vec{I}) = -\alpha\vec{I_o} + (\vec{H} - \alpha\,\delta\vec{I})$$



(Fig. 6.57)

A mathematical analysis shows that the susceptibility for parallel field χ_{11} decreases to zero as the temperature falls to zero.

If the external field is applied perpendicular to $\vec{I_0}$, then the magnetisations $\vec{I_A}$ and $\vec{I_B}$ of each sub-lattice turn through an angle ϕ towards H as shown in fig. 6·56 and the susceptibility in this case comes out to be $\chi_\perp = \frac{\lambda}{2T_n}$

Thus $\chi_\perp$ should be constant below the Neel temperature.

Fig. 6·56 represents the variation of χ_{11} and $\chi_\perp$ with temperature below and above Neel temperature. The results are in qualitative agreement with the theory.

6.29. Ferrimagnetism; Ferrites :

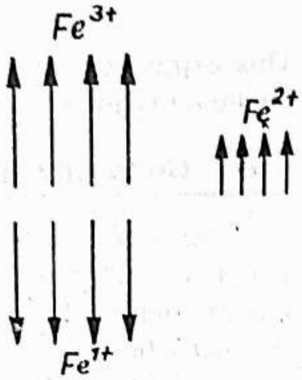
In antiferromagnetics the magnetic moments of the sublattices are equal and opposite and hence the net spontaneous magnetisation is zero. But in certain substances the magnetic moments of two sub-lattices are not equal in magnitude because of difference in number of atomic dipoles or the nature of atoms. This results in small finite spontaneous magnetisation of the crystal. Such an uncompensated antiferromagnetism is called **ferri-magnetism.**

The external behaviour of ferrimagnetics is similar to ferromagnetic substances but the temperature dependence of their spontaneous magnetisation may be quite different. For example the magnetisation of a ferrimagnetic does not necessarily decrease monotonically with the rise in temperature but can pass through zero even before the **Neel temperature.** The familiar example of ferrimagnetic is magnetite Fe_3O_4 or FeO.

Fe_2O_3 ferric ions Fe^{3+} are in a state with spin $S = 5/2$ and zero orbital magnetic moment. Thus each ion should contribute $5\,\mu_B$ to the saturation moment. The ferrous ions (Fe^{2+}) have a spin 2 and should contribute $4\,\mu_B$ apart from any residual orbital magnetic contribution. Thus the effective number of Bohr magnetons should be about $2 \times 5 + 4 = 14$ per Fe_3O_4 unit if all spins were parallel. But the observed value is $4{\cdot}1\,\mu_B$. The difference in the two values is accounted for if the moments of Fe^{3+} ions are antiparallel to each other and then the observed moment arises only due to Fe^{2+} ion. Neutron diffraction verifies this model (fig. 6·58).

The substances having this type of spin order are called **Ferrites.** The usual chemical formula of a ferrite is $MO.Fe_2O_3$, where M is a divalent cation and stands for Zn, Cd, Fe, Ni, Cu, Co or Mg.

A remarkable property of ferrites is the combination of extension of magnetic parameters (high magnetic permeability, small coersive force, high saturation magnetisation etc.) with high resistivity of the order of 10^3 Ω–m. This property of ferrities is useful in the field of high and ultra high frequency electronics.



(Fig. 6.58)

6.30. Boundary Conditions on the Field Vectors

Let us know how the field vector **B** and **H** change in passing an interface between two media. The two media may be two materials with different magnetic properties or a material medium and vacuum. The method employed is similar to that used in studying the behaviour of **D** and **E** at an interface.

Consider two magnetically isotropic media 1 and 2 with permeabilities μ_1 and μ_2 respectively, in contact, as shown in figu. 6.59. $\mathbf{B}_1$ (and $\mathbf{H}_1$) and $\mathbf{B}_2$ (and $\mathbf{H}_2$) make angles θ_1 and θ_2 respectively with the normal to the interface.

We first, consider the flux of **B** through a Gaussian surface in the shape of a shallow pillbox enclosing a region of the boundary as shown in the left of fig. 6.59. Since the sides perpendicular to the boundary can be made of negligible area, the flux is

$$\oint \mathbf{B} \cdot d\,\mathbf{a} = -B_{1n}\,\Delta S + B_{2n}\,\Delta S = 0, \qquad \text{or} \qquad B_{1n} = B_{2n}$$

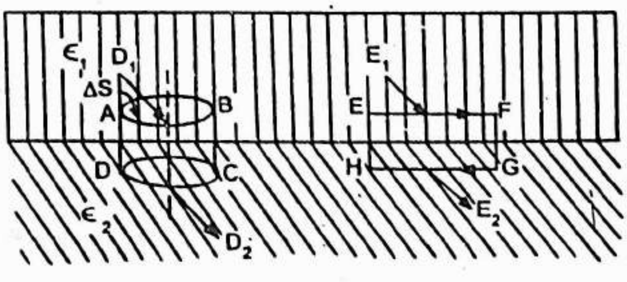
or

$$B_1 \cos\theta_1 = B_2 \cos\theta_2 \qquad \text{...(1)}$$

where B_{1n}, B_{2n} stand for the normal components of **B** in the two media. From eq. (1) it is clear that the normal component of induction is continuous across the boundary.

The boundary condition for **H** is obtained with reference to the rectangular path shown on the right of fig. 6.48. Since the closed path encloses zero free current, the line integral of **H** around the path is zero

$$\oint \mathbf{H} \cdot dl = 0.$$



(Fig. 6.59)

The sides of the path parallel to the surface in the two media can be made very long in comparison with the portions perpendicular to the surface and the above integral then becomes.

$$(H_{1t} - H_{2t})\,\Delta l = 0$$

i.e.

$$H_{1t} = H_{2t} \quad \text{or } H_1 \sin\theta_1 = H_2 \sin\theta \qquad \text{...(2)}$$

where H_{1t} and H_{2t} represent respectively the tangential components of **H** in the two media. Thus the tangential component of **H** is continuous across the boundary.

If eq. (2) is divided by eq. (1) and the result is rearranged, we obtain,

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{B_1 H_2}{B_2 H_1} = \frac{\mu_1}{\mu_2} \quad ...(3)$$

This expression describes, in terms of their directions, the behaviour of **B** and **H** in passing from one medium to another.

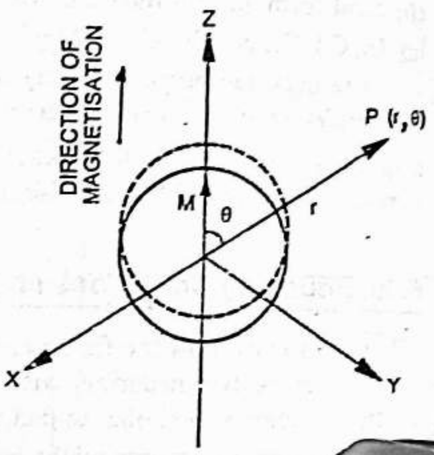
5.31. Uniformly Magnetised Sphere in an External Field.

The problem of uniformly magnetised sphere in an external magnetic field is analogous to that of a dielectric sphere in an external electric field. Let us first consider a sphere of radius R with a uniform magnetisation **M** along z-axis without an external magnetic field (such as we might have from a permanent magnet).

We can later superpose a uniform external magnetic field. Let us now find the field inside and outside the sphere as follows :

As in electrostatic case the magnetised sphere may be thought to be equivalent to two spheres one having a north polarity and the another south polarity and the two spheres displaced by a very small distance. The two sphere system is equivalent to a short magnetic dipole of moment $\mathbf{m} = \left(\frac{4}{3}\pi R^3\right)\mathbf{M}$

[since magnetisation = magnetic moment per unit volume = **M** and volume of sphere $= \frac{4}{3}\pi R^3$, hence dipole moment of sphere $= \frac{4}{3}\pi R^3 \mathbf{M}$].



(Fig. 6.60)

This equivalency holds for the calculations of magnetic scalar potential and field intensity only at external distant points.

The magnetic scalar potential at a distant point (r, θ) is given by

$$\phi = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{r}}{\mathbf{r}^3} = \frac{\mu_0}{4\pi} \frac{m \cos \theta}{r^2}$$

$$= \frac{\mu_0}{4\pi} \frac{(\frac{4}{3}\pi R^3 M) \cos \theta.}{r^2} = \frac{\mu_0 M R^3 \cos \theta}{3r^2} \quad ...(1)$$

and the *magnetic induction* at the outside point (r, θ) is obtained as

$$\mathbf{B}_{out} = -\text{grad}\,\phi = -\text{grad}\left(\frac{\mu_0 M R^3 \cos \theta}{3r^2}\right) \quad ...(2a)$$

As $$\mathbf{B} = \mu_0 \mathbf{H} \quad i.e. \quad \mathbf{H} = \frac{\mathbf{B}}{\mu_0}$$

Hence the magnetic field intensity **H** at outside point is given by

$$\mathbf{H}_{out} = \frac{\mathbf{B}_{out}}{\mu_0} = -\text{grad}\left(\frac{M R^3 \cos \theta}{3r^2}\right) \quad ...(2b)$$

Thus the radial and transverse components of magnetic field intensity are

$$H_r = -\frac{\partial}{\partial r}\left(\frac{Mr^2 \cos\theta}{3r^2}\right) = \frac{2}{3}\frac{MR^2 \cos\theta}{r^3} \qquad ...(3)$$

and

$$H_\theta = -\frac{1}{r}\left(\frac{MR^3 \cos\theta}{3r^2}\right) = \frac{1}{3}\frac{MR^3 \sin\theta}{r^3} \qquad ...(4)$$

Thus

$$\mathbf{H} = \mathbf{H}_r + \mathbf{H}_\theta$$

$$= \frac{2}{3}\frac{MR^3 \cos\theta}{r^3}\hat{\mathbf{r}} + \frac{1}{3}\frac{MR^3 \sin\theta}{r^3}\hat{\theta}$$

$$= \frac{MR^3}{3r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] \qquad ...(5)$$

Thus

$$H = |\mathbf{H}| = \frac{MR^3}{3r^3}\sqrt{[(2\cos\theta)^2 + (\sin\theta)^2]}$$

$$= \frac{MR^3}{3r^3}\sqrt{(1 + 3\cos^2\theta)} \qquad ...(6)$$

Now let us find the potential and field at a point inside the magnetised sphere. This is obtained by putting $r = R$ in equation (1). Then

$$\phi_{(R,\theta)} = \frac{\mu_0 MR\cos\theta}{3} \qquad ...(7)$$

$$= \frac{\mu_0 Mz}{3} \quad (\text{since } z = R\cos\theta) \qquad ...(8)$$

Since no free poles exist inside the magnetised sphere the potential ϕ_{in} must satisfy Laplace's equation $\nabla^2 \phi_{in} = 0$. According to uniqueness theorem there can be only one solution of Laplace's equation and that it must be the same as given by equation (8)

i.e.

$$\phi_{in} = (\mu_0 Mz/3) \qquad ...(9)$$

Therefore the field induction inside the sphere may be obtained as

$$(\mathbf{B}_{in}) = -\,\text{grad}\,(\phi_{in}) = -\,\text{grad}\left(\frac{\mu_0 Mz}{3}\right) \qquad ...(10)$$

Hence the *magnetic field intensity inside* the sphere is

$$(\mathbf{H}_{in}) = \frac{B_{in}}{\mu_0} = -\,\text{grad}\left(\frac{Mz}{3}\right)$$

or in terms of components

$$(H_x)_{in} = -\frac{\partial}{\partial x}(Mz/3) = 0$$

$$(H_y)_{in} = -\frac{\partial\,(Mz/3)}{\partial y} = 0$$

and

$$(H_x)_{in} = -\frac{\partial}{\partial z}(Mz/3) = -\frac{M}{3}$$

Since the direction of **M** is along z-axis, we can write

$$\mathbf{H}_{in} = -\frac{\mathbf{M}}{3} \qquad ...(11)$$

Evidently it is constant throughout the magnetised sphere and points opposite to the direction of **M**.

Let us now assume the existence of uniform external magnetic field $\mathbf{H}_0$ along z-axis, then the resultant field intensity at external and internal points may be written as

$$(\mathbf{H})_{out} = \left(H_0 \cos\theta + \frac{2}{3}\frac{MR^2 \cos\theta}{r^3}\right)\hat{\mathbf{r}} + \left(-H_0 \sin\theta + \frac{MR^3 \sin\theta}{3r^2}\right)\hat{\boldsymbol{\theta}} \qquad ...(12)$$

and

$$(\mathbf{H})_{in} = \mathbf{H}_0 - \frac{\mathbf{M}}{3} \qquad ...(13)$$

Equation (13) shows that the magnetisation produces a reverse field inside the sphere, known as *demagnetising field,* which is proportional to $\mathbf{M}$; the factor $(\frac{1}{2})$ is known as the *demagnetising factor.*

For a paramagnetic or diamagnetic substance, we know

$$\mathbf{M} = \chi_{in}(\mathbf{H})_{in}$$

$$= \left(\frac{\mu}{\mu_0} - 1\right)\mathbf{H}_{in} \qquad ...(14)$$

[Since $\mu_0 = \mu_0(1+\chi_m)$]

Substituting this value in equation (13), we get

$$\mathbf{H}_{in} = \mathbf{H}_0 - \frac{1}{3}\left(\frac{\mu}{\mu_0} - 1\right)\mathbf{H}_{in} = \mathbf{H}_0 - \frac{1}{3}(K_m - 1)\,H_{in}\left(\text{since } K_m = \frac{\mu}{\mu_0}\right).$$

This yields

$$\mathbf{H}_{in} = \frac{3}{K_m + 2}\mathbf{H}_0 \qquad ...(15)$$

and

$$\mathbf{M} = \left(\frac{\mu}{\mu_0} - 1\right)\left(\frac{3}{K_m + 2}\right)\mathbf{H}_0$$

i.e.

$$\mathbf{M} = \frac{3(K_m - 1)}{(K_m + 2)}\mathbf{H}_0 \qquad ...(17)$$

Also

$$\mathbf{B}_{in} = \mu\,\mathbf{H}_{in} = \mu\left(\frac{3}{K_m + 2}\right)\mathbf{H}_0 = \left(\frac{3\mu}{\frac{\mu}{\mu_0} + 2}\right)\mathbf{H}_0$$

$$= \left(\frac{3\mu}{\mu + 2\mu_0}\right)\mu_0\,\mathbf{H}_0 \qquad ...(18)$$

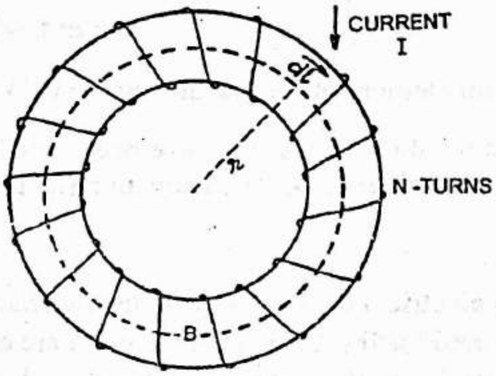
For a paramagnetic substance, $\mu/\mu_0 = K_m$ is slightly greater than unity, so the H_{in} is slightly less than H_0 and induction $\mathbf{B}_{in}$ is greater than the free space value $\mu_0\,\mathbf{H}_0$.

Therefore the lines of induction are crowded together in a paramagnetic sphere. In a diamagnetic substance $\mu/\mu_0 = K_m < 1$ and the reverse is the case. As in the electrostatic case, the field outside the sphere is equal to $\mathbf{H}_0$ plus the field of a dipole at the centre of the sphere of magnitude equal to the total moment of the sphere. For a ferromagnetic substance we consider the case when there is no external field (*i.e.* $\mathbf{H}_0 = 0$); but a finite magnetisation $\mathbf{M}$ corresponds to a spherical permanent magnet. The external field is then purely that of a point dipole given by equation (5) and the internal field is just the demagnetising field $(-\mathbf{M}/3)$.

6.32. The Magnetic Circuit

It is familiar that the magnetic flux lines form closed loops. If all the magnetic flux (or substantially all of it) associated with a particular distribution of currents is confined to a rather well defined path, then we may speak of a *magnetic circuit.* Kirchoff's second law, $\Sigma\,\xi = \Sigma IR$ for a closed path, relates electric current I, e.m.f. ξ and resistance R in an electric circuit. Quantities analogous to these exist in a *magnetic circuit* and is related through a similar type of law. Here we shall develop this law and its utility in some simple cases.

As an example of a magnetic circuit let us consider the arrangement shown in fig. 6.61. Here a ring of iron of uniform permeability μ is surrounded by a close fitting toroidal coil of N turns carrying a current of I amperes. The cross-sectional area of iron ring is A. In such a circuit the magnetic flux ϕ_m is confined to the iron ring and has the same value through each cross-sectional area of the ring. In this respect it resembles with an electric current that is confined to a closed conducting loop and has the same value around the loop. Thus, in drawing a parallelism between magnetic and electric circuits, *the magnetic flux is taken as the magnetic counterpart of the electric current.*



(Fig. 6.61)

The magnetic analogue to the electromotive force (e.m.f.) is the *magnetomotive force* (m.m.f).

We know that the e.m.f. in an electric circuit is given by the line integral of electric field strength around the circuit viz.,

e.m.f. $$\xi = \oint \mathbf{E} \cdot d\,\boldsymbol{l} \qquad \text{...(1)}$$

Thus by analogy, we define the m.m.f. is the line integral of magnetising force H around the magnetic circuit *i.e.*,

$$\text{m.m.f.} = \oint \mathbf{H} \cdot d\,\boldsymbol{l} \qquad \text{...(2)}$$

But Ampere' circuital law for a path following the circuit (e.g.) dotted path in fig.) gives

$$\oint \mathbf{H} \cdot d\mathbf{l} = NI \qquad \text{...(3)}$$

comparing (2) and (3), we get

$$\text{m.m.f.} = \oint \mathbf{H} \cdot d\,\boldsymbol{l} = NI \qquad \text{...(4)}$$

At each point of the dotted path **H** and $d\,\boldsymbol{l}$ are parallel and we also have $\phi_m = BA$ and therefore

$$H = \frac{B}{\mu} = \frac{(\phi_m/A)}{\mu} = \frac{\phi_m}{\mu A} \qquad \text{...(5)}$$

Then the line integral (2) may be expressed as

$$\text{m.m.f.} = \oint \mathbf{H} \cdot d\,\boldsymbol{l} = \oint H\,dl = \oint \frac{\phi_m}{\mu A}\,dl$$

Since we are dealing with a magnetic circuit, we expect ϕ_m to be essentially constant at all points in the circuit; hence we may take ϕ_m outside the integral

$$\text{m.m.f.} = \phi_m \oint \frac{dl}{\mu A} \qquad \text{...(6a)}$$

i.e. $$\phi_m = \frac{\text{m.m.f.}}{\oint \frac{dl}{\mu A}} \qquad \text{...(6b)}$$

Since the flux ϕ_m of a magnetic circuit is equivalent to electric current of an electric circuit and m.m.f. of a magnetic circuit is equivalent to e.m.f. of electric circuit, hence comparing equation (6b) with Kirchoff's second law $\left(I = \frac{\text{e.m.f.}}{R}\right)$, we get, by analogy, the ***magnetic resistance*** known as ***reluctance*** of the magnetic circuit as

$$R = \oint \frac{dl}{\mu A} \qquad \text{...(7)}$$

The equation (6a) may be written as

$$\text{m.m.f.} = \text{flux } (\phi_m) \times \text{ reluctance } (R) \qquad \text{...(6c)}$$

for an element of the circuit equation (7) gives $dR = \dfrac{dl}{\mu A}$ ...(8)

The parallelism that we have been drawing between electric and magnetic circuits is confirmed by the striking resemblence between equation (8) and the corresponding expression for an element of resistance

$$dR = \frac{dl}{\sigma A}, \qquad \text{...(9)}$$

σ being electrical conductivity of the material.

In the circuit of fig. 6.50, where μ and A are constant, the *reluctance* takes the simple form

$$R = \frac{1}{\mu A} \oint dl = \frac{1}{\mu A} \times 2\pi r = \frac{2\pi r}{\mu A}.$$

If the circuit is not uniform but consists of series combination of two or more dissimilar portions, the *total reluctance* of the circuit may be approximated as

$$R = \sum_i \frac{l_i}{\mu_i A_i} = \sum_i R_i \qquad \text{...(10)}$$

i.e., in the series the net reluctance is the sum of reluctances of separate portions.

Similarly, the *reluctance in parallel* are also combined in accordance with rules for resistances *i.e.*,

$$\frac{1}{R} = \sum_i \frac{1}{R_i}. \qquad \text{...(11)}$$

The reciprocal of reluctance is known as permeance and is denoted by A. The M.K.S. unit of reluctance is ampere turn/weber while that of permeance is weber/ampere-turn.

Neglecting leakage of flux through the air evidently introduces an error of same sort as neglecting leakage of current from an electric circuit which is placed in a slighly conducting solution. This error is not very great except perhaps at an air gap, where unless the gap is very small compared with the linear dimensions of the circuit at this point, there is some spreading of flux.

The following restrictions on the analogy should, however, be noted :

(a) The absolute permeability μ apparently corresponds to conductivity σ. But μ, the absolute permeability in magnetism really corresponds with ε, the absolute permittivity in electricity. The real analogue of a piece of magnetized material is a polarized dielectric.

(b) The resistance of a conductor is independent of current at a given temperature. The reluctance of a magnetic circuit varies considerably with the magnetic flux in as much as μ depends upon magnetisation.

(c) In a magnetic circuit, energy is only required to establish the magnetic flux and not to maintain it while the electric circuit involves the expenditure of energy during the entire period of current flow.

(d) In an electric circuit the current indicates the flow of charge carriers, while in magnetic circuit there is no such flow of particles in magnetic flux.

In spite of these limitations the magnetic circuit concept is of most use when applied to circuits containing ferromagnetic materials. As illustration of the utility of magnetic circuit let us discuss some simple cases :

(1) Rowland ring : In the case of a Rowland ring, shown in the circuit arrangement of fig. 6.50, having mean radius, r cross-sectional area A and number of turns per unit length as n, the reluctance is given by

$$\begin{aligned} R &= \oint \frac{dl}{\mu A} \\ &= \frac{1}{\mu A} \int dl = \frac{2\pi r}{\mu A} \qquad \text{...(12)} \end{aligned}$$

$$\text{m.m.f.} = NI = 2\pi r n I. \qquad \text{...(13)}$$

Here $N (= 2\pi rn)$ represents the total number of turns.

According to equation [6(c)] m.m.f. is equal to the product of flux reluctance, so that

$$\phi_m = \frac{\text{m.m.f.}}{\text{reluctance}}$$

$$= \frac{2\pi rnI}{2\pi r/\mu A} = \mu nAI, \qquad ...(14)$$

and

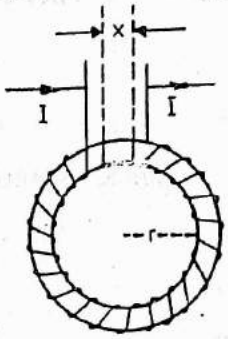
$$B = \frac{\phi_m}{A} = \frac{\mu nAI}{A} = \mu nI. \qquad ...(15)$$

Since

$$B = \mu H, \text{ we have} \qquad ...(16)$$

$$H = nI.$$

(2) Ring with a small air gap : If a narrow (gap) of width x is cut in the uniform ring such as shown in fig. 6.62 then the flux throughout the iron portion of the circuit is confined to the iron itself but in the air gap it spreads somewhat, occupying a cross-sectional area A, which is greater than the cross-sectional area A' of the iron. But in each case the flux has the same value.



(Fig. 6.62)

The reluctance of the circuit is given by

$$R = R_{iron} + R_{air\ gap}$$

$$= \frac{2\pi r - x}{\mu_0 A} + \frac{x}{\mu_0 A'}$$

$$\text{(since for a ir } \mu = \mu_0\text{).} \qquad ...(17)$$

The flux throughout the circuit is given by

$$\phi_m = \frac{\text{m.m.f.}}{R}$$

$$= \frac{2\pi rnI}{\left[\frac{\pi r - x}{\mu A} + \frac{x}{\mu_0 A'}\right]} \qquad ...(18)$$

The flux density in air gap is then given by $B_{air} = \dfrac{\phi_m}{A'}$ ...(19)

When the thickness x of the air gap is very small compared to the diameter of the pole faces, the spreading of the flux in the air gap is negligible, in which case $A \approx A'$ and the flux density at any point in the magnetic circuit is given by

$$B \approx \frac{2\pi rnI}{\left(\frac{2\pi r - x}{\mu}\right) + \frac{x}{\mu_0}} \qquad ...(2)$$

Since $\mu \ll \mu_0$ for iron, the reluctance of circuit of fig. 6.51 is largely provided by the air gap, unless the air gap is very narrow.

(3) Magnetic circuit of an Electromagnet : An electromagnet consists of a yoke, the two limbs, the two pole pieces and an air gap (fig. 6.63). Let l_1 be effective length, A_1 cross sectional area of yoke and μ_1 the permeability of material of yoke.

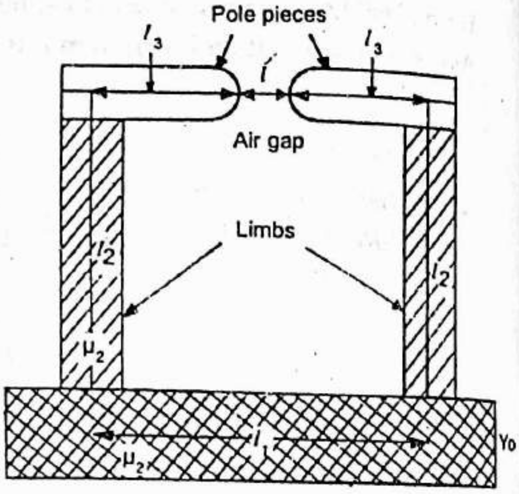
The reluctance of yoke $= \dfrac{l_1}{\mu_1 A_1}$

If l_2 is length of each limb, A_2 its cross-sectional area and μ_2, the permeability of the material, then reluctance of limbs $= \dfrac{2l_2}{\mu_2 A_2}$.

If l_3 is length of each pole piece, A_3 its cross-sectional area and μ_3 the permeability of its material, then reluctance of pole pieces $= \dfrac{2l_3}{\mu_3 A_3}$.

If l is length, of air gap A its permeability reluctance of air gap $= \dfrac{l}{\mu_0 A}$

$\therefore$ Net reluctance of magnetic circuit



(Fig. 6.63)

$$= \frac{l_1}{\mu_1 A_1} + \frac{2l_2}{\mu_2 A_2} + \frac{2l_3}{\mu_3 A_3} + \frac{l}{\mu_0 A}$$

Hence magnetic flux

$$\phi_B = \frac{\text{m.m.f. (Ni)}}{\text{Net Reluctance}} = \frac{\text{Ni}}{\dfrac{l_1}{\mu_1 A_1} + \dfrac{2l_2}{\mu_2 A_2} + \dfrac{2l_3}{\mu_3 A_3} + \dfrac{l}{\mu_0 A}}$$

This calculation of ϕ_B is only approximate due to leakage of magnetic flux through air gap is very small, the leakage is negligible and the calculations are more correct.

6.33. Magnetic Energy

It follows from Faraday's Law of induction that establishing a magnetic field requires the expenditure of energy. If an external source of e.m.f. E is applied to a circuit, then the current I through the circuit may be expressed, in general, as

$$E - e = IR, \qquad ...(1)$$

where e is the *induced e.m.f.* and R is the resistance of the current circuit. The work done by applied e.m.f. in moving the charge increment $dq = Idt$ through the circuit is

$$Edq = -edq + IRdq, \quad i.e. \quad EIdt = -eIdt + I^2Rdt. \qquad ...(2)$$

But from Faraday's law of induction $e = -d\phi/dt$, therefore equation (2) may be expressed as

$$EIdt = Id\phi + I^2Rdt. \qquad ...(3)$$

In this equation the last term $I^2 R\, dt$ represents the irreversible conversion of electrical energy into heat by the circuit, but this term absorbs the entire work input only in cases where the flux change $d\phi$ is zero. The additional term $Id\phi$ is the work done against the induced e.m.f. in the circuit or alternatively $Id\phi$ represent the *work done by applied e.m.f. E in altering the magnetic field structure*. Hence *for a rigid stationary circuit,* where there is no loss of energy except joule heat loss, the term $Id\phi$ represents the change in *magnetic energy of the circuit i.e.*

$$\delta U_m = Id\phi \qquad ...(4)$$

If *a system contains a group of n rigid current carrying circuits*, none of which extends to infinity, immersed in a medium, with linear magnetic properties, then the magnetic energy of the system may be expressed as

$$U_m = \sum_{i=1}^{n} \int I_i d\phi_i. \qquad ...(5)$$

The integration extends from zero flux situation (corresponding to all $(I_i = 0)$ to final set of flux values.

For a group of rigid circuits in linear magnetic media the ϕ_i are linearly related to currents in the circuits and the magnetic energy is independent of the way in which these currents are brought to their final sets of values. Let us assume that all currents are brought to their final sets of values in concert *i.e.*, at any instant of time all currents (and all fluxes) will be at the same fraction of their final values. Let this fraction be represented by α. If final values of currents are given by symbols $I_i^{(f)}$, then $I_i = \alpha I_i^{(f)}$, furthermore $d\phi_i = \phi_i^{(f)}\, d\alpha$,

so that equation (5) may be written in the form

$$U_m = \sum_{i=1}^{n} I_i^{(f)} \phi_i^{(f)} \int_0^1 \alpha\, d\alpha = \frac{1}{2}\sum_{i=1}^{n} I_i^{(f)} \phi_i^{(f)}.$$

We have used the superscript (f) merely to designate a quantity which remains constant when α varies, we now find it. It is convenient to drop the superscript and write

$$U_m = \frac{1}{2}\sum_{i=1}^{n} I_i \phi_i. \qquad \text{...(6)}$$

For convenience let us now assume that each circuit consists of only a single loop, then the flux ϕ_i may be expressed as $\phi_i = \int_{S_i} \mathbf{B} \cdot \mathbf{n}\, da$.

Now if $\mathbf{A}$ is the local vector potential, then we may write

$$\phi_i = \int_{S_i} \mathbf{B} \cdot \mathbf{n}\, da = \int_{C_i} \mathbf{A} \cdot d\,\mathbf{l}_i. \qquad \text{...(7)}$$

Substituting values of ϕ_i from (7) in equation (6), we get

$$U_m = \frac{1}{2}\sum_i \int_{C_i} I_i\, \mathbf{A} \cdot d\,\mathbf{l}_i. \qquad \text{...(8)}$$

Now if we do not have well defined current circuits, but instead each circuit is a closed path in the medium (which we take to be conducting), then we may approximate our situation very closely by choosing a large number of continuous circuits (C_i), therefore replacing $I_i\, d\,\mathbf{l}_i \to \mathbf{J}\, dv$ and $\sum_i \int_{C_i} \to \int_V$ in equation (8), we get

$$U_m = \frac{1}{2}\int_V \mathbf{J} \cdot \mathbf{A}\, dv \qquad \text{...(9)}$$

$$= \frac{1}{2}\int_V (\text{curl } \mathbf{H}) \cdot \mathbf{A}\, dv \quad (\text{since curl } \mathbf{H} = \mathbf{J}). \qquad \text{...(10)}$$

Now using vector identity,

$$\text{div}\,(\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{H}$$

i.e. $\mathbf{A} \cdot \text{curl } \mathbf{H} = \mathbf{H} \cdot \text{curl } \mathbf{A} - \text{div}\,(\mathbf{A} \times \mathbf{H})$

using this substitution equation (10) takes the form

$$U_m = \frac{1}{2}\int_V \mathbf{H} \cdot \text{curl } \mathbf{A}\, dv - \frac{1}{2}\int_V \text{div}\,(\mathbf{A} \times \mathbf{H})\, dv. \qquad \text{...(11)}$$

Now applying Gauss divergence theorem to last integral, we get

$$U_m = \frac{1}{2}\int_V \mathbf{H} \cdot \text{curl } \mathbf{A}\, dv - \frac{1}{2}\int (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{n}\, da, \qquad \text{...(12)}$$

where S is the surface which bounds volume V. Since, we have assumed that none of the ***current circuits*** extends to infinity, it is convenient to move the surface S out to a very large distance so that all parts of this surface are far from currents. Of course the volume of the system must be increased accordingly. Now $\mathbf{H}$

varies as $1/r^2$, A varies as $1/r$ and the surface area is proportional to r^2, where r is the distance from an origin near middle to the current distribution to a characteristic point on the surface S. Thus the contribution from the surface integral

$$\int_S (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{n}\, da$$

falls off as $1/r$ or faster and when S is moved on to infinity *i.e.* $r \to \infty$ this contribution vanishes. Thus, we have

$$\int_S (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{n}\, da = 0 \text{ as } r \to \infty.$$

Using this equation (12) gives

$$U_m = \frac{1}{2} \int_V \mathbf{H} \cdot \operatorname{curl} \mathbf{A}\, dv = \frac{1}{2} \int_V \mathbf{H} \cdot \mathbf{B}\, dv. \qquad \text{...(13)}$$

Since $\mathbf{B} = \operatorname{curl} \mathbf{A}$.

This result is completely analogous to the expression for electrostatic energy, equation (9) of section 3.8, the expression (13) for magnetic energy, is restricted to systems containing linear magnetic media.

Hence *the energy density* (*i.e.* energy per unit volume) of magnetic field is given by

$$u_m = \frac{U_m}{V} = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{1}{2} \mu H^2 \qquad \text{...(14)}$$

(since $\mathbf{B} = \mu \mathbf{H}$)

Questions and Problems

1. Explain Biot-Savart law and from this deduce Ampere's circuital law for steady current. Define the magnetic vector potential for steady current and show that it satisfies the Poisson's equation

   $$\nabla^2 A = -\mu_0 J,$$

   where J is current density. **(Meerut 1979)**
2. Explain Biot-Savart law. Find the element of force dF_{12} caused by current element $I_2 dS_2$ on $I_1 dS_1$ and also the force dF_{12} on the current element $I_2 dS_2$ produced by $I_1 dS_1$. Show that they are unequal. How do you recouncile your results with Newton's third law ? **(Meerut 1988, 83; Rohilkhand 83, 81)**
3. State Biot-Savart law and relate it to Ampere's law. Show that the divergence of the magnetic induction is always zero. **(Meerut 1981, 96)**
4. State Ampere's circuital law. Apply this to calculate the magnetic field due to solenoid. **(Rohilkhand 1999)**
5. State Ampere's circuital law and. Discuss why and how it was modified to include the displacement current. **(Meerut 1999)**
6. State Ampere's law and show that it leads to a definition of magnetic field induction B. Prove that this field can be derived from a vector potential. Expand this potential in a multiple potentials and thus prove that the leading term in the potential is a dipole term. **(Rohilkhand 1998; Agra 1997; Merut 1992)**
7. (a) State Ampere's line integral theorem.
   (b) Show mathematically how a circular coil having n turns and cross-sectional area A and carrying current I can be taken equivalent to a magnetic dipole of moment $\vec{\mu} = n I A$. **(Meerut 1980)**
8. What do you mean by magnetic scalar potential ? Obtain an expression for the same due to a current loop. **(Rohilkhand 1999)**
9. (a) Define magnetic vector potential and discuss its utility in magnetostatics.
   (b) Obtain expression for magnetic vector potential at a point quite far away from a stationary current distribution. Define herefrom the magnetic moment characterising a given current distribution. **(Rohilkhand 1980)**
10. (a) What do you understand by permeability and susceptibility of a magnetic substance ? Discuss about their variation wih the intensity and frequency of the applied magnetic field.
    (b) Discuss Langevin's theory of paramagnetism in gas. **(Rohilkhand 79; Agra 75, 74)**
11. (a) What do you understand by scalar and vector potential ? What is their utility ?
    (b) Discuss Langevin's theory of paramagnetism in gases. **(Agra 1997)**
12. Differentiate between magnetic vector potential and magnetic scalar potential. Derive expression for the magnetic vector potential and magnetic induction due to a small element of a current carrying current at a large distance. **(Meerut 1991)**

13. What do you understand by magnetic permeability and susceptibility of a substance ? Discuss with required theory a suitable method of measurement of susceptibility of a solid. **(Rohilkhand 87, 82)**
14. Distinguish between dia, para and ferromagnetic substances. Given an account of Langevin's theory of diamagnetism and show that diamagnetic susceptibility is independent of temperature. **(Meerut 1972)**
15. Distinguish between para, dia and ferromagnetism. Describe Langevin's theory of paramagnetism and discuss Wiess modification to explain the phenomenon of spontaneous magnetism in ferromagnetic material. **(Rohilkhand 1988)**
16. What is a magnetic dipole ? Derive an expression for the vector potential and magnetic field due to a small loop of current. **(Meerut 1987)**
17. (a) How do you explain diamagnetism with the help of Faraday's laws of electromagnetic induction ? Derive an expression for the change in magnetic moment of an orbit due to an applied magnetic field.

    (b) Show in what features a magnetic circuit is analogous to an electric circuits. In what respect does the analogy fail ? **(Meerut 1987)**
18. Given an account of Langevin's theory of paramagnetism and point out its limitations. Show how this has been extended to account for the behaviour of solid paramagnetics. **(Rohilkhand 1976; Meerut 72)**
19. Give Langevin's theory of paramagnetism and discuss Weiss' modifications to explain the phenomenon of spontaneous magnetisation in ferromagnetic substances.
20. What is hysteresis ? Discuss in brief a laboratory method for determining the hysteresis curve of a sample or iron. What is the importance of this curve ? Derive an expression for the energy dissipated in hysteresis cycle. **(Rohilkhand 1985; Meerut 1989, 82)**
21. (a) Define the magnetic moment of an enclosure containing a current.

    (b) What are various kinds of currents produced in a material medium. Show that $\partial P/\partial t$ represents the space-time average of molecular currents produced due to varying polarisaion. **(Agra 1973)**
22. (a) What do you understand by the intensity of magnetization m. Establish the relation $J = \text{curl } m$ where J is current density vector in a non-uniformly magnetised material at a point where intensity of magnetization vector is m.

    (b) Show that at the interface of the media.

    $$B_{1n} = B_{2n} \text{ and } H_{1t} = H_{2t}$$

    where B_{1n}, B_{2n} stand for the normal component of magnetic induction B and H_{1t} and H_{2t} that for tangential component of magnetic intensity H,
23. A magnetised sphere of radius R is placed in uniform external magnetic field H_0. Find out the potential and field inside and outside the sphere.
24. Discuss the nature of magnetic field in magnetic materials.

    Describe Langevin's theory of paramagnetism. **(Agra 1983)**
25. Determine an expression for the magnetic interaction between two steady current loops. Hence obtain an expression for the magnetic induction intensity due to steady current. Derive an expression for the scalar potential of this field and discuss its nature. **(Agra 1982)**
26. Show the magnetic energy U_m within a magnetic material when placed in a magnetic field H is given by

    $$U = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} \, dV$$

    where the symbols their usual meaning.
27. Write short notes on :
    - (i) Biot Savart Law, **(Rohilkhand 1987)**
    - (ii) Hysteresis, **(Rohilkhand 1987)**
    - (iii) Displacement current. **(Agra 1976)**
    - (iv) Magnetic circuits,
    - (v) Magnetic reluctance. **(Agra 1976)**
28. Define or repute the following statement with reason :

    The magnetic scalar potential can be used to compute the magnetic fields within current carrying media. **(Rohilkhand 1986, 88)**

10

Applications of Electromagnetic Waves II : Dispersion and Scattering

10.1. Introduction

Maxwell's theory and Maxwell's equations are based on the assumption of dielectrics of constant permittivity ε, magnetic substances of constant permeability μ, conductors with conductivity σ. These assumptions are unsatisfactory for two reasons :

(i) There are cases in which the three constants mentioned are not really constants. For example the permeability of iron depends on the field strength. The permittivity (or dielectric constant) of almost all substances depend on the frequency; as we have seen. The refractive index n is given by the relation $n = \sqrt{(\mu\varepsilon/\mu_0\varepsilon_0)} = \sqrt{(K_m K_e)}$ and the well known phenomenon of *dispersion* shows that the refractive index varies with wavelength of frequency.

(ii) Matter is known to be composed of electric charges, with negative charged around atomic-nuclei with positive charges. It ought to be possible to explain these typically electrical properties of matter directly in terms of the electronic structure, wihout variable dielectric constant.

These reasons imply that we must consider the interaction between electromagnetic radiation and microscopic matter (*i.e.* individual electrons, nuclei etc.) for explaining various optical and electrical properties. Such an interaction is properly described by quantum theory. However before the advent of quantum theory a variety of *interesting* effects were explained in terms of *classical electron theory'* which treats the interaction of electromagnetic radiation with discrete charges classically.

When a beam of electromagnetic radiation is propagated in a material medium, its velocity is less than that in vacuum and its intensity gradually decreases as it advances into the medium. The velocity of electromagnetic radiations in a material medium varies with wavelength due to variation of refractive index of medium with wavelength. The phenomenon of variation of refractive index with wavelength is called *dispersion.* The reduction in the intensity of a beam traversing a medium is partly due to scattering and partly due to absorption of electromagnetic radiation. In true absorption the electromagnetic radiation disappears, the energy being converted into heat. The phenomenon of scattering, absorption and dispersion, are inseparably connected.

We shall now examine the way in which simple electronic models of dielectrics and conductors can lead to variations of dielectric constant and conductivity with frequency which are in at least qualitative agreement with thé observations.

10.2. Dispersion : Normal and Anomalous

The variation of refractive index of a medium with wavelength is known as *dispersion.* Mathematically it is represented by $dn/d\lambda$. In general the refractive index decreases with wavelength. This effect is known as *normal dispersion.* But over small wavelength ranges, there is often increase of refractive index due to an increased absorprion of the radiation passing through the medium. This effect is known as *anomalous dispersion.* Fig. 10.1. shows graphically normal and anomalous dispersions.

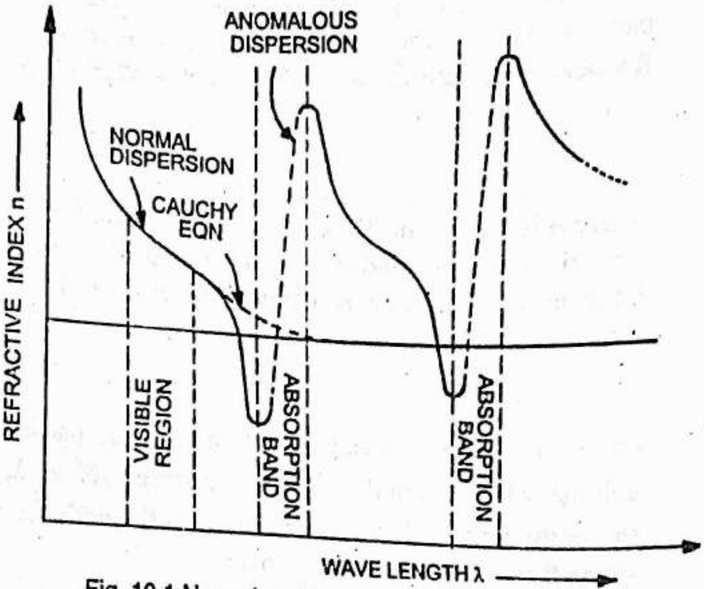


Fig. 10.1 Normal and Anomalous dispersion.

The characteristics of normal dispersion

(i) *The refractive index decreases as wavelength increases.*

(ii) The rate of increase of refractive index with wavelength *i.e.* $dn/d\lambda$ (*i.e.* slope of curve) *is greater at shorter wavelengths.* In other words *dispersion increases as the wavelength decreases.*

Cauchy's Equation. The normal dispersion may be explained by *Cauchy's* empirical equation expressing refractive index as a function of wavelength given by

$$n^2 = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} \qquad ...(1)$$

where *A, B,C* are constants, depending on the medium and decreasing rapidly in magnitude as we proceed to the higher order terms. For some purposes the variation of refractive index with wavelength may be represented with sufficient accuracy by the first two terms of the power series, *i.e.,*

$$n^2 = A + \frac{B}{\lambda^2} \qquad ...(2)$$

Since *A* and *B* are positive, it follows that the refractive index *n* decreases as λ increases *i.e., n* is maximum at the violet end and least at the red end of the visible region of the spectrum.

Differentiating above equation, we get for the dispersion

$$\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3}\cdot\frac{1}{2n} = -\frac{B}{\lambda^3}\left(A + \frac{B}{\lambda^2}\right)^{-1/2} \approx -\frac{AB}{\lambda^3} \qquad ...(3)$$

This indicates that to a close approximation the dispersion varies inversely as the cube of wavelength. The negative sign indicates that the slope of dispersion curve is negative and the magnitude of slope *i.e.* dispersion decreases as λ increases.

Hartmann's Formula. If the wavelength range is not too large the *normal dispersion* may be explained by *Hartmann's empirical formula*

$$n = n_0 + \frac{b}{\lambda - \lambda_0} \qquad ...(4)$$

where n_0, λ_0 and *a* are constants to the determined from the observations.

Sellmeier's Formula. The first dispersion formula of general application was given by Sellmeier in 1871, on the basis of the elastic solid theory of light. He assumed that the medium contains elastically bound particles capable of vibrating with a natural frequency of vibration say ν_0 in the absence of any periodic force and if the radiation of this frequency is passed through the medium, the particles resonate and the energy is absorbed. For radiation of other frequencies the particles execute forced oscillations, the amplitudes increasing as the frequency of the radiation approaches the resonant frequency. These vibrations cause a change in the velocity of radiation. If all the elastically bound particles in the medium have a natural frequency corresponding to radiation of wavelength λ_0 in a vacuum, then Sellmeier's theory gives

$$n^2 = 1 + \frac{A\lambda^2}{\lambda^2 - \lambda_0^2} \quad ...(5)$$

where A is a constant. This equation corresponds to a medium having one absorption band. The existence of several absorption bands is explained by postulating the existence of particles with several different natural frequencies. In this case equation (5) takes the form

$$n^2 = 1 + \sum_p \frac{A_p\lambda^2}{\lambda^2 - \lambda_p^2} \quad ...(6)$$

where A_p is proportional to the number of particles, per unit volume whose natural frequency corresponds to λ_p. Fig. 10.2 shows the form of dispersion curve by Sellmeier's formula in case where there are two absorption bands.

Although Sellmeier's formula represents a considerable improvement over Cauchy's formula, it fails in the region of absorption band; because it indicates that n is infinite and discontinuous at λ_1, λ_2 etc. However in regions remote from an absorption band it gives better results than a formula of Cauchy type having same number of constants. It is easily seen that Sellmeier's formula reduces to Cauchy's formula when certain approximations are made. If there is one absorption band and this is at a considerable distance to a short wavelength side of the region where dispersion is required *i.e.* if $\lambda_0 \ll \lambda$, then equation (5) gives

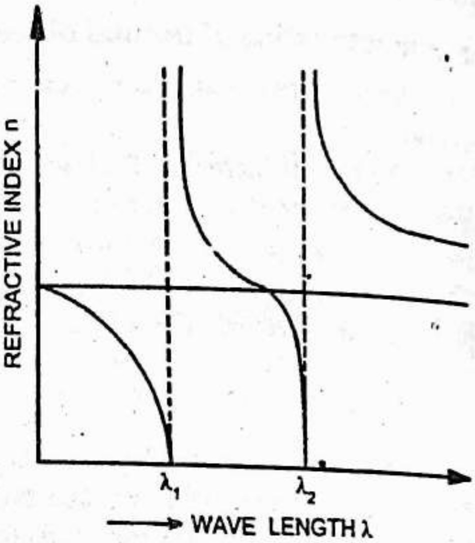


Fig. 10.2. Dispersion curve given by Sellmeier's formula for absorption bands.

$$n^2 = 1 + A\left(1 - \frac{\lambda_0^2}{\lambda^2}\right)^{-2} = 1 + A + \frac{A\lambda_0^2}{\lambda^2} + \frac{A\lambda_0^4}{\lambda^4} \quad ...(7)$$

which has the same form as Cauchy's formula (1)

10.3. Dispersion in Gases

The adequate theory of dispersion based on electromagnetic theory was given by H.A. Lorentz. In the case of gases, he assumed :

1. When an electromangetic wave passes through a medium, the atomic (or molecular) dipoles are created in the medium due to relative displacement of electrons and nuclei of neutral atoms due to electric field of the wave. Thus a dipole moment is induced in the gas molecule.

2. These atomic (or molecular) dipoles execute forced vibrations under the oscillatory electric field of the passing wave.

3. The molecules of a gas are far apart; therefore the interactions between them may be neglected. Moreover the local polarisation intensity $\mathbf{P}/3\varepsilon_0$ due to orientation of dipoles, already existing may be

neglected. Therefore the electric force on the electron is $e\mathbf{E}$ where $\mathbf{E}$ denotes the electric intensity in the electromagnetic wave and e is charge on electron (e being a negative quantity $= -1{\cdot}6 \times 10^{-19}$ coul). The magnetic force on the electron on account of its motion may be neglected.

4. The electrons are bound to the nucleus by a *linear restoring force* proportional to displacement $\mathbf{r}$ from the equilibrium position given by $F_r \propto \mathbf{r}$ or $\mathbf{F}_r = -K\,\mathbf{r}$, where K is a certain positive constant which depends on the structure and properties of particles.

5. The electrons experience a *damping or frictional* force proportional to velocity to account for the absorption; given by

$$\mathbf{F}_d = -b\frac{d\mathbf{r}}{dt}$$

where b is a new positive constant called the resistance constant.

6. The nucleus is very heavy as compared to electron; hence reduced mass of the dipole approaches the mass of the electron

$$\left\{ m = \frac{m_e m_n}{m_e + m_n} = \frac{m_e}{1 + (m_e/m_n)} \to m_e \right\}$$

7. The electric field $\mathbf{E}$ is constant in space over an atom or molecule, thus

$$\mathbf{E} = E_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \approx E_0\, e^{-i\omega t}$$

In the light or above assumptions, the total force acting on the electron

$$= e\mathbf{E} - k\mathbf{r} - b\,(d\mathbf{r}/dt):$$

hence according to Newton second law, the equation of motion of the electron is

$$m\frac{d^2\mathbf{r}}{dt^2} = e\mathbf{E} - K\mathbf{r} - b\frac{d\mathbf{r}}{dt}$$

or

$$m\frac{d^2\mathbf{r}}{dt^2} + b\frac{b\mathbf{r}}{dt} + K\mathbf{r} = e\mathbf{E} \qquad \text{...(1a)}$$

In an *isotropic medium* electric field $\mathbf{E}$ and the displacement $\mathbf{r}$ of the charge are in same direction, therefore we may use the magnitude of vectors and rewrite equation (1a) as

$$m\frac{d^2 r}{dt^2} + b\frac{dr}{dt} + Kr = eE \qquad \text{...(1b)}$$

or

$$\frac{d^2 r}{dt^2} + \frac{b}{m}\frac{dr}{dt} + \frac{K}{m}r = \frac{eE}{m} \qquad \text{...(2)}$$

Now substituting

$$\frac{b}{m} = g \text{ and } \frac{K}{m} = \omega_0^2 \text{ and } E = E_0\, e^{-i\omega t}$$

in equation (2), we get

$$\frac{d^2 r}{dt^2} + g\frac{dr}{dt} + \omega_0^2\, r = \frac{e}{m}E_0 e^{-i\omega t} \qquad \text{...(4)}$$

The equation represents the equation of damped harmonic oscillator in forced oscillation, with ω_0 as natural frequency of oscillator. The solution of this differential equation consists of two parts namely (i) the complementary function and (ii) the particular integral.

(i) Complementary function : The complimentary function is the solution of equation (4) in the absence of electric field E *i.e.*

$$\frac{d^2r}{dt^2}+g\frac{dr}{dt}+\omega_0^2 r = 0 \qquad ...(5)$$

Let the solution of above equation be

$$r = A^{\alpha t}$$

so that $$\frac{dr}{dt} = Ae^{\alpha t} \text{ and } \frac{d^2r}{dt^2} = A\alpha^2 e^{\alpha t}.$$

Making these substitution, equation (5) gives

$$\alpha^2+g\alpha+\omega_0^2 = 0$$

This gives $$\alpha = \frac{-g\pm\sqrt{(g^2-4\omega_0^2)}}{2} = -\frac{g}{2}\pm\sqrt{\left\{\frac{g^2}{4}-\omega_0^2\right\}}$$

Now assuming that damping is small so that $g^2/4 \ll \omega_0^2$: hence we may write

$$\alpha = -\frac{g}{2}\pm i\sqrt{\left\{\omega_0^2-\frac{g^2}{4}\right\}} = +\frac{g}{2}\pm i\beta \text{ where } \beta = \sqrt{\left(\omega_0^2-\frac{g^2}{4}\right)}$$

Therefore the solution (6) may be expressed as

$$r = A_1 e^{[(-g/2)+i\beta]t}+Ae^{[(-g/2)-i\beta]t}$$
$$= e^{-gt/2}\,[A_1 e^{i\beta t}+A_2 e^{-i\beta t}]$$

The constants A_1 and A_2 may be determined by the initial conditions. ...(7)

(ii) The particular integral. The particular integral is obtained by solving the equation (4) *i.e.*

$$\frac{d^2r}{dt^2}+g\frac{dr}{dt}+\omega_0^2 r = \frac{e}{m}E_0 r^{-i\omega t} \qquad ...(8)$$

Let its solution be given by

$$r = Be^{-\omega t}$$

i.e. $$\frac{dr}{dt} = -i\omega Be^{-\omega t} \text{ and } \frac{d^2r}{dt^2} = -\omega^2 Be^{-i\omega t}$$

Making these substitution in equation (8), we get

$$-\omega^2 Be^{-i\omega t}-i\omega g\,Be^{-i\omega t}+\omega_C^2 Be^{-i\omega t} = (e/m)\,E_0 e^{-i\omega t}$$

For all values of t, this gives .

$$B = \frac{eE_0}{m\,(\omega_0^2-\omega^2-ig\omega)} \qquad ...(10)$$

Thus the solution (2) becomes

$$r = \frac{eE_0}{m\,(\omega_0^2-\omega^2-ig\omega)}\,e^{-i\omega t} \qquad ...(11)$$

Thus the general solution which is the sum of complementary function and particular integral may be expressed from (7) and (11) as

$$r = e^{-gt/2}\,(A_1 e^{i\beta t}+A_2 e^{-i\beta t})+\frac{eE_0}{m\,(\omega_0^2-\omega^2-ig\omega)}\,e^{-i\omega t} \qquad ...(12)$$

In this solution the first term on R.H.S. represents free damped vibrations which soon disappear due to the factor $e^{-gt/2}$: hence while considering the first solution : This term may be neglected. Then the solution (10) takes the form

$$r = \frac{eE_0e^{-i\omega t}}{m(\omega_0^2-\omega^2-ig\omega)} = \frac{eE}{m(\omega_0^2-\omega^2-ig\omega)} \qquad ...(13a)$$

or in vector form

$$\mathbf{r} = \frac{e\mathbf{E}}{m(\omega_0^2-\omega^2-ig\omega)} \qquad ...(13b)$$

Now the *dipole moment* **p** which results due to the displacement of electron may be expressed as

$$\mathbf{p} = e\mathbf{r} \qquad ...(14)$$

If there are N dispersion electrons per unit volume, then polarisation **P**, which is equal, to the dipole moment per unit volume, is given by

$$\mathbf{P} = N\mathbf{p} = Ne\mathbf{r} \qquad ...(15)$$

using equation (13b), we get

$$\mathbf{P} = \frac{N(e^2/m)\,\mathbf{E}}{\omega_0^2-\omega^2-ig\omega} \qquad ...(16)$$

In deriving this equation we have assumed that the all oscillators of the medium have the same natural frequency. This would correspond to the existance of only one *absorption line*.To take into account the possibility of other absorption lines we suppose that there are N dispersion electrons per unit volume such that N_1 electrons are characterised by constants ω_1 and g_1 : N_2 electrons by constants ω_2 and g_2 and so on. Then equation (15) becomes

$$\mathbf{P} = e\,\Sigma\, N_k\mathbf{r}_k \qquad ...(17)$$

and from equation (13b), we have

$$\mathbf{r}_k = \frac{eE}{m(\omega^2-\omega^2-ig_k\omega)}$$

so that equation (16) takes the form

$$\mathbf{P} = \mathbf{E}\sum_k \frac{N_k(e^2/m)}{(\omega_k^2-\omega^2-ig_k\omega)} \qquad ...(18)$$

The *electrical polarizability* α is now given by

$$\alpha = \frac{\mathbf{P}}{\mathbf{E}} = \sum_k \frac{N_k}{(\omega_k^2-\omega^2-ig_k\omega)} \qquad ...(19)$$

∴ *Dielectric constant*

$$K_e = 1+\frac{1}{\varepsilon_0}\left\{\frac{\mathbf{P}}{\mathbf{E}}\right\}\left\{\text{as } K_e = 1+\chi_e = 1+\frac{\alpha}{\varepsilon_0} = 1+\frac{1}{\varepsilon_0}\frac{\mathbf{P}}{\mathbf{E}}\right\}$$

$$= 1+\frac{1}{\varepsilon_0}\sum_k \frac{N_k(e^2/m)}{(\omega_k^2-\omega^2-ig\omega)} \qquad ...(20)$$

For a *non-magnetic substance*, magnetic relative permeability $K_m = 1$

$$n = \sqrt{K_eK_m} = \sqrt{K_e} \qquad ...(21)$$

Therefore equation (20) gives

$$n^2 = 1+\frac{1}{\varepsilon_0}\sum_k \frac{N_k(e^2/m)}{(\omega_k^2-\omega^2-ig_k\omega)} \qquad ...(22)$$

This equation indicates that the *refractive index is complex and frequency dependent.* The signifieance of complex refractive index is that there is an absorption of energy in the medium; while the frequency dependence of refractive index explains the well known phenomenon of dispersion. Equation (22) is known as *Lorentz dispersion equation.* Now let us consider the following special cases :

Case (i) Static Case. In the limit of low frequencies (long wavelengths of waves where ($\omega \ll \omega_k$), we may neglect the last two terms in denominator and find

$$n^2 = K_e = 1+\frac{1}{\varepsilon_0}\sum_k \frac{N(e^2/m)}{\omega_k^2} \qquad ...(23)$$

This equation gives the *static values of refractive index and dielectric constant of insulators.*

Case (ii). Normal dispersion. In the region remote from natural frequencies of electrons, the damping is very small *i.e.* the term containing g_k may be neglected in comparsion with $\omega_k^2-\omega^2$ and we find a *refractive index given by*

$$n^2 = 1+\frac{1}{\varepsilon_0}\sum_k \frac{N_k(e^2/m)}{\omega_k^2-\omega^2} \qquad ..(24)$$

This equation indicates that the refractive index increases with increase in frquencey (or decrease in wavelength) of incident waves. This is *normal dispersion.* If we consider that the natural frequencies of electrons lie in ultraviolet, (or in infrared) region, then for optical frequencies (visible region), the refractive index increases with increase of frequency, the normal behaviour, *i.e.* for a given medium the red light has the lowest while the violet light has the largest refractive index for optical range of frequencies.

Sellmeier's and Cauchy's Formulae from Lorentz dispersion equation

The wavelength λ in free space is given by

$$\lambda = \frac{2\pi c}{\omega} \text{ and if } \lambda_k = \frac{2\pi c}{\omega_k}, \qquad ...(25)$$

then equation (24) may be expressed as

$$\begin{aligned} n^2 &= 1+\frac{1}{\varepsilon_0}\sum_k \frac{N_k\left(\frac{e^2}{m}\right)}{\left(\frac{4\pi^2c^2}{\lambda_k^2}-\frac{4\pi^2c^2}{\lambda^2}\right)} \\ &= 1+\frac{1}{\varepsilon_0}\sum_k \frac{(e^2/m)\,N_k\lambda^2\lambda_k^2}{4\pi^2c^2(\lambda^2-\lambda_k^2)} \\ &= 1+\sum_k \frac{A_k\lambda^2}{\lambda^2-\lambda_k^2} \qquad ...(26) \end{aligned}$$

where $$A_k = \frac{1}{\varepsilon_0}\frac{N_k\lambda_k^2(e^2/m)}{4\pi^2c^2} = \frac{1}{4\pi^2\varepsilon_0}\frac{e^2}{mc^2}N_k\lambda_k^2.$$

Equation (26) represents well known *Sellmeier's formula.*

Further if we consider that $\lambda \gg \lambda_k$, then equation (26) reduces to

$$n^2 = 1+\sum_k A_k\left[1-\frac{\lambda_k}{\lambda}\right]^{-1} = 1+\sum_k A_k\left[1+\frac{\lambda_k^2}{\lambda^2}+\frac{\lambda_k^4}{\lambda^4}+\ldots\right]$$

$$= 1+\sum_k A_k+\sum_k \frac{A_k\lambda_k^2}{\lambda^2}+\sum_k \frac{A_k\lambda_k^4}{\lambda^4}+\ldots$$

$$= A+\frac{B}{\lambda^2}+\frac{C}{\lambda^4}+\ldots \qquad \ldots(27)$$

where

$$A = 1+\sum_k A_k;\quad B = \sum_k A_k\lambda_k^2;\quad C = \sum_k A_k\lambda_k^4$$

Equation (27) represents required *Cauchy's formula.*

Case (iii). Anomalous dispersion. If the frequency of the electromagnetic wave lie near a natural frequency, we obtain the phenomenon of anomalous dispersion. For simplicity let us assume that there is only one natural frequency *i.e.* $\omega_k = \omega_0$; then equation (22) retains complex refractive index say n^* and takes the form

$$n^{*2} = 1+\frac{1}{\varepsilon_0^2}\left(\frac{e^2}{m}\right)\frac{N}{\omega_0^2-\omega^2-ig\omega} \qquad \ldots(28)$$

or

$$n^* = \left[1+\frac{1}{\varepsilon_0^2}\left(\frac{c^2}{m}\right)\frac{N}{(\omega_0^2-\omega^2-ig\omega)}\right]^{1/2} \qquad \ldots(29)$$

Remembering that for gases the refractive index $|n^*|$ is very nearly unity, therefore equation (29) may be expressed as

$$n^* = 1+\frac{1}{2\varepsilon_0}\left(\frac{c^2}{m}\right)\frac{N}{(\omega_0^2-\omega^2-ig\omega)} \qquad \ldots(30)$$

Multiplying numerator and denominator by $(\omega_0^2-\omega^2+ig\omega)$ we get

$$n^* = 1+\frac{1}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\left[\frac{N(\omega_0^2-\omega^2+ig\omega)}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right]$$

$$= \left[1+\frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right]+i\left[\frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{g\omega}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right] \qquad \ldots(31)$$

Replacing complex n^* by $n(1+i\kappa)$, and comparing real and imaginary parts, we get

$$\left.\begin{aligned} n &= 1+\frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+g^2\omega^2} \\ n\kappa &= \frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{g\,\omega}{(\omega_0^2-\omega^2)^2+g^2\omega^2} \end{aligned}\right\} \qquad \ldots(32)$$

Fig. 10·3 represents variation of real and imaginary parts of n^* (*i.e.* n and $n\kappa$) as a function of frequency ω. It is clear that at low frequencies, the (real) refractive index is slightly greater than unity and increases as ω increases reaches a maximum and in the neighbourhood of resonance freqeuncy ω_0, the refractive index decreases rapidly and at $\omega = \omega_0$ it approaches unity and continues to decrease until it becomes minimum at certain frequency-greater than but in the neighbourhood of resonance frequency. When frequency further increases, n again increases and approaches a value slightly less than unity. for every large value of frequency. Thus we see that in the neighbourhood of absorption region the refractive index decreases with increasing frequency and this is the anomalous behaviour giving rise to the term *anomalous dispersion.*

The imaginary part $n\kappa$ of n^* corresponds to absorption of electromagnetic waves propagating through the gas. It has a typical resonance shape. It is obvious that the absorption coefficient $n\kappa$ is small everywhere except near the resonance frequency where n varies rapidly. Thus in the region of rapidly varying n, the gas is relatively highly absorbing.

For a real gas there exist many resonant frequencies ω_k and corresponding damping constant g_k, so that

$$n^* = 1+\frac{1}{2\varepsilon_0}\frac{e^2}{m}\sum_k \frac{N_k}{(\omega_k^2-\omega^2-ig_k\omega)}$$

$$= 1+\frac{1}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\sum_k \frac{N_k(\omega_k^2-\omega^2+ig_k\omega)}{(\omega_k^2-\omega^2)^2+g_k^2\omega^2} \quad ...(33)$$

and if we separate into real and imaginary parts by putting $n^* = n(1+i\kappa)$, we obtain

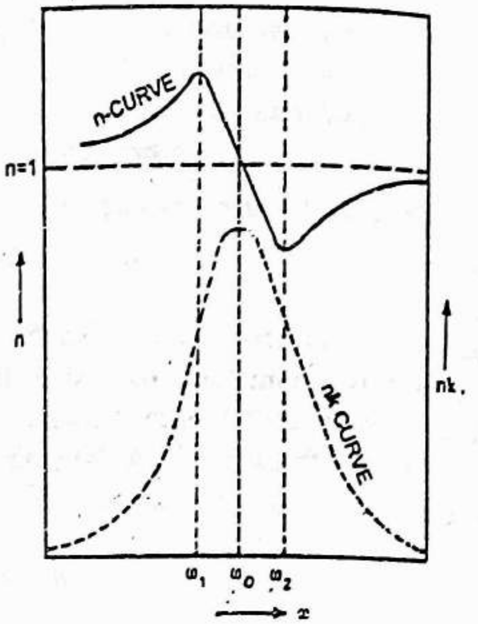


Fig. 10.3. Plot of n_k and n as a function of ω.

$$\left.\begin{aligned} n &= 1+\frac{i}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\sum_k \frac{N_k(\omega_k^2-\omega^2)}{(\omega_k^2-\omega^2)^2+\omega^2 g_k^2} \\ \text{and} \qquad n\kappa &= \frac{1}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\sum_k \frac{N_k\,\omega g_k}{(\omega_k^2-\omega^2)^2+\omega^2 g_k^2} \end{aligned}\right\} \quad ...(34)$$

Ex. 1. *Prove that in the case of gases the maximum and minimum values of refractivne index (n) occur at the positions where the absorption coefficient reaches half its maximum value.*

Solution. We know that

$$n^* = n+in\kappa = \left[1+\frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\left\{\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right\}\right]+i\left[\frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{g\omega}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right] \quad ...(1)$$

This gives *refractinve index,*

$$n = 1+\frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2+g^2\omega^2}$$

and *absorption coeffcieint,*

$$n\kappa = \frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\frac{g\omega}{(\omega_0^2-\omega^2)^2+g^2\omega^2} \quad ...(2)$$

It is obvious that absorption coefficient is maximum when $\omega = \omega_0$ and this maximum value is given by

$$(n\kappa)_{maximum} = \frac{N}{2\varepsilon_0}\left(\frac{e^2}{m}\right)\cdot\frac{1}{g\omega_0} \qquad (\text{since } \omega = \omega_0)$$

i.e.

$$(n\kappa)_{maximum} = \frac{A}{g\omega_0} \quad \text{when } A = \frac{Ne^2}{2\varepsilon_0 m} \quad ...(3)$$

The refractinve index n is maximum or minimum when

$$\frac{\partial n}{\partial\omega} = 0$$

i.e.
$$\frac{\partial}{\partial\omega}\left[1+\frac{Ne^2}{2\varepsilon_0 m}\left(\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right)\right]=0$$

i.e.
$$\frac{\partial}{\partial\omega}\left[\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+g^2\omega^2}\right]=0. \qquad \text{...(4)}$$

Now near absorption band $\omega \to \omega_0$, so that

$$\omega_0^2-\omega^2 = (\omega_0-\omega)(\omega_0+\omega) = 2\omega_0(\omega_0-\omega)$$

and
$$\omega^2 = \omega_0^2$$

Therefore equation (4) becomes

$$\frac{\partial}{\partial\omega}\left[\frac{2\omega_0(\omega_0-\omega)}{4\omega_0^2(\omega_0-\omega)^2+g^2\omega_0^2}\right]=0$$

or
$$\frac{-2\omega_0\{4\omega_0^2(\omega_0^2-\omega)^2+g^2\omega_0^2\}-\{-4\omega_0^2(\omega_0-\omega)\,2\omega_0(\omega_0-\omega)\}}{[4\omega_0^2(\omega_0-\omega)^2+g^2\omega_0^2]^2}=0$$

i.e.
$$(\omega_0-\omega)^2\,.\,4\omega_0^2+g^2\omega_0^2-(\omega_0-\omega)^2\,.\,8\omega_0^2=0$$

i.e.
$$-4(\omega_0-\omega)^2\omega_0^2+g^2\omega_0^2=0$$

i.e.
$$(\omega_0-\omega)^2=g^2/4$$

i.e.
$$\omega_0-\omega=\pm g/2 \qquad \text{...(5)}$$

If ω_1 and ω_2 correspond to maximum and minimum values of refractive index, then we may write

$$\omega_0-\omega_1 = g/2 \quad \text{and} \quad \omega_0-\omega_2=-g/2$$

$$\omega_1=\omega_0-g/2 \quad \text{and} \quad \omega_2=\omega_0+g/2$$

$$\omega_2-\omega_1 = g \qquad \text{...(6)}$$

equation expresses an important result that the *two frequencies at which refractive index is minimum and maximum are separated by a distance g on the frequency axis.*

Now Let us find values of absorption coefficient at ω_1 and ω_2

$$(n\kappa)_{\omega_1}=\left(\frac{Ne^2}{2\varepsilon_0 m}\right)\frac{g\,\omega_1}{(\omega_0^2-\omega_1^2)^2+g^2\omega_1^2}$$

Remembering that ω_1 lies near ω_0 or $g/\omega_0 \ll 1$, we get

$$(n\kappa)_{\omega_1}=\frac{Ne^2}{2\varepsilon_0 m}\frac{g\omega_2}{4\omega_0^2\cdot(\omega_0-\omega_1)^2+g^2\omega_0^2} \qquad \text{...(7)}$$

Using $\omega_0-\omega_1=g/2$, we get

$$(n\kappa)_{\omega_1}=\frac{Ne^2}{2\varepsilon_0 m}\frac{g\omega_0}{4\omega_0^2(g^2/4)+g^2\omega_0^2}$$

$$=\frac{Ne^2}{2\varepsilon_0 m}\cdot\frac{1}{2g\omega_0}=\frac{A}{2g\omega_0} \qquad \text{...(8a)}$$

Similarly
$$(n\kappa)_{\omega_2}=\frac{Ne^2}{2\varepsilon_0 m}\frac{1}{2g\omega_0}=\frac{A}{2g\omega_0} \qquad \text{...(8b)}$$

Thus comparing equations (3) and (8), we get

$$(n\kappa)_{\omega_1 \text{ or } \omega_2}=\tfrac{1}{2}(n\kappa)_{maximum}. \qquad \text{...(9)}$$

Thus for gases the maximum and minimum values of refractive index occur at the position where the absorption coefficient reaches half its maximum value.

10.4. Experimental Demonstration of Anomalous Dispersion in Gases.

The experimental demonstration of anomalous dispersion of sodium vapour in the neighbourhood of yellow *D*-line was given by R.W. Wood in 1904. The experimental arrangement is shown in fig. 10.4. Pieces of sodium are placed along the bottom of steel tube *T* which is partially evaluated to a pressure about 1 to 3 cm of Hg and fitted with water cooled glass window at each end. When the tube is heated from the bottom, sodium vapour diffuses upward across the tube. Thus the density of sodium vapour gradually decreases from bottom to top hence it is optically equivalent to a prism whose thickness increases downwards, resting upon its base, the refracting edge being perpendicular to the axis of tube. While light from the arc is focussed on a horizontal slit S_1 and is collimated by passing through the lens L_1. The parallel beam is refracted on passing through the sodium vapour prism and finally brought to focus on the vertical slit S_2 of the spectrometer by the use of lens L_2.

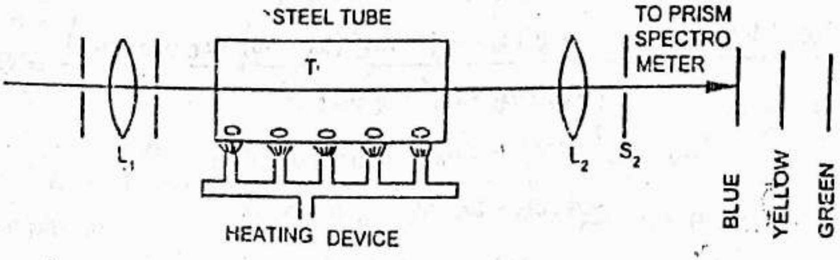


Fig. 10.4 The experimental arrangement fordemonstrating anomalous dispersion.

Initially when the tube is cold, the gas is homogeneous and a white image of slit S_1 is produced on S_2 giving rise to a continuous spectrum in the plane of the spectrometer. On account of heating when the sodium is vaporised, the prismatic action of sodium vapour causes dispersion so that the images of S_1 corresponding to different wavelengths are dispersed across the length of slit S_2. For the spectral region remote from *D*-lines (absorption lines) the refractive index of sodium vapour is unity and therefore the horizontal continuous spectrum in the spectrometer remains unaffected. But as *D*-lines are approached from the short wavelength side (*i.e.* from violet region), the refractive index for the vapour becomes less than one and the light is deviated upwards in passing through the sodium vapours. Since the spectrometer inverts the slit image, therefore the spectrum on the short wavelength side of *D*-lines the spectrum is bent down. Similarly for long wavelength side of *D*-lines the spectrum is bent up as shown on R.H.S. of fig. 10.4. This illustrates the anomalous dispersion for *D*-lines of sodium vapour.

10.5. Dispersion in Solids and Liquids.

In the case of solids and liquids the molecules are close to each other and hence the interactions among themselves can not be neglected. Since the material is polarised, therefore we may no longer make the approximation that the force acting on the electron is simply by due to electric vector of electromagnetic wave in free space, but we must take into account the added force on the electron due to the polarisation of material. In other words the effective electric

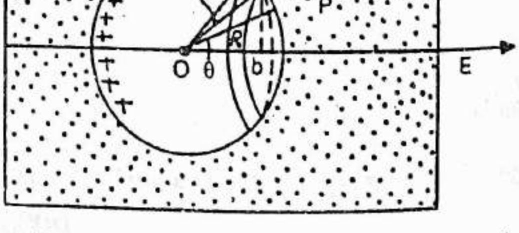


Fig. 10.5. Field in spherical cavity in material medium.

field strength is changed and is the sum of electric field vector **E** of electromagentic wave and the *electric field contribution* $\mathbf{E}_p$ due to polarisation. To calculate $\mathbf{E}_p$, consider a small sphere of radius R, with its centre at the position of the electron in question, cut out of the distribution of polarisation outside the sphere, we have induced charges on the surface of spherical volume. The surface charge density of these induced charges on *spherical ring* at an angle θ to the direction of field is

$$\sigma = \mathbf{P} \cdot \mathbf{n} = P \cos\theta .$$

The area of the ring

$$= 2\pi\,(ab)\,R\,d\theta = 2\pi R \sin\theta\, R\, d\theta$$
$$= 2\pi R^2 \sin\theta\, d\theta.$$

∴ The charge on the ring is

$$dq = 2\pi R^2 \sin\theta\, d\theta\, \sigma$$
$$= 2\pi R^2 \sin\theta\, d\theta\, P\cos\theta$$
$$= 2\pi R^2 P \sin\theta \cos\theta\, d\theta.$$

This charge produces field at the centre of the sphere given by

$$d\mathbf{E}_p = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\mathbf{n} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R^2 P\sin\theta\cos\theta\, d\theta}{R^2}\mathbf{n}.$$

By symmetry the components of field perpendicular to the direction of **P** or **E** cancel out ; thus we have to find simply the local field $\mathbf{E}_p$ along the electric field **E** of electromagnetic wave.

The component of $\mathbf{E}_p$ parallel to **E** is given by

$$d\mathbf{E}_p = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R^2 P\sin\theta\cos\theta\, d\theta}{R^2}\cos\theta .$$

Therefore the total field due to all the induced charges on the sphere parallel to **E** may be obtained by integrating over all such rings and is given by

$$\mathbf{E}_p = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0}\frac{2\pi R^2 \mathbf{P}\sin\theta\cos^2\theta\, d\theta}{R^2}$$

$$= \frac{\mathbf{P}}{2\varepsilon_0}\int_0^{2\pi} \sin\theta\cos\theta\, d\theta = \frac{\mathbf{P}}{2\varepsilon_0}\cdot\frac{2}{3} = \frac{\mathbf{P}}{3\varepsilon_0}. \qquad ...(1)$$

∴ The *total effective electric field* at the centre of the sphere is

$$\mathbf{E}_{eff} = \mathbf{E} + \mathbf{E}_p = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}. \qquad ...(2)$$

Of course there is still the contribution to the effective field due to atoms inside the small sphere in question, but in an isotropic medium this contribution averages to zero.

Therefore taking charge on an electron as (e), the equation of motion of the electron in this case will be

$$m\frac{d^2\mathbf{r}}{dt^2} = e\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right) - K\mathbf{r} - b\frac{d\mathbf{r}}{dt}$$

or

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{b}{m}\frac{d\mathbf{r}}{dt} + \frac{K}{m}\mathbf{r} = \frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right)$$

or

$$\frac{d^2\mathbf{r}}{dt^2} + g\frac{d\mathbf{r}}{dt} + \omega_0^2\mathbf{r} = \frac{e}{m}\left[\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right]\left[\text{where } \frac{b}{m} = g \text{ and } \frac{K}{m} = \omega_0^2\right]$$

As $\mathbf{P} = Ne\mathbf{r}$, we get

$$\frac{d^2\mathbf{r}}{dt^2}+g\frac{d\mathbf{r}}{dt}+\omega_0^2\mathbf{r}=\frac{e}{m}\left[\mathbf{E}-\frac{Ne\mathbf{r}}{3\varepsilon_0}\right]$$

or

$$\frac{d^2\mathbf{r}}{dt^2}+g\frac{d\mathbf{r}}{dt}+\left[\omega_0^2-\frac{Ne^2}{3\varepsilon_0 m}\right]\mathbf{r}=\frac{e}{m}\mathbf{E}. \qquad \text{...(3)}$$

In an isotropic medium **E** and **r** point in same direction, therefore we may use magnitude of vectors and rewrite equation (3) as

$$\frac{d^2r}{dt^2}+g\frac{dr}{dt}+\left[\omega_0^2-\frac{Ne^2}{3\varepsilon_0 m}\right]r=\frac{e}{m}E. \qquad \text{...(4)}$$

The solution of equation (4) taking $E = E_0\, e^{-i\omega t}$ is

$$r=\frac{(eE/m)}{\left[\left\{\omega_0^2-\frac{Ne^2}{3\varepsilon_0 m}\right\}-\omega^2-ig\omega\right]} \qquad \text{...(5a)}$$

In vector form this equation takes the form

$$\mathbf{r}=\frac{(e\mathbf{E}/m)}{\left[\left(\omega_0^2-\frac{Ne^2}{3\varepsilon_0 m}\right)-\omega^2-ig\omega\right]} \qquad \text{...(5b)}$$

As there are N dispersion electrons per unit volume, therefore the polarisation **P**, which is equal to dipole moment per unit volume, may be expressed as

$$\mathbf{P}=N\mathbf{p}=Ne\mathbf{r} \qquad \text{...(6)}$$

(since $\mathbf{p} = e\mathbf{r}$)

Using (5*b*), we obtain

$$\mathbf{p}=\frac{(Ne^2/m)\,\mathbf{E}}{\omega_0^2-\frac{Ne^2}{3\varepsilon_0 m}-\omega^2-ig\omega} \qquad \text{...(7)}$$

In deriving this we have assumed that all electrons are in identical situations having the same natural frequency. This would correspond to existence of only one absorption line. To take into account the possibility of other absorption lines we suppose that there are N dispersion electrons per unit volume such that N_1 electrons are characterised by constants ω_1 and g_1, N_2 electrons by constants ω_2 and g_2 and so on. Then equation (6) becomes

$$\mathbf{P}=e\sum_k N_k\,\mathbf{r} \qquad \text{...(8)}$$

and from (5b), we have

$$\mathbf{r}_k=\frac{(e\mathbf{E}/m)}{\omega_k^2-\frac{N_k e^2}{3\varepsilon_0 m}-\omega^2-ig_k\omega} \qquad \text{...(9)}$$

Therefore we get

$$\mathbf{P}=\frac{e^2}{m}\mathbf{E}\sum_k\frac{N_k}{\omega_k^2-\left(\frac{N_k e^2}{3\varepsilon_0 m}\right)-\omega^2-ig_k\omega} \qquad \text{...(10)}$$

$$\therefore \quad \text{Polarizability } \alpha = \frac{\mathbf{P}}{\mathbf{E}} = \frac{e^2}{m}\sum_k \frac{N_k}{\omega_k^2 - \left(\dfrac{N_k e^2}{3\varepsilon_0 m}\right) - \omega^2 - ig_k\omega} \qquad ...(11)$$

$\therefore$ *Dielectric constant* or relative permittivity of medium is given by

$$K_e = 1 + \chi_e = 1 + \frac{1}{\varepsilon_0}\frac{\mathbf{P}}{\mathbf{E}}$$

$$= 1 + \frac{e^2}{\varepsilon_0 m}\sum_k \frac{N_k}{\omega_k^2 - \left(\dfrac{N_k e^2}{3\varepsilon_0 m}\right) - \omega^2 - ig_k\omega}$$

or

$$(K_e - 1) = \frac{e^2}{\varepsilon_0 m}\sum_k \frac{N_k}{\omega_k^2 - \left(\dfrac{N_k e^2}{3\varepsilon_0 m}\right) - \omega^2 - ig_k\omega}$$

or

$$(K_e - 1)\sum_k (\omega_k^2 - \omega^2 - ig_k\omega) - \sum_k \frac{N_k e^2}{3\varepsilon_0 m}(K_e - 1) = \sum_k \frac{N_k e^2}{\varepsilon_0 m}$$

$$(K_e - 1)\sum_k (\omega_k^2 - \omega^2 - ig_k\omega) = \sum_k \frac{N_k e^2}{\varepsilon_0 m}\left(1 + \frac{K_e - 1}{3}\right)$$

or

$$= \sum_k \frac{N_k e^2}{3\varepsilon_0 m}(K_e + 2)$$

or

$$\frac{K_e - 1}{K_e + 2} = \frac{1}{3\varepsilon_0}\left(\frac{e^2}{m}\right)\sum_k \frac{N_k}{(\omega_k^2 - \omega^2 - ig_k\omega)} \qquad ...(12)$$

If N represents the number of molecules (or atoms) per unit volume and if f_k is the fraction of molecules (or atoms) of kth type, so that

$$N_k = f_k N. \qquad ...(13)$$

The equation (12) may be written as

$$\frac{K_e - 1}{K_e + 2} = \frac{e^2}{3\varepsilon_0 m} N \sum_k \frac{f_k}{\omega_k^2 - \omega^2 - ig_k\omega} \qquad ...(14)$$

In this formula the quantities f_k, which are independent of number of molecules (or atoms) per unit volume, are often called the *oscillator strengths* of various resonances.

Equation (14) represents the dispersion equation for solids and liquids and indicates that dielectric constant K_e (and hence refractive index n) is complex and is frequency dependent. Writing again $K_e = n^{*2} = n(1 + i\kappa)$, equation (14) takes the form

$$\frac{n^2(1 - i\kappa)^2 + 1}{n^2(1 + i\kappa)^2 + 2} = \frac{1}{3\varepsilon_0}\frac{e^2}{m} N \sum_k \frac{f_k}{\omega_k^2 - \omega^2 - ig_k\omega} \qquad ...(15)$$

Particular Cases :

Case (i). Clausius Mossotti relation

If M is the molecular weight of the substance, ρ its density and N_0 the Avogadro number, then

number of molecules per unit volume,
$$N = \frac{N_0 \rho}{M} \qquad \text{...(16)}$$

Substituting this in equation (14), we get

$$\frac{K_e - 1}{K_e + 2} = \frac{1}{3} \cdot \frac{e^2}{\varepsilon_0 m} \frac{N_0 \rho}{M} \sum_k \frac{f_k}{\omega_k^2 - \omega^2 - i g_k \omega}$$

or
$$\frac{K_e - 1}{K_e + 2} \frac{M}{\rho} = \frac{1}{3\varepsilon_0} \left(\frac{e^2}{m} \right) N_0 \sum_k \frac{f_k}{\omega_k^2 - \omega^2 - i g_k \omega}$$

For a given frequency R.H.S. is constant, therefore

$$\frac{K_e - 1}{K_e + 2} \frac{M}{\rho} = \text{constant.} \qquad \text{...(17)}$$

This result is known as *Clausius-Mossotti relation.*

Case (ii) Lorentz-Lorentz formula. For *transparent materials* we can neglect the damping force *i.e.* $g_k \to 0$; so that the refractive index is real ($\kappa = 0$) and may be given by

$$K_e = n^2.$$

Using this, equation (15) [or (17) for particular frequency] gives

$$\frac{n^2 - 1}{n^2 + 2} \frac{M}{\rho} = \text{constant.} \qquad \text{...(18)}$$

This law is known as *Lorentz-Lorentz formula.* This formula represents the dependence of refractive index on the density of the medium and it surprisingly holds for many substances.

Note. For gases n is nearly equal to 1 and so we may approximate

$$n^2 - 1 = (n-1)(n+1) \approx 2(n-1) \text{ and } (n^2 + 2) \approx 3;$$

therefore for gases the Lorentz-Lorentz formula takes the approximate form

$$\frac{2}{3}(n - 1) \frac{M}{\rho} = \text{constant.} \qquad \text{...(19)}$$

10.6. Scattering and Scattering Parameters

Whenever an electromagnetic wave is directed at matter, the electrons of the matter are set into periodic motion due to periodically vibrating electric and magnetic fields of the incident electromagnetic wave. Since a periodically vibrating electron is equivalent to accelerated electron and an accelerated charge always radiates out energy, therefore the electron in turn emits electromagnetic wave into space. This phenomenon is known as *scattering* and the re-emitted electromagnetic wave is known as scattered *radiation.*

If the energies of the incident and scattered radiations are equal, the phenomenon is knwon as *elastic scattering* : but if the energies of incident and scattered radiations are different, the phenomenon is known as *inelastic scattering.*

The scattering process is described in terms of a quantity known as *scattering cross-section.* To do this let us note that each molecule of the matter presents to the on-coming radiation a target area πd^2, d being diameter of molecule. This target area is just a cross-section of the region within which an interaction of electromagnetic wave and molecule of target, can take place as viewed along the direction of incident beam. This is where the name scattering cross-section comes from. Now we shall define differential and total scattering cross-sections.

Differential Scattering Cross-section. The differential scattering cross-section in any direction is defined as the ratio of the amount of energy scattered by the target in this direction per unit solid angle per second to the incident energy flux density (*i.e.* incident energy per unit area normal to direction of incident radiation) per second *i.e.*

$$\sigma(\Omega) = \frac{\text{amount of energy scattered per unit solid angle per second}}{\text{incident energy flux density per second}}$$

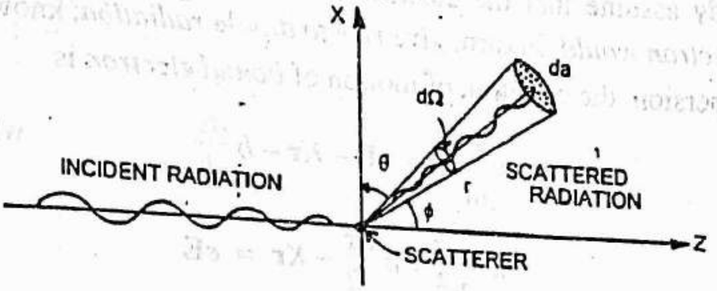


Fig. 10.6

Let $d\Omega$ be the solid angle substended by an element of area da at the scatterer as shown in fig. 10.6. If S_S is the intensity of scattered radiation, then amount of energy scattered per second in the direction $d\Omega$ is given by

$$dP_S = S_S\, da.$$

∴ The amount of energy scattered per second per unit solid angle is given by

$$\frac{dP_S}{d\Omega} = \frac{S_S\, da}{d\Omega} = S_S r^2 \left(\text{since } d\Omega = \frac{da}{r^2} \right)$$

Hence the differential scattering cross-section $\sigma(\Omega)$ is given by

$$\sigma(\Omega) = \frac{\text{scattered energy per unit solid angle per second } (dP_S/d\Omega)}{\text{incident energy density per second } (S_i)}$$

$$= \frac{S_S \cdot r^2}{S_i} \qquad \text{...(2)}$$

From equation (2) it is clear that the differential scattering cross-section has the dimensions of cross-section (area).

Total scattering cross-section. The total scattering cross-section (σ_T) is obtained by integrating differential scattering cross-section over all angles *i.e.*

$$\sigma_T = \int \sigma(\Omega)\, d\Omega = \int \frac{S_S}{S_i} r^2\, d\Omega \qquad \text{using (2)}$$

$$= \int \frac{S_S}{S_i} r^2 \cdot \frac{da}{r^2} \qquad \text{since } d\Omega = \frac{da}{r^2}$$

$$= \int \frac{S_S\, da}{S_i} = \frac{P_S}{S_i} \qquad \text{...(3)}$$

where $P_S = \int S_S\, da =$ total energy scattered per second (or total power scattered).

Thus the total scattering cross-section $n(\sigma_{\perp})$ *is defined as the ratio of total power* (*i.e.* energy per second) *scattering in all directions to the intensity* (*i.e.* energy per unit area per second) *of incident radiations.*

10.7. Theory of Scattering of Electromagnetic Waves

When an electromagnetic wave passes through a medium. The atomic (or molecular) dipoles are created in the medium due to relative displacement of electrons and nuclei of neutral atoms (or molecules due to electric field* of the wave. Thus a dipole moment is induced in the atom. These atomic dipoles execute forced vibrations under the action of the oscillatory electric field of the incident wave. Since nucleus is heavy, we may assume that the electron could be set into periodic motion by the field. The *periodic motion of the electron would*, in turn, *give rise to dipole radiation*; known as scattered radiation.

As in the case of dispersion, the equation of motion of *bound electron* is

$$m\frac{d^2\mathbf{r}}{dt^2} = e\mathbf{E} - K\mathbf{r} - b\frac{d\mathbf{r}}{dt} \qquad \text{where } \mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

or

$$m\frac{d^2\mathbf{r}}{dt^2} + b\frac{d\mathbf{r}}{dt} + K\mathbf{r} = e\mathbf{E}$$

or

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{b}{m}\frac{d\mathbf{r}}{dt} + \frac{K}{m}\mathbf{r} = \frac{e}{m}\mathbf{E}$$

Substituting $\frac{b}{m} = g$ and $\frac{K}{m} = \omega_0^2$; we get

$$\frac{d^2\mathbf{r}}{dt^2} + g\frac{d\mathbf{r}}{dt} + \omega_0^2\mathbf{r} = \frac{e}{m}\mathbf{E}. \qquad \text{...(1a)}$$

This equation represents the equation of damped harmonic oscillator in forced oscillation with ω_0 as natural frequency of oscillator. From equation (11) it is obvious that **E** and **r** are along the same direction; hence we may use magnitudes of vectors and write equation (1a) as

$$\frac{d^2\mathbf{r}}{dt^2} + g\frac{d\mathbf{r}}{dt^2} + \omega_0^2 r - \frac{e}{m}E = 0 \qquad \text{...(1b)}$$

Now let us consider that incident electromagnetic wave is a *plane polarised* having its electric vector along *X*-axis and is propagating along *Z*-direction; then acceleration of the charge (*e*) will be along *X*-axis (*i.e.* along **E**) and

$$E = E_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} = E_0 e^{-i(\omega t - kz)}. \qquad \text{...(2)}$$

so that equation of motion of electron may be written as

$$\frac{d^2x}{dt^2} + g\frac{dx}{dt} + \omega_0^2 x = \frac{e}{m}E_0 e^{-i(\omega t - kz)} \qquad \text{...(3)}$$

The solution of this equation has been obtained in section 10·3 and is given by

$$x = \frac{[e/m]\,E_0\,e^{-i(\omega t - kz)}}{\omega_g^2 - \omega^2 - ig\omega} \qquad \text{...(4)}$$

---

* The net force on charge q is $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ ...(a)

In a plane electromagnetic wave $\mathbf{B} = \frac{\mathbf{n}\times\mathbf{E}}{c}$

∴ Equations (a) becomes $\mathbf{F} = q[\mathbf{E} + \frac{\mathbf{v}}{c}\times(\mathbf{n}\times E)] \approx q\mathbf{E}$ (as $v/c \ll 1$)

Multiplying and dividing by $(\omega_0^2 - \omega^2 + ig\omega)$, we get

$$x = \frac{\left(\frac{e}{m}\right)(\omega_0^2 - \omega^2 + i\omega g)\, E_0 e^{-i(\omega t - kz)}}{(\omega_0^2 - \omega)^2 + g^2\omega^2}$$

$$= \frac{(eE_0/m)\, e^{-i(\omega t - kz)}}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}} \left[\frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}} + i\,\frac{\omega g}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}}\right]$$

$$= \left(\frac{eE_0}{m}\right)\frac{e^{-i(\omega t - kz)}}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]}(\cos\delta + i\sin\delta) \qquad \text{...(5a)}$$

where
$$\cos\delta = \frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}}$$

and
$$\sin\delta = \frac{g\omega}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}}$$

So that
$$\tan\delta = \frac{\sin\delta}{\cos\delta}$$

$$= \frac{g\omega}{\omega_0^2 - \omega^2} \qquad \text{...(6)}$$

$$x = \frac{\left(\frac{eE_0}{m}\right)e^{-i(\omega t - kz)}}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}}\, e^{i\delta} \qquad \text{...(5b)}$$

$$= \frac{\left(\frac{eE_0}{m}\right)e^{-i(\omega t - kz - \delta)}}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]^{1/2}} \qquad \text{...(7)}$$

Now as oscillatory electron is equivalent to an induced oscillating electric dipole of moment

$$p = ex \qquad \text{...(8)}$$

Substituting value of x, we get

$$p = \frac{\left(\frac{e^2}{m}\right)E_0 e^{-i(\omega t - kz - \delta)}}{(\omega_0^2 - \omega^2)^2 + g^2\omega^2} \qquad \text{...(9)}$$

$$= |p_0|\, e^{-i(\omega t - kz - \delta)}$$

where
$$|p_0| = \frac{\left(\frac{e^2}{m}E_0\right)}{[(\omega_0^2 - \omega)^2 + g^2\omega^2]^{1/2}} \qquad \text{...(10)}$$

But the *average energy radiated per second per unit area in a normal direction by an oscillating dipole is given* by*

---

* Refer section 12·11

$$S_s = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\omega^4|p_0|^2}{8\pi c^3 r^2}\sin^2\theta \qquad ...(11)$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{\omega^4}{8\pi c^3 r^2}\frac{(e^2E_0/m)^2\sin^2\theta}{(\omega_0^2-\omega^2)^2+g^2\omega^2} \qquad \text{[using (10)]}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\omega^4}{8\pi c^3 r^2}\cdot\frac{(e^4E_0^2/m^2)\sin^2\theta}{(\omega_0^2-\omega^2)^2+g^2\omega^2} \qquad ...(12)$$

Further for a plane wave the incident intensity is given by Poynting vector

$$\mathbf{S}_i = \mathbf{E}\times\mathbf{H} = \tfrac{1}{2}E_0H_0 \qquad \text{(since } \mathbf{E} \text{ is perpendicular to } \mathbf{H})$$

$$= \tfrac{1}{2}\varepsilon_0 c E_0^2$$

$$\left(\text{as } H_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_0 = (c\varepsilon_0E_0)\right) \qquad ...(13)$$

So the *differential scattering cross-section*

$$\sigma(\Omega) = \frac{S_s}{S_i}r^2 = \frac{\left\{\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega^4}{8\pi c^3r^2}\right)(e^4E_0^2/m^2)\sin^2\theta\right\}\Big/\left\{(\omega_0^2-\omega^2)^2+g^2\omega^2\right\}}{\frac{1}{2}\varepsilon_0 cE_0^2}$$

$$= \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2\frac{\omega^4\sin^2\theta}{[(\omega_0^2-\omega^2)^2+g^2\omega^2]}$$

$$= r_0^2\,\frac{\omega^4\sin^2\theta}{[(\omega_0^2-\omega^2)^2+g^2\omega^2]} \qquad ...(14)$$

where $r_0 = \dfrac{e^2}{4\pi\varepsilon_0 mc^2}$ and is known as the *classical radius of the electron.*

Sometimes is it desirable to express the differential scattering cross-section in terms of angle of scattering ϕ and polarisation angle α shown in fig. 10 instead of angle θ, the angle between **E** & **r**. To find a relation between α, ϕ and θ, we draw a plane *PMN* perpendicular to plane containing **E**, then $ON = r\cos\theta$

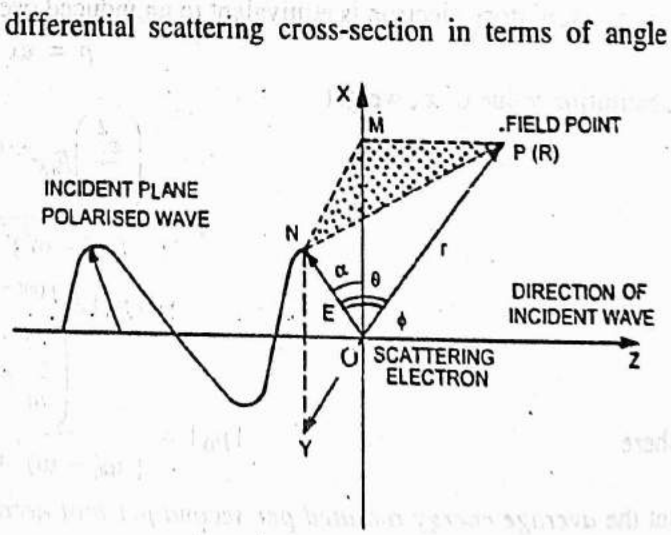


Fig. 10.7

$OM = r\sin\phi$ and again $ON = OM\cos\alpha = r\sin\phi\cos\alpha$; thereby giving

$$\cos\theta = \sin\phi\cos\alpha$$

or $$\cos^2\theta = \sin^2\phi\cos^2\alpha$$

$\therefore$ $$\sin^2\theta = 1-\cos^2\theta$$

$$= 1-\sin^2\phi\cos^2\alpha$$

$$= 1-\cos^2\alpha\,(1-\cos^2\phi) \qquad ...(15a)$$

For plane polarised wave $\alpha = 0$;

$$\therefore \quad \sin^2\theta = \cos^2\phi \qquad \text{...(15b)}$$

If the primary wave is unpolarised (*i.e. randomly polarised*), we must average over θ, and thus we obtain

$$\begin{aligned}\overline{\sin^2\theta} &= 1 - \overline{\cos^2\alpha}\,(1 - \cos^2\phi)\\ &= 1 - \tfrac{1}{2}(1 - \cos^2\phi) \text{ since } \overline{\cos^2\alpha} = \tfrac{1}{2}\\ &= \tfrac{1}{2}(1 + \cos^2\phi)\end{aligned} \qquad \text{...(16)}$$

Making this substitution, equation (14) gives

$$\sigma(\Omega) = \frac{r_0^2\omega^4}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]} \cdot \left\{\tfrac{1}{2}(1 + \cos^2\phi)\right\} \qquad \text{...(17)}$$

The factor $\frac{1}{2}(1 + \cos^2\phi)$ is the *polarisation factor known as* degree of *depolarisation.* From equations (17) it is clear that

(i) Scattering depends on the freqeuncy (ω) of incident radiation.

(ii) Scattering depends on the angle of scattering ϕ .

(iii) Scattering depends on the nature of scatterer (due to factors ω_0 and g).

The total *scattering cross-section* is obtained by integrating differential scattering cross-section over all angles *i.e.*

$$\begin{aligned}\sigma_T &= \int \sigma(\Omega)\, d\Omega\\ &= \frac{r_0^2\omega^4}{[(\omega_0^2 - \omega^2)^2 + g^2\omega^2]}\int_0^{\pi} \tfrac{1}{2}(1 + \cos^2\phi)\, 2\pi \sin\phi\, d\phi\\ &= \frac{r_0^2\omega^4}{(\omega_0^2 - \omega^2)^2 + g^2\omega^2} \cdot \frac{8}{3}\pi = \frac{8}{3}\pi \frac{r_0^2\omega^4}{(\omega_0^2 - \omega^2)^2 + g^2\omega^2}\end{aligned} \qquad \text{...(18)}$$

This equation shows that total scattering cross-section, in general, depen upon the frequency (ω) of incident radiation. Fig. 10·8 represent a plot of σ_T against ω.

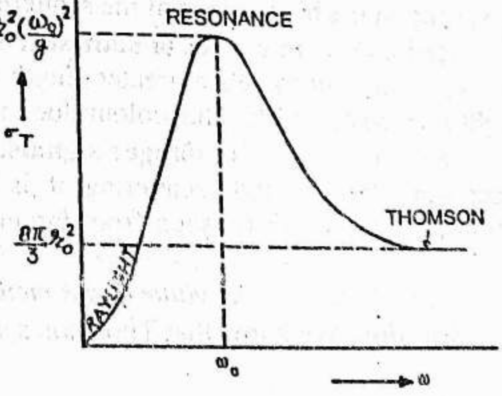


Fig. 10.8

Now let us consider the particular cases :

Case (i) Scattering by a free electron : Thomson scattering.

When the electron is free $\omega \gg \omega_0$ (since $K \to 0$) and damping constant $g = 0$,

therefore equations (17) and (18) for differential scattering cross-section and total scattering cross-section take the form

$$\sigma(\Omega) = r_0^2\, \tfrac{1}{2}(1 + \cos^2\phi) \qquad \text{...(19)}$$

and

$$\sigma_T = \tfrac{8}{3}\pi r_0^2 \qquad \text{...(20)}$$

These equation represents *Thomson Scattering formulae* for differential and total scattering cross-sections. These formulae apply only to free electrons and are appropriate for scattering of X-rays by electrons or γ-rays by protons. From these formulae it is clear that

(i) Thomson scattering of electromagnetic waves is independent of frequency or wavelength of incident waves,

(ii) Scattering depends on the classical radius of electron *i.e.* the nature of scatterer.

(iii) Scattering occurs in all directions and is maximum for $\phi = 0$ and π (*i.e.* in the forward and backward directions) while it is minimum for $\phi = \pi/2$ or $3\pi/2$ (*i.e.* in the side way directions). Moreover scattering is symmetrical about the line $\phi = \pi/2$.

Fig. 10.9 represents the plot of differential cross-section against scattering angle ϕ according to classical and quantum mechanical theories. The quantum mechanical formula approaches Thomson formula as the frequency goes to zero.

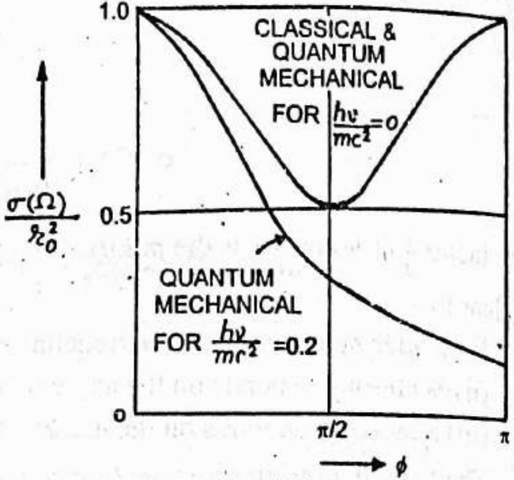


Fig. 10.9

Case (ii) Resonant Scattering. When $\omega \approx \omega_0$, then equation (18) gives

$$\sigma_T = \frac{8}{3}\pi r_0^2 \left(\frac{\omega_0}{\omega}\right)^2 \qquad ...(21)$$

In this case the total scattering cross-section is maximum, thus resulting in very large scattering known as *resonant scattering.* The example of resonant scattering is that a bulb filled with sodium vapour which has a natural frequency in the visible region, illuminated with light of this colour, will scatter so much that it appears luminous.

Case (iii). Rayleigh scattering. If $\omega \ll \omega_0$, then

$$\sigma_T = \frac{8}{3}\pi r_0^2 \left(\frac{\omega}{\omega_0}\right)^2 \qquad ...(22)$$

The *scattering is proportional* to ω^4 *i.e. fourth power of frequency of incident radiation* or inversely proportional to fourth power of wavelength of incident radiation (*i.e.* $\sigma \propto 1/\lambda^4$). The scattering is known as *Rayleigh scattering.* This is likely to occur when ω corresponds to frequencies of visible light and ω_0 to that of ultraviolet.

Reyleigh scattering is capable of explaining the phenomenon like blue colour of sky, red colour of sunset or sunrise, use of red light for danger signals too.

Blue colour of sky. According to Rayleigh scattering $\sigma_T \propto 1/\lambda^4$ *i.e.* shorter is the wavelength of incident radiation, stronger is the scattering by the air molecules much more than the long red ones, resulting in the blue colour of the scattered light from the sky.

Red colour of sunset of sunrise. At sunset or sunrise, the sun and its neighbouring portion appear red because sunlight travels a greater length of earth's atmosphere and reaches directly the observer's eye. The light is deprived of the blue colour due to scattering and the remaining colour appears red.

Red light used for danger signals. Red light has longest wavelength in the visible region; therefore. according to Rayleigh scattering it is scattered least. Consequently it can travel longer distance in atmosphere and can be seen from larger distances in comparison to other wavelengths of visible region. *That is why red light is used for danger signals even through our eye is most sensitive to yellow green.*

Ex. 2. *Calculate the value and dimensions of Thomson scattering cross-section.* **(Meerut 1969)**

Solution. We know that Thomson scattering cross-section is given by

$$\sigma_T = \frac{8\pi}{3} r_0^2 = \frac{8\pi}{3}\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2}\right)^2$$

Hence $e = 1{\cdot}6 \times 10^{-19}$ coulomb, $m = 9{\cdot}1 \times 10^{-31}$ kg. : $c = 3 \times 10^{8}$ m/sec.

$$\therefore \quad \sigma_T = \frac{8}{3} \times 3{\cdot}14 \times \left[9 \times 10^{9} \times \frac{(1{\cdot}6 \times 10^{-19})^2}{9{\cdot}1 \times 10^{-31} \times (3 \times 10^{8})^2} \right]^2$$

$$= 6{\cdot}65 \times 10^{-20}\ \text{m}^2$$

Obviously the dimensions of σ_T are the same as that of

$$r_0^2 = \left(\frac{1}{4\pi\varepsilon_0} \frac{e^2}{mc^2} \right)^2$$

But $r_0 = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{mc^2}$ and $F = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2}$ (Coulomb's law)

$$\therefore \quad [r_0] = \left[\frac{Fr^2}{mc^2} \right] = \frac{[MLT^{-2}]\,[L^2]}{M\,[(LT^{-1})^2]} = \frac{[ML^3T^{-2}]}{[ML^2T^{-2}]} = [L]$$

As dimensions of r_0 are $[L]$, this implies that dimensions of σ_T are $[L^2]$ *i.e.* that of area.

10.8. Polarisation of Scattered Light

We know that atomic or molecular dipoles induced due to incident electromagnetic wave have their axes parallel to the electric field of the incident wave and that the power radiated in a given direction is directly proportional to $\sin^2\theta$, so that we have

(i) For *plane polarised incident wave*

$$\sigma(\Omega) \propto \sin^2\theta$$

$$\propto \cos^2\phi \qquad [\text{since } \theta = \pi/2 - \phi]$$

For *unpolarised incident wave*

$$\sigma(\Omega) \propto \sin^2\theta$$

$$\propto \tfrac{1}{2}(1 + \cos^2\phi) \qquad [\text{since } \sin^2\theta = \tfrac{1}{2}(1 + \cos^2\phi)]$$

Case (i) *If the incident radiation is plane polarised,* the dipoles will all vibrate along the direction of electric vector. Along the direction of dipole axis $\phi = \pi/2$, $\sigma(\Omega) \propto \cos^2\phi = 0$, hence there will be no scattered light along the direction of dipole axis. The scattering will have a maximum intensity at right angles and it will be plane polarised. For example if an incident electromagnetic beam propagating along Z-axis is linearly polarised with its electric vector along X-axis, then there will be no scattered light along X-axis and the scattered waves propagating along Y-axis will have maximum intensity and will be plane polarised with its electric vector still along X-axis (fig. 10·10).

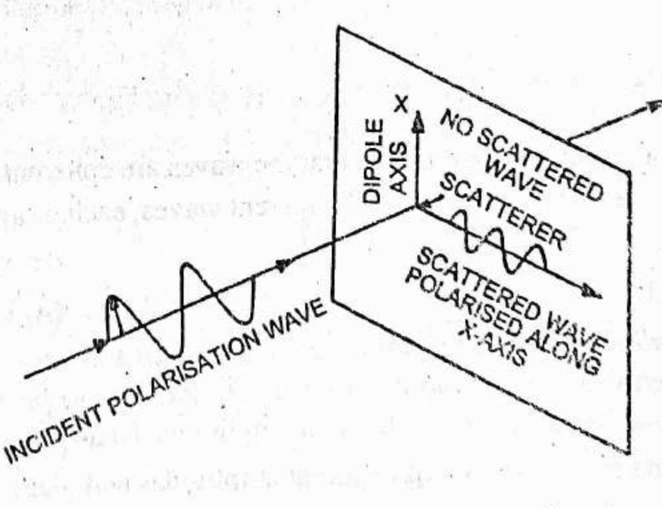

X
DIPOLE AXIS
NO SCATTERED WAVE
Z
SCATTERER
SCATTERED WAVE POLARISED ALONG X-AXIS
INCIDENT POLARISATION WAVE

Fig. 10.10 If a plane polarised wave with its E vector oscillating along X-direction is incident on a dipole, then there is no scattered wave along X-axis.

Case (ii). *When incident wave is unpolarised,* the incident electric field possess all possible orientations, therefore the dipoles will vibrate along all possible directions but they all will lie in the plane perpendicular to the incident direction. If incident electromagnetic wave is travelling along Z-axis, all electric vectors will lie in x-y plane. The scattered radiation is still polarised with the electric field in a plane normal to the direction of incidence. From *degree of depolarisation* $\rho = \frac{1}{2}(1 + \cos^2\phi)$, it is obvious that ρ is minimum ($\rho = 0{\cdot}5$) for $\phi = \pi/2$ and maximum ($\rho = 1$) for $\phi = 0$. This means that degree of depolarisation is never 1 unless $\phi = 0$. This implies that scattered radiations are at least partially polarised and that the polarisation is maximum when ρ is minimum *i.e.* for $\phi = \pi/2$. Fig. 10.11 represents that scattered waves along *X* and *Y* axes ($\phi = \pi/2$) are plane polarised with electric field along *Y* and *X*-axis respectively.

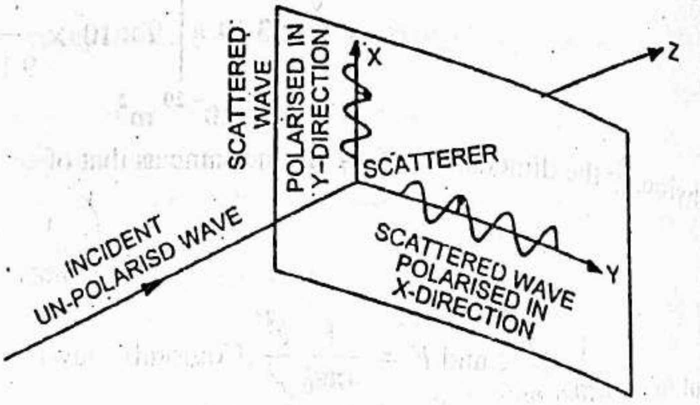

SCATTERED WAVE
POLARISED IN Y-DIRECTION
X
Z
SCATTERER
INCIDENT UN-POLARISD WAVE
SCATTERED WAVE POLARISED IN X-DIRECTION
Y

Fig. 10.11. If an unpolarised electromagnetic wave is propagating along Z-axis, then the scattered wave along any direction perpendicular to Z-axis is linearly polarised.

By means of scattering experiment based on these facts the poalrisation of *X*-rays was first found indicating thereby that *X*-rays are also electromagnetic waves.

10.9. Coherence and Incoherence of Scattered Light

In section 10·11, we have calaculated the scattering from a single dipole or a single atom, when a plane wave is incident on it. But a target contains a large number of particles and hence the resultant intensity of scattered beam must be found by adding contributions from a large number of particles. Since Maxwell's equations are linear, the field vectors **E** and **B** obey the principle of superposition, so that the total amplitude must be sum of amplitudes in the various waves taking into consideration the phases of the waves. Finally the resultant intensity is obtained by squaring the resultant amplitude. *If all the waves have fixed phases relative to each other; they are said to be coherent waves.* For such waves the resultant amplitude is the sum of amplitudes of individual waves (since no phase variations are to be taken into account *i.e. Amplitude,*

$$A_C = A_1 + A_2 + A_3 = \ldots = \sum_k A_k \qquad \ldots(1)$$

hence *intensity,*

$$I_C = A_C^2 = (A_1^2 + A_2 + A_3 + \ldots)^2 = (\sum_k A_k)^2 \qquad \ldots(2)$$

where subscript *c* reminds that the waves are coherent.

If there are *N* identical coherent waves, each of amplitude *A*, then

$$A_C = NA \qquad \ldots(3)$$

and

$$I_C = (A_C)^2 = N^2A^2. \qquad \ldots(4)$$

If all the waves have phase relations relative to each other, they are said to be **incoherent waves.** For such waves the phase variations of amplitudes are to be taken into account and then the resultant intensity is found by squaring the resultant amplitude. To do this let us consider a number of sinusoidal waves, all of the same frequency, but of different amplitudes and phases. Let k^{th} wave be represented by

$$E = \Sigma A_k \cos(\omega t - \delta_k).$$

Then to consider the sum of amplitudes, we write

$$E = \sum_k A_k \cos(\omega t - \delta_k) = \Sigma (A_k \sin \delta_k) \cos \omega t + \sum_k (A_k \sin \delta_k) \sin \omega t \qquad ...(5)$$

For incoherent sourses δ's are completely independent of each other, this means that each δ is equally likely to have any value between 0 to 2π, independent of others. Thus we note that $\sum_k A_k \cos \delta_k$ will be far less than $\sum_k A_k$, because $\cos \delta_k$ will be positive for large number of terms and negative for another large number of terms, thus the effect of many terms will just cancel out. Therefore to find the intensity we must compute the square of summations.

The square of first summation in (5) is

$$(\sum_k A_k \cos \delta_k)^2 = \sum_k A_k^2 \cos^2 \delta_k + \sum_{k \neq l} A_k A_l \cos \delta_k \cos \delta_l . \qquad ...(6)$$

Since δ's are independent, therefore we must average this summation for values of δ between 0 to 2π. Remembering that

$$\overline{\cos^2 \delta_k} = \overline{\sin^2 \delta_k} = \tfrac{1}{2} \text{ and } \overline{\cos \delta_k \cos \delta_l} = \overline{\sin \delta_k \sin \delta_l} = 0$$

where bars denote average values. Therefore we get

$$\overline{(\Sigma A_k \cos \delta_k)^2} = \tfrac{1}{2} \sum_k A_k^2 .$$

Similarly the square of second summation gives

$$\overline{(\sum_k A_k \sin \delta_k)^2} = \sum_k A_k^2 \overline{\sin^2 \delta_k} + \overline{\sum_{k \neq l} A_k A_l \sin \delta_k \sin \delta_l} = \tfrac{1}{2} \sum_k A_k^2$$

Thus we get

intenstiy of incoherent wave $\qquad I_{ic} = \sum_k A_k^2 = A_1^2 + A_2^2 + A_3^2 + \ldots \qquad ...(7)$

where the subscript "*ic*" denotes incoherence.

Thus in *the case of complete incoherence we find that the mean square amplitude, or mean intensity, averaged over phase is the sum of the individual intensities.*

If there are N identical incoherent waves each of amplitude A, then

$$I_{ic} = NA^2 , \qquad ...(8)$$

We can apply the idea of coherence to the scattering of waves by gas molecules. The phase of the wave at point P scattered by an atom 'a' [fig. (10·12 *a*)] depends on the total path the wave has traversed from source to a and from a to P. Since molecules of a gas have no fixed positions relative to each other, these paths are randomly related to each other, phases are incoherent and hence the resultant intensity will be obtained by adding intensities of individual waves according to equation (8). Equation (8) can not be applied in the case of scattering of X-rays by crystals, where the various atoms are in fixed lattice positions, and in the same direction as the incident radiation as shown in fig. 10·12 (*b*).

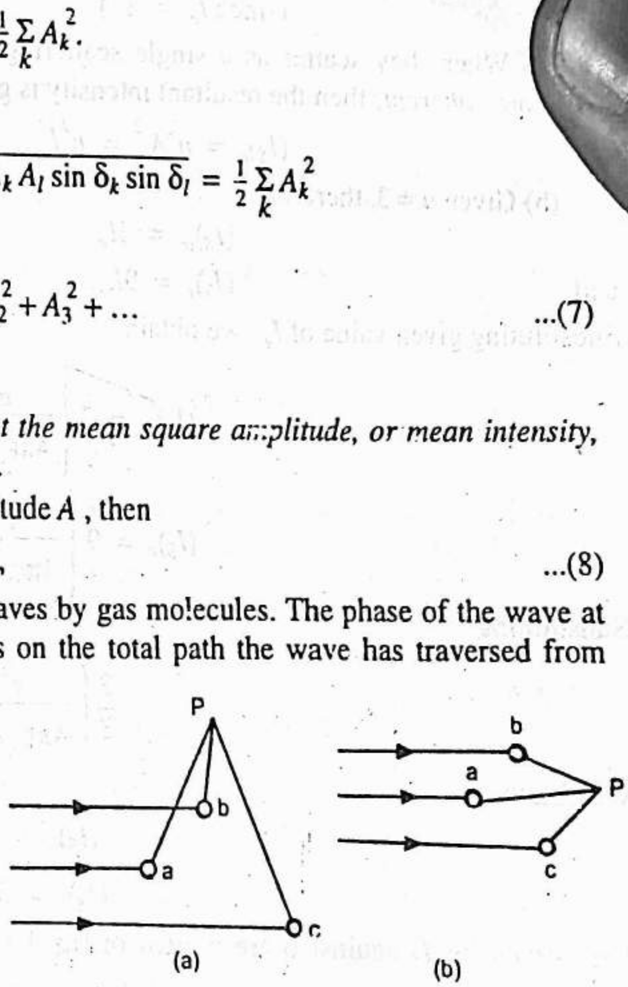


Fig. 10.12

Ex. 3. *The intensity of X-rays scattered by an electron at a distance r from it and making an angle ϕ with the original direction is given by*

$$I_e = \left[\frac{e^2}{4\pi\varepsilon_0 mc^2}\right]^2 \frac{I\frac{1}{2}(1+\cos^2\phi)}{r^2}$$

where I is the intensity of the incident beam. Use the above equation to.

(a) calculate the intensity of scattered radiations I_s, in terms of I_e for an electrons (i) when they scatter independently, (ii) when all of them act as a single scattering centre.

(b) plot schematic curves to illustrate the variations in I_s with ϕ in the two cases in (a) above for n = 3.

(Meerut 1969)

Solution. (a) (i) When n electrons scatter independently the scattered waves are incoherent; hence the resultant intensity of scattered radiation is given by [equation (8)]

i.e.

$$(I_S)_{ic} = nA^2 = nI_e$$

$$(\text{since } I_e = A^2) \qquad \text{...(1)}$$

(ii) When they scatter as a single scattering centre, the scattered waves are *coherent;* then the resultant intensity is given by

$$(I_S)_c = n^2A^2 = n^2I_c \qquad \text{...(2)}$$

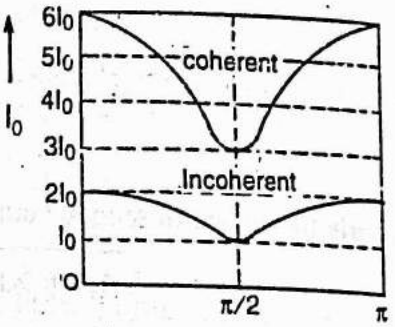


Fig. 10.13

(b) Given $n = 3$, therefore

$$(I_S)_{ic} = 3I_e \qquad \text{...(3)}$$

and

$$(I_S)_c = 9I_e. \qquad \text{...(4)}$$

Substituting given value of I_e, we obtain

$$(I_S)_{ic} = 3\left\{\frac{e^2}{4\pi\varepsilon_0 mc^2}\right\}^2 \frac{I\frac{1}{2}(1+\cos^2\phi)}{r^2} \qquad \text{...(5)}$$

$$(I_S)_c = 9\left\{\frac{e^2}{4\pi\varepsilon_0 mc^2}\right\}^2 \frac{I\frac{1}{2}(1+\cos^2\phi)}{r^2} \qquad \text{...(6)}$$

Substituting

$$\frac{2}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \cdot \frac{I}{r^2} = I_0 \qquad \text{...(7)}$$

We obtain

$$(I_S)_{ic} = I_0(1+\cos^2\phi) \qquad \text{...(8)}$$

$$(I_S)_c = 3I_0(1+\cos^2\phi). \qquad \text{...(9)}$$

The curves for I_S against ϕ are plotted in fig. 10·13. These curves indicate that the intensity of scattered radiation in case of coherence [case (ii)] is three times greater than that of in case of incoherence [case (i)].

QUESTIONS AND PROBLEMS

1. What is normal and anomalous dispersion ? Derive Sellmeier's equation for the refractive index of a dielectric medium. **(Rohilkhand 1984; Agra 1997)**
2. What do you mean by dispersion ? Discuss it in the case of gaseous medium having both real and complex refractive index. **(Bombay 1999; Agra 91)**
3. Give the theory of normal and anomalous dispersion. Describe how the latter has been studied in the case of sodium vapour. Hence show that the refractive index of glass for X-rays is less than 1 and discuss its significance. **(Agra 1997, 80)**

   [**Hint :** For X-rays $\omega >> \omega_k >> g_k$ therefore equation (22) of section 10·3 gives.

   $$n^2 = 1 - \frac{i}{\varepsilon_0} \sum_k \frac{N_k (e^2/m)}{\omega^2} = 1 - \frac{Ne^2}{\varepsilon_0 m\omega^2} \text{ (since } \Sigma N_k = N)$$

   Obviously refractive index of glass for X-rays is less than one. This signifies the possibility of total external reflection at glancing angles, actually verified by Comption.]
4. What is anomalous dispersion and how is it explained on the basis of electromagnetic theory ? **(Meerut 1998)**
5. From the theory of dispersion show that in the case of transparent gaseous substance in the region remote from the natural frequencies the index of refraction is given approximately by formula

   $$n^2 = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$$

   where A, B, C are constants, λ is the wavelength of incident light. Obtain also an expression for the dispersive power of the medium under consideration.
6. For the D-lines of sodium the following values of constants in the dispersion formula are found

   $$\omega_0 = 3 \times 10^{16}, G = 2 \times 10^{10}, \frac{Ne^2}{m\varepsilon_0} = 10^{23}$$

   Plot the refractive index (n) and absorption coefficient ($n\kappa$) as a function of frequency of electromagnetic waves. Find the maximum and minimum values of the refractive index n. Find the maximum values of absorption coefficient ($n\kappa$) and the half width of the absorption band in angstrom units.
7. What is dispersion ? Give Lorentz theory of dispersion and show that in the case of non-polar dielectrics.

   $$\frac{n^2 - 1}{n^2 + 2} \frac{M}{\rho} = \text{constant.}$$

   where M is the molecular weight of the dielectric, ρ its density and n the refractive index.
8. Give an account of electromagnetic theory of dispersion. **(Rohilkhand 1986, 82)**
9. (a) Differentiate between Thomson and Rayleigh scattering.

   (b) Derive the Thomson scattering formula for the scattering of a plane polarised electromagnetic wave by a free electron and discuss its success and limitations. **(Agra 1975; Meerut 1992, 86)**
10. What do you mean by scattering cross-section ? Discuss Rayleigh scattering and its cross-section. **(Meerut 1992)**
11. Explain Rayleigh scattering on the basis of theory of scattering of electromagnetic wave by bound electrons. **(Meerut 1998)**
12. Define scattering cross-section. Discuss different types of scattering by considering the incidence of electromagnetic wave on bound electrons. Why does the sky appears blue. **(Rohilkhand 1985)**
13. What is difference between Thomson, Rayleigh and resonacne scattering of radiation ? Derive the expression for the differential cross-sections for Thomson scattering of radiation by electrons. **(Rohilkhand 1999, 92, 90; Agra 1989)**
14. Derive the scattering cross-section for the scattering of a plane monochromatic electromagnetic radiation by a free charged particle. **(Meerut 1997)**
15. The intensity of X-rays scattered by a electron at a distance r from it and making an angle ϕ with the original direction is given by

   $$I_C = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \frac{I}{r^2} \frac{1}{2} (1 + \cos^2 \phi)$$

   Use the above result to

(a) Obtain an expression for the coefficient of scattering and its dimensions.

(b) Calculate the intensity of scattered radiation I_S in terms of I_e for n-electrons

(i) when they scatter independently (ii) when all of them act as a single scattering centre.

(c) Plot schematic curves to illustrate the variations in I_s with ϕ in the two cases in (b) above for $n = 3$.

(Meerut 1969)

16. (a) What is the effect of radiation damping on the monochromatic nature of radiations emitted by a harmonic oscillator ?

(b) Show that the scattering cross-section of electromagnetic radiation by a bound electron depends upon the inverse fourth power of the incident wavelength. **(Agra 1984)**

17. On the basis of Rayleigh scattering explain the following :

(a) Red light is used for danger signals while the eye is most sensitive to yellow green.

(b) The clear sky appears blue.

(c) The rising or setting sun appears red.

18. Write short notes on the following :

(i) Normal and anomalous dispersion

(ii) Rayleigh scattering. **(Rohilkhand 1997, 96, 87, 85)**

(iii) Thomson scattering. **(Meerut 1991; Agra 1973, 71)**

(iv) Scattering of electromagnetic waves. **(Agra 1972)**

(Rohilkhand 1991, 84)

11

Electromagnetic Potentials and Spherical Waves

11.1. Electromagnetic Vector and Scalar Potentials :

Maxwell's first order differential equation relating the various components of electric and magnetic fields can be solved as they stand in certain simple cases. It is often convenient to introduce electromagnetic potentials, obtaining a small number of second order equations, while satisfying same of Maxwell's equations identically. Electromagnetic field can be derived in terms of these potentials. We are already familiar with the concept in electrostatics and magnetostatics, where we used the scalar potential ϕ and vector potentials **A**.

Maxwell's field equations are

$$\left.\begin{aligned} \text{div } \mathbf{D} &= \nabla \cdot \mathbf{D} = \rho && \text{...(a)} \\ \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 && \text{...(b)} \\ \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{...(c)} \\ \text{and} \quad \text{Curl } \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && \text{...(d)} \end{aligned}\right\} \quad \text{...(1)}$$

According to equation (1b) div **B** $= 0$ *i.e.* field vector **B** is solenoidal) and we know that div of curl of any vector is always zero; hence **B** can be expressed as the curl of a vector *i.e.*

$$\mathbf{B} = \text{curl } \mathbf{A} = \nabla \times \mathbf{A} \qquad \text{...(2)}$$

where A is known as electromagnetic vector potential.

Substituting value of **B** from (2) in (1c), we get

$$\text{curl } \mathbf{E} = -\frac{\partial}{\partial t}(\text{curl } \mathbf{A})$$

i.e.

$$\text{curl}\left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right] = 0.$$

This implies that the vector $\left\{\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right\}$ has zero curl *i.e.* $\left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right.$ is irrotational$\left.\right]$ and we know that the curl of gradient of any scalar function is always zero; hence the vector $\left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right]$, can be expressed as the gradient of a scalar function '*s*'

i.e.

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = \text{grad } s$$

or equivalently in conventional form $\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t} = -\text{ grad } \phi,$

or $$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi \qquad \text{...(3)}$$

where ϕ is called the **scalar potential function.**

Equation (2) and (3) express electromagnetic field vectors **E** and **B** in terms of electromagnetic scalar and vector potentials ϕ and **A**. These electromagnetic potentials play a very important role in electrodynamics particularly in relativistic electrodynamics.

Now Maxwell's equation (1d) is

$$\text{curl } \mathbf{H} = \mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}.$$

In term of **B** and **E** (using $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D}=\varepsilon\mathbf{E}$), this equation takes the form

$$\text{curl } \mathbf{B} = \mu\mathbf{J}+\mu\varepsilon\frac{\partial \mathbf{E}}{\partial t} \qquad \text{...(4)}$$

Using equations (2) and (3), we get

$$\text{curl curl } \mathbf{A} = \mu\mathbf{J}+\mu\varepsilon\frac{\partial}{\partial t}\left[-\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi\right]$$

i.e. $$\text{grad div } \mathbf{A} - \nabla^2\mathbf{A} = \mu\mathbf{J} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\varepsilon\frac{\partial}{\partial t}(\text{grad } \phi)$$

Rearranging, we get $$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left[\text{div } \mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t}\right] = -\mu\mathbf{J} \qquad \text{...(5)}$$

Similarly equation (1a) in terms of **E** takes the form

$$\text{div}\mathbf{E} = \frac{\rho}{\varepsilon}$$

Using (5), we get $$\text{div }\left[-\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi\right] = \frac{\rho}{\varepsilon}$$

i.e. $$\text{div grad } \phi + \frac{\partial}{\partial t}(\text{div } \mathbf{A}) = -\frac{\rho}{\varepsilon}$$

Adding and subtracting $\mu\varepsilon\frac{\partial^2 \phi}{\partial t^2}$ and rearranging, we get

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left[\text{div } \mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t}\right] = -\frac{\rho}{\varepsilon} \qquad \text{...(6)}$$

Equations (4) and (5) represent two inhomogeneous wave equations in terms of electromagnetic potentials **A** and ϕ in place of four Maxwell's equations. These two equations contain **A** and ϕ coupled with each other. These inhomogeneous wave equations can be uncoupled by using the concept that **A** and ϕ are arbitrary.

Generally **A** and ϕ are chosen subject to the condition.

$$\text{div } \mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t} = 0. \qquad \text{...(7)}$$

Since div **A** is so far arbitary, we can do this. This condition is known as the ***Lorentz condition***. Then the equations (2) and (6) for electromagnetic potentials become.

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \qquad \text{...(8)}$$

and

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \qquad ...(9)$$

If A and ϕ sastisfy these equations, then the fields determined by equations (2) and (3) satisfy Maxwell's equations. Equations (7), (8) and (9) form a set of equations equivalent in all respects to Maxwell's equations.

Equations (8) and (9) for A and ϕ take prescribed values of charge and current density. Sometimes, however, we wish to assume that the *current density obeys Ohm's* ($\mathbf{J} = \sigma\mathbf{E}$) and that the charge density is zero, as we should have in the interior of a conductor. In that case proceeding in a similar manner, we find that in place of *Lorentz condition* (7) we should assume

$$\text{div}\,\mathbf{A} + \mu\varepsilon\frac{\partial\phi}{\partial t} + \mu\sigma\phi = 0 \qquad ...(10)$$

and in place of equations (8) and (9). we have

$$\nabla^2\mathbf{A} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \qquad ...(11)$$

and

$$\nabla^2\phi - \mu\sigma\frac{\partial\phi}{\partial t} - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = 0 \qquad ...(12)$$

If the sources **J** and ρ are absent, then equations (8) and (9) or (11) and (12) reduce to homogeneous equations

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{dt^2} = 0 \qquad ...(13)$$

and

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = 0 \qquad ...(14)$$

These equations represent an extension of Poisson's equation, obtained by adding the time derivatives and are known as *D' Alembert equations*. Introducing *D' Alembertian or D' Alembert's operator*

$$\Box^2 = \nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2} \qquad ...(15)$$

equations (13) and (14) take the form similar to Poisson's equations

$$\left.\begin{aligned} \Box^2\mathbf{A} &= 0 \\ \text{and} \quad \Box^2\phi &= 0 \end{aligned}\right\} \qquad ...(16)$$

In terms of D' Alembertian, equations (8) and (9) take the form

$$\left.\begin{aligned} \Box^2\mathbf{A} &= -\mu\mathbf{J} \\ \text{and} \quad \Box^2\phi &= -\frac{\rho}{\varepsilon} \end{aligned}\right\} \qquad ...(17)$$

In free space $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ so that equations (17) take the form

$$\left.\begin{aligned} \Box^2\mathbf{A} &= -\mu_0\mathbf{J} \\ \text{and} \quad \Box^2\phi &= -\frac{\rho}{\varepsilon_0} \end{aligned}\right\} \qquad ...(18)$$

where $\Box^2 = \nabla^2 - \mu_0\varepsilon_0\dfrac{\partial^2}{\partial t^2} = \nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}$

$$= \sum_{\alpha=1}^{4} \frac{\partial^2}{\partial x_\alpha^2}.$$

Ex. 1. *Show that the electromagnetic potentials in uniform electric and magnetic field may be expressed as*

$$\phi = -\mathbf{E} \cdot \mathbf{r} \text{ and } \mathbf{A} = \tfrac{1}{2}(\mathbf{B} \times \mathbf{r})$$

where **r** *is the position vector of the point under consideration.*

Solution. If **E** is electric field, then

$$\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z = (\mathbf{E} \cdot \nabla)\,\mathbf{r}$$

For uniform electric field; the above result may be expressed as

$$\mathbf{E} = \nabla(\mathbf{E} \cdot \mathbf{r}) \qquad ...(1)$$

Also for electrostatic field

$$\mathbf{E} = -\nabla\phi \qquad ...(2)$$

Comparing equation (1) and (2), we get

$$\phi = -(\mathbf{E} \cdot \mathbf{r}) \qquad ...(3)$$

This is the first result.

For second result let use vector identity

$$\text{curl}\,(\mathbf{a} \times \mathbf{b}) = \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a} + (\mathbf{b} \cdot \text{grad})\,\mathbf{a} - (\mathbf{a} \cdot \text{grad})\,\mathbf{b}$$

If we write **a** = **B** and **b** = **r** in above equation, then

$$\text{curl}\,(\mathbf{B} \times \mathbf{r}) = \mathbf{B}\,\text{div}\,\mathbf{r} - \mathbf{r}\,\text{div}\,\mathbf{B} + (\mathbf{r} \cdot \text{grad}\,)\,\mathbf{B} - (\mathbf{B} \cdot \text{grad})\,\mathbf{r}$$

For uniform field **B**, div **B** = 0 and (**r** • grad) **B** = 0; so above equation takes the form

$$\text{curl}\,(\mathbf{B} \times \mathbf{r}) = \mathbf{B}\,\text{div}\,\mathbf{r} - (\mathbf{B} \cdot \text{grad})\,\mathbf{r} \qquad ...(4)$$

But div **r** = 3 and (**B** • grad) **r** = **B**

Therefore equation (4) gives

$$\text{curl}\,(\mathbf{B} \times \mathbf{r}) = 3\mathbf{B} - \mathbf{B} = 2\mathbf{B}$$

This implies

$$\mathbf{B} = \text{curl}\,(\tfrac{1}{2}\,\mathbf{B} \times \mathbf{r}) \qquad ...(5)$$

But

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad ...(6)$$

Comparing (5) and (6), we obtain

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{B} \times \mathbf{r}) \qquad ...(7)$$

This is the second result.

11.2. Non-uniqueness of Electromagnetic Potentials and Concept of Gauge.

If **A** and ϕ are electromagnetic potentials, then field vectors **B** and **E** are given by

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad ...(1)$$

$$\mathbf{E} = -\,\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t}$$

From these equations it is clear that for given set of A and ϕ, the field vectors **E** and **B** have unique solution. But the converse is not true *i.e.* if **E** and **B** are given, the potentials *A* and ϕ are not unique. This may be seen as follows :

Since vector **A** defined by equation (1) is arbitarary to the extent that the gradient of some scalar function Λ can be added to *A* without affecting **B** (since curlΛ = 0 always) *i.e.*

$$\mathbf{A} \rightarrow \mathbf{A}' + \text{grad}\,\Lambda \qquad ...(3)$$

Then equation (2) takes the form

$$\mathbf{E} = -\,\text{grad}\,\phi - \frac{\partial}{\partial t}(\mathbf{A}' - \text{grad}\,\Lambda)$$

$$= -\text{grad}\left(\phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \mathbf{A}'}{\partial t}$$

This implies that if we adopt the transformation (3), then in order that the electric field (2) be unchanged as well, the scalar potential ϕ must be expressed simultaneously as

$$\phi \rightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t} \qquad ...(4)$$

Then expressions for **B** and **E** remain unchanged under transformation (3) and (4) *viz.*

$$\mathbf{B} = \text{curl } \mathbf{A} = \text{curl } (\mathbf{A}' - \text{grad } \Lambda) = \text{curl } \mathbf{A} \qquad ...(5a)$$

and

$$\mathbf{E} = -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$= -\text{grad}\left(\phi' - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}(\mathbf{A}' - \text{grad } \Lambda)$$

$$= -\text{grad } \phi' - \frac{\partial \mathbf{A}'}{\partial t} \qquad ...(5b)$$

Equations (5a) and (5b) indicate that we get the same field vectors whether we use the set ($\mathbf{A}, \phi$) of ($\mathbf{A}', \phi'$). Thus we conclude that electomagnetic potentials **A** and ϕ define the field vectors uniquely : while they themselves are not unique. The transformations given by equation (3) and (4) are known as *gauge transformations* and the arbitrary scalar function Λ is scalled the *gauge function.* Since electromagnetic potentials **A** and ϕ have been introduced for convenience, it is the field vectors and not the electromagnetic potentials that have physical meaning. Thus we can say that through **A** and ϕ are not unique, the field vectors **E** and **B** are invariant under gauge transformation. ***The invariance of fields under gauge transformations is known as gauge invariance.***

Becuase of non-uniqueness of electromagnetic potentials **A** and ϕ, we are free to impose an additional condition on **A**. Alternatively we may say that to specify a vector uniquely, we must specify both divergence and curl of given vector instead of specifying its curl only. The div **A** may be chosen in a convenient manner to simplify a given problem. Generally div **A** is chosen in two ways as described earlier according as the field contains sources or is free from sources. Let us now discuss various gauge transformations.

11.3. Lorentz Gauge

Maxwell's field equation in terms of electromagnetic potentials are given by

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\text{div } \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J} \qquad ...(1)$$

and

$$\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} - \text{grad}\left(\text{div } \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\varepsilon} \qquad ...(2)$$

Since div **A** is arbitrary, we may impose the condition known as *Lorentz condition by choosing*

$$\text{div } \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0, \qquad ...(3)$$

When the vector and scalar potentials satisfy this condition; the gauge is known as *Lorentz gauge.*

Utilizing Lorentz condition (3), the field equations (1) and (2) reduce to

$$\left.\begin{aligned} \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu \mathbf{J} \\ \text{and} \qquad \nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\varepsilon} \end{aligned}\right] \qquad \text{...(4)}$$

But $\mu\varepsilon = \frac{1}{v^2}$ where v is the phase velocity of wave.

∴ Equations (4) take the form

$$\left.\begin{aligned} \nabla^2 \mathbf{A} - \frac{1\,\partial^2 \mathbf{A}}{v^2\, t^2} &= -\mu \mathbf{J} \\ \text{and} \qquad \nabla^2 \phi - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\varepsilon} \end{aligned}\right] \qquad \text{...(5)}$$

Now defining Alembertion $\square^2$ as

$$\square^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} \qquad \text{...(6)}$$

equations (5) take the form

$$\left.\begin{aligned} \square^2 \mathbf{A} &= -\mu \mathbf{J} \quad \text{...}(a) \\ \text{and} \qquad \square^2 \phi &= -\frac{\rho}{\varepsilon} \quad \text{...}(b) \end{aligned}\right\} \qquad \text{...(7)}$$

Equation (7) are similar to Poisson's equations; hence these equations may be solved by a similar method.

Now let us examine the requirement imposed by Lorentz condition on the gauge function Λ.

Substituting $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \text{grad } \Lambda$

and
$$\phi \to \phi' = \phi - \frac{\partial \Lambda}{\partial t}$$

in Lorentz condition (3), we obtain

$$\text{div}\,(\mathbf{A}' - \text{grad } \Lambda) - \mu\varepsilon \frac{\partial}{\partial t}\left(\phi' + \frac{\partial \Lambda}{\partial t}\right) = 0$$

i.e.
$$\text{div}\,\mathbf{A}' + \mu\varepsilon \frac{\partial \phi'}{\partial t} = 0 \text{ only if}$$

$$\nabla^2 \Lambda - \mu\varepsilon \frac{\partial^2 \Lambda}{\partial t^2} = 0 \text{ or } \square^2 \Lambda = 0 \qquad \text{...(8)}$$

i.e. Lorentz condition is invariant under those gauge transformations for which the gauge function satisfies equation (8). All potentials in this restricted class are said to belong to the *Lorentz gauge*. The Lorentz gauge is commonly used because of

(i) It leads to uncoupled wave equations (7a) and (7b) for potentials A and ϕ.

(ii) It leads A and ϕ on equivalent footings since equations (7a) and (7b) have similar form.

(iii) It is a concept which is independent of co-ordinate-system chosen and so is convenient for the consideration of special relativity.

11.4. Coulomb or Transverse Gauge

The wave equations in terms of electromagnetic potentials are

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\text{div}\,\mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J} \qquad \text{...(1)}$$

and

$$\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\text{div}\,\mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\varepsilon} \qquad \text{...(2a)}$$

Equation (2a) is equivalent to

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\text{div}\,\mathbf{A}) = -\frac{\rho}{\varepsilon} \qquad \text{...(2b)}$$

The Coulomb gauge restricts the divergence of **A** as

$$\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = 0 \qquad \text{...(3)}$$

Using this, equation (2b) for scalar potential ϕ reduces to Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon} \qquad \text{...(4)}$$

whose solution* is

$$\phi(r, t) = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dv' \qquad \text{...(5)}$$

Thus the scalar potential is just the instantaneous Coulomb potential due to the charge density $\rho(\mathbf{r}, t)$. *This gives the origin of the name Coulomb gauge.*

Using Coulomb gauge condition (3), equation of the vector potential reduces to

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J}$$

Remembering that $\mu\varepsilon = 1/v^2$, v being phase velocity of the wave, above equation may be expressed as

$$\nabla^2 \mathbf{A} - \frac{1}{v^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \mu\varepsilon\, \text{grad}\left(\frac{\partial \phi}{\partial t}\right) \qquad \text{...(6)}$$

This equation may be put in a convenient from by using Poisson's equation (4) which with the help of (5) may be expressed as

$$\nabla^2 \left[\frac{I}{4\pi\varepsilon} \int_V \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dv'\right] = -\frac{\rho(\mathbf{r}', t)}{\varepsilon} \qquad \text{...(7)}$$

Now since Poisson's equation holds for scalars and vectors both; therefore replacing *scalar potential source* $\rho(\mathbf{r}', t)$ by vector potential source $\mathbf{J}(\mathbf{r}', t)$: we obtain

$$\nabla^2 \left[\frac{1}{4\pi\varepsilon} \int_V \frac{\mathbf{J}(\mathbf{r}' t)}{|\mathbf{r} - \mathbf{r}'|} dv'\right] = -\frac{\mathbf{J}(\mathbf{r}', t)}{\varepsilon} \qquad \text{...(8a)}$$

Now substituting

$$\left[\int_V \frac{\mathbf{J}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dv'\right] = \mathbf{G}(\mathbf{r}', t) \qquad \text{...(A)}$$

We obtain

$$\nabla^2 G(\mathbf{r}', t) = -4\pi \mathbf{J}(\mathbf{r}', t) \qquad \text{...(8b)}$$

Now using the vector identity

$$\nabla \times \nabla \times \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G}$$

i.e. $\nabla^2 \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G}$, we may write equation (8b) as

$$\nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G} = -4\pi \mathbf{J}$$

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Substituting **G** from equation (*A*), we get

$$\nabla\left[\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right]-\nabla\times\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = -4\pi\mathbf{J}(\mathbf{r}',t)$$

i.e.

$$\mathbf{J}(\mathbf{r}',t) = -\frac{1}{4\pi}\nabla\left[\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right]+\frac{1}{4\pi}\nabla\times\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' \qquad ...(9)$$

Now considering $\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{(\mathbf{r}-\mathbf{r}')}\,dv'$, we may write

$$\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r},t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = \int_V \nabla\cdot\frac{(\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = -\int_V \nabla'\cdot\mathbf{J}\,\frac{(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' \qquad ...(10)$$

$$\left[\text{since } \nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right]$$

Now using vector identity

$$\nabla\cdot(\phi\mathbf{a}) = \phi\nabla\cdot\mathbf{a}+\mathbf{a}\cdot(\nabla\phi)$$

equation (10) may be written as [taking $\mathbf{a} = \nabla'\cdot\mathbf{J}(\mathbf{r}',t)$ and $\phi = 1/|\mathbf{r}-\mathbf{r}'|$]

$$\nabla\cdot\int_V \frac{J(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = -\int_V\left[\nabla'\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)-\frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right]dv'$$

$$= -\int_V \nabla'\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)dv'+\int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'$$

Using Gauss divergence theorem to change volume integral in first term on R.H.S. of above equation into surface integral, we get

$$\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = -\oint_S \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\cdot d\mathbf{S}+\int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}'\cdot t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'$$

Noting that $\oint_S \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\cdot d\mathbf{S}\to 0$ since **J** vanishes on the surface, above equation takes the form

$$\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = \int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' \qquad ...(11)$$

Using this substitution equation (9) becomes

$$\mathbf{J}(\mathbf{r}',t) = -\frac{1}{4\pi}\nabla\int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',l)}{|\mathbf{r}-\mathbf{r}'|}\,dv'+\frac{1}{4\pi}\nabla\times\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'$$

$$= \mathbf{J}_l+\mathbf{J}_t \qquad ...(12)$$

where

$$\mathbf{J}_l = -\frac{1}{4\pi}\nabla\int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' \qquad ...(13a)$$

and

$$\mathbf{J}_t = \frac{1}{4\pi}\nabla\times\nabla\times\int \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' \qquad ...(13b)$$

We note that

$$\nabla\times\mathbf{J}_l = -\frac{1}{4\pi}\nabla\times\nabla\int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' \qquad ...(14a)$$

$$= -\frac{1}{4\pi}\,\text{curl grad}\int_V \frac{\nabla'\,\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv' = 0$$

(Since curl of gradient of any scalar function always vanishes)

and

$$\nabla\cdot\mathbf{J}_t = \frac{1}{4\pi}\nabla\cdot\left[\nabla\times\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right]$$

$$= \frac{1}{4\pi}\,\text{div curl}\left(\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right) = 0 \qquad ...(14b)$$

Since divergence of curl of any vector always vanishes. From (14a) and (14b) $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$. That is why $\mathbf{J}_l$ and $\mathbf{J}_t$ are referred as longitudinal and transverse respectively.

Now substituting values of ϕ and J from (5) and (12) in equation (6), we obtain

$$\nabla^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\,(\mathbf{J}_l + \mathbf{J}_t) + \varepsilon \nabla \left[\frac{\partial}{\partial t}\left(\frac{1}{4\pi\varepsilon}\int_V \frac{\rho\,(\mathbf{r}'.\,t)}{|\mathbf{r}-\mathbf{r}'|}\right) dv'\right]$$

Now using equation of continuitty

$$\nabla' \cdot \mathbf{J}\,(\mathbf{r}', t) + \frac{\partial}{\partial t}\,\rho\,(\mathbf{r}, t) = 0.$$

i.e.

$$\frac{\partial \rho\,(\mathbf{r}', t)}{\partial t} = -\nabla' \cdot \mathbf{J}\,(\mathbf{r}', t)$$

in the last term, we obtain

$$\nabla^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\,(\mathbf{J}_l + \mathbf{J}_t) - \frac{\mu}{4\pi}\int_V \frac{\nabla^2 .\, \mathbf{J}(\mathbf{r}'.\,t)}{|\mathbf{r}-\mathbf{r}'|}\,dv'$$

Keeping in view equation (13a) to write the last term, above equation takes the form

$$\nabla^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_l - \mu \mathbf{J}_t + \mu \mathbf{J}_l$$

i.e.

$$\nabla^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_t$$

or

$$\Box^2 \mathbf{A} = -\mu \mathbf{J}_t \qquad ...(15)$$

This equation indicates that the source of wave equation for **A** can be expressed entirely in terms of the transverse **J**. This gives the origin of the name *transverse gauge.*

The coulumb or transverse gauge has the advantage that the scalar potential is exactly the *electrostatic potential.* This indicates the electric field vector **E** given by relation

$$E = -\text{grad}\ \phi - \frac{\partial \mathbf{A}}{\partial t} \qquad ...(16)$$

may be considered to be formed of electrostatic field (contribution due to term–grad ϕ) and a wave field (contribution due to term $-\partial \mathbf{A}/\partial t$).

This gauge is often used when no sources are present. Then $\phi = 0$, and A sutisfies the homogeneous wave equations (15). The fields are given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad ...(17)$$

1.5. Hertz Vector

The whole electromagnetic field may be represented by a single vector **Z**, known as the *Hertz vector*, after, the name of H Hertz. We shall now see that the set of four Maxwell's equations is equivalent to a single wave equation in terms of **Z.**

Maxwell's equations are

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \nabla \cdot \mathbf{D} = \rho && ...(a)\\ \text{div}\,\mathbf{B} &= \nabla \cdot \mathbf{B} = 0 && ...(b)\\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && ...(c)\\ \text{and}\quad \text{curl}\,\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && ...(d) \end{aligned}\right\} \qquad ...(1)$$

Let us consider a homogeneous linear non-conducting medium, then,

$$\rho = 0,\ \mathbf{J} = \sigma \mathbf{E} = 0,\ \mathbf{B} = \mu \mathbf{H} \ \text{ and } \ \mathbf{D} = \varepsilon \mathbf{E}. \qquad ...(2)$$

So the Maxwell's equations in terms of **E** and **B** take the form

$$\left.\begin{aligned} \text{div}\,\mathbf{E} &= \nabla \cdot \mathbf{E} = 0 && ...(a)\\ \text{div}\,\mathbf{B} &= \nabla \cdot \mathbf{B} = 0 && ...(b)\\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}. && ...(c)\\ \text{and} \qquad \text{curl}\,\mathbf{B} &= \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}. && ...(d) \end{aligned}\right\} \qquad ...(2)$$

Since div **B** = 0, div curl **A** = 0 always; this implies

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad ...(3)$$

where **A** is electromagnetic vector potential. The formula becomes appreciably simpler if we choose vector potential **A** as proportional to the *time derivative of an auxiliary vector* **Z**, known as *Hertz's vector* as

$$\mathbf{A} = \mu\varepsilon \frac{\partial \mathbf{Z}}{\partial t}. \qquad ...(4)$$

$$\therefore \qquad \mathbf{B} = \text{curl}\,\mathbf{A} = \text{curl}\left(\mu\varepsilon \frac{\partial \mathbf{Z}}{\partial t}\right) = \mu\varepsilon \frac{\partial}{\partial t}(\text{curl}\,\mathbf{Z}). \qquad ...(5)$$

Substituting this in (2, d), we get

$$\text{curl}\left\{\mu\varepsilon \frac{\partial}{\partial t}(\text{curl}\,\mathbf{Z})\right\} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}.$$

i.e.

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}(\text{curl curl}\,\mathbf{Z}).$$

Integrating, we get

$$\mathbf{E} = \text{curl curl}\,\mathbf{Z} = \nabla \times \nabla \times \mathbf{Z} \qquad ...(6a)$$

$$= \text{grad div}\,\mathbf{Z} - \nabla^2 \mathbf{Z}. \qquad ...(6b)$$

In order that equation (1c) is satisfied, we have

$$\text{curl}\,\mathbf{E} = -\frac{\partial}{\partial t}\left\{\mu\varepsilon \frac{\partial}{\partial t}\,\text{curl}\,\mathbf{Z}\right\} \qquad \text{using (5)}$$

$$= -\,\text{curl}\left(\mu\varepsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2}\right)$$

This gives

$$\mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} - \text{grad}\,\phi = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\,\phi \qquad ...(7)$$

The arbitrary vector (– grad ϕ) appears in place of a constant of integration; since the equally of curls of two vectors implies that these vectors differ by an irrotational (curlless) vector (– grad ϕ). The two expressions (6b) and (7) for **E** will be identical if

$$\phi = -\,\text{div}\,\mathbf{Z} = -\,\nabla \cdot \mathbf{Z} \qquad ...(8)$$

and

$$\nabla^2 \mathbf{Z} = \mu\varepsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} \ \text{or} \ \nabla^2 \mathbf{Z} - \mu\varepsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} = 0. \qquad ...(9a)$$

Equation (9a) may also be expressed as

$$\nabla^2 \mathbf{Z} - \frac{1}{v^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2} = 0 \qquad ...(9b)$$

(since $\mu\varepsilon = \dfrac{1}{v^2}$, v being phase velocity of the wave)

or

$$\Box^2 \mathbf{Z} = 0 \qquad \text{...(9c)}$$

Thus we get a single equation in terms of ***Hertz vector* Z** in place of a set of four Maxwell's equation (2).

· It may be seen that in this gauge known as ***electro-gauge,*** the electromagnetic field vectors **A** and ϕ satisfy

Lorentz condition

$$\text{div}\,\mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0 \qquad \text{...(10)}$$

and Coulomb gauze condition

$$\nabla \cdot \mathbf{B} = 0 \text{ and } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\,\phi \qquad \text{...(11)}$$

Thus electro-gauge defined in terms of Hertz vector is quite general and Lorentz's and Coulomb's gauges are its particular cases.

If the *medium is characterised by a conductivity* σ,

$$\left.\begin{aligned} \text{then} \quad \mathbf{A} &= \mu\sigma\mathbf{Z} + \mu\varepsilon\frac{\partial \mathbf{Z}}{\partial t} \\ \phi &= -\nabla\cdot\mathbf{Z} \end{aligned}\right] \qquad \text{...(12)}$$

These equations satisfy (11) and the general Lorentz condition viz.

$$\nabla\cdot\mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t} + \mu\sigma\,\phi = 0 \qquad \text{...(13)}$$

If we take

$$\mathbf{E} = \nabla \times (\nabla \times \mathbf{Z}) \qquad \text{...(14)}$$

ans

$$\nabla^2 \mathbf{Z} - \mu\sigma\frac{\partial \mathbf{Z}}{\partial t} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{...(15)}$$

So that magnetic field induction is given by

$$\mathbf{B} = \mu\sigma\,\nabla\times\mathbf{Z} + \mu\varepsilon\frac{\partial(\nabla\times\mathbf{Z})}{\partial t} \qquad \text{...(16)}$$

Equation (15) is a single wave equation in place of a set of four Maxwell's equations

$$\left.\begin{aligned} &\nabla\cdot\mathbf{D} = 0;\ \nabla\cdot\mathbf{B} = 0 \\ &\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ and curl}\,\mathbf{H} = \sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t} \end{aligned}\right]$$

and contains all the properties of the electromagnetic wave.

11.6. Lorentz Force in Terms of Electromagnetic Potentials.

The Lorentz force on a change q moving with velocity **v** in an electromagnetic field, characterised by electric and magnetic field vectors **E** and **B** is given by

$$\mathbf{F} = q\,[\mathbf{E} + \mathbf{v}\times\mathbf{B}] \qquad \text{...(1)}$$

The field vectors **E** and **B** in terms of electromanetic potentials are expressed as

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} \text{ and } \mathbf{E} = -\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{...(2)}$$

Making these substitutions in equation (1), we get

$$\begin{aligned} \mathbf{F} &= q\left[-\text{grad}\ \phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A})\right] \\ &= q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\,\mathbf{A}\right] \\ &= q\left[-\nabla\phi - \left\{\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}\right\} + \nabla(\mathbf{v}\cdot\mathbf{A})\right] \end{aligned} \qquad \text{...(3)}$$

The vector potential function **A** depends on space and time coordinates, therefore $\mathbf{A} = \mathbf{A}\,(x, y, z, t)$

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{A}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{A}}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \mathbf{A}}{\partial t}$$

$$= \frac{\partial \mathbf{A}}{\partial x}v_x + \frac{\partial \mathbf{A}}{\partial y}v_y + \frac{\partial \mathbf{A}}{\partial z}v_z + \frac{\partial \mathbf{A}}{\partial t}$$

$$= (\mathbf{v}\bullet\nabla)\,\mathbf{A} + \frac{\partial \mathbf{A}}{\partial t} \qquad ...(4)$$

Using (4), equation (3) becomes

$$\mathbf{F} = q\left[-\nabla\phi - \frac{d\mathbf{A}}{dt} + \nabla(\mathbf{v}\bullet\mathbf{A})\right]$$

or

$$\mathbf{F} = q\left[-\nabla(\phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right] \qquad ...(5)$$

This equation expresses the Lorentz force in terms of electromagnetic potentials **A** and ϕ.

Ex. 2. *Show that*

(i) The momentum of a charged particle in an electomagnetic field is given by

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}.$$

(ii) The Lagrangian and Hamiltonian of a charged particle in an electromagnetic field are given by

$$L = \tfrac{1}{2}mv^2 - q(\phi - \mathbf{v}\bullet\mathbf{A})$$

$$H = \frac{1}{2m}(\mathbf{v} - q\mathbf{A})^2 + q\phi.$$

Solution. Lorentz force in terms of electromagnetic potentials is

$$\mathbf{F} = q\left[-\nabla(\phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right] \qquad ...(1)$$

According to Newton's second law $\mathbf{F} = \dfrac{d\,(m\mathbf{v})}{dt}$

Substituting this in (1), we get

$$\frac{d}{dt}(m\mathbf{v}) = q\left[-\nabla(\phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right]$$

i.e.

$$\frac{d}{dt}(m\mathbf{v} + q\mathbf{A}) + \nabla\{q(\phi - \mathbf{v}\bullet\mathbf{A})\} = 0 \qquad ...(2)$$

This equation has the general form of a set of Lagrangian equation given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha} = 0. \qquad ...(3)$$

Comparing equation (2) and (3), we get

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{x}_\alpha} &= mv_\alpha + q\mathbf{A}_\alpha & ...(a) \\ \frac{\partial L}{\partial x_\alpha} &= \frac{\partial}{\partial x_\alpha}\{-q(\phi_\alpha - \mathbf{v}_\alpha\bullet\mathbf{A})\} & ...(b) \end{aligned}\right\} \qquad ...(4) \qquad \left(\text{where } \dot{x}_\alpha = \frac{\partial x_\alpha}{\partial t}\right)$$

From classical mechanics we know that $\partial L/\partial x_\alpha$ represents α^{th} component of momentum, *i.e.* $\mathbf{p}_\alpha = \partial L/\partial \dot{x}_\alpha$; hence equation (4a) gives

$$\mathbf{p}_\alpha = \frac{\partial L}{\partial \dot{x}_\alpha} = m\mathbf{v}_\alpha + q\mathbf{A}_\alpha$$

This implies

momentum of charged particle,

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \qquad ...(5)$$

(b) Integrating equations (4a) and (4b), we get

$$L = \tfrac{1}{2}mv^2 + q\,(\mathbf{v} \cdot \mathbf{A}) + c_1 \qquad ...(6)$$

$$L = -\,q\phi + q\,(\mathbf{v} \cdot \mathbf{A}) + c_2 \qquad ...(7)$$

where c_1 and c_2 are constants of integration such that c_1 is independent of position; while c_2 is independent of velocity.

If we combine equations (6) and (7), we obtain for the Lagrangian

$$L = \tfrac{1}{2}\,mv^2 + q\,(\mathbf{v} \cdot \mathbf{A}) - q\phi = \tfrac{1}{2}\,mv^2 - q\,(\phi - \mathbf{v} \cdot \mathbf{A}) \qquad ...(8)$$

Now Hamiltonian,

$$\begin{aligned} H &= \sum_i p_\alpha \dot{x}_\alpha - L = \mathbf{p} \cdot \mathbf{v} - L \\ &= (m\mathbf{v} + q\mathbf{A}) \cdot \mathbf{v} - [\tfrac{1}{2}mv^2 + q\,(\mathbf{v} \cdot \mathbf{A}) - q\phi] \\ &= \tfrac{1}{2}mv^2 + q\phi \\ &= \frac{(m\mathbf{v})^2}{2m} + q\phi \\ &= \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi \text{ using (5).} \end{aligned}$$

$$H = \frac{1}{2m}\,(\mathbf{p} - q\mathbf{A})^2 + q\phi.$$

11.7. Hertz's Solution of Wave Equation : Hertzian Oscillator.

In section 11.5, we have seen that the set of four Maxwell's equations may be reduced to a *single equation* in terms of Hertz vector **Z** viz.

$$\nabla^2 \mathbf{Z} - \mu\varepsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} = 0 \qquad ...(1)$$

with the conditions

$$\left.\begin{aligned} \phi &= -\nabla \cdot \mathbf{Z} && ...(a) \\ \mathbf{A} &= \mu\varepsilon \frac{\partial \mathbf{Z}}{\partial t} && ...(b) \\ \mathbf{E} &= \text{grad div } \mathbf{Z} - \nabla^2 \mathbf{Z} && ...(c) \\ \text{and} \quad \mathbf{B} &= \text{curl } \mathbf{A} = \mu\varepsilon \frac{(\partial \text{ curl } \mathbf{Z})}{\partial t} && ...(d) \end{aligned}\right\} \qquad ...(2)$$

Now we seek to find the solution of field equations in *free space* where $\mu\varepsilon = \mu_0\,\varepsilon_0 = 1/c^2$, so that the equivalent equations (1) and (2) take the form

$$\nabla^2 \mathbf{Z} - \frac{1}{c^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2} = 0 \qquad ...(3)$$

$$\left.\begin{aligned} \phi &= -\nabla \cdot \mathbf{Z} && ...(a) \\ \mathbf{A} &= \frac{1}{c^2}\frac{\partial \mathbf{Z}}{\partial t} && ...(b) \\ \mathbf{E} &= \text{grad div}\mathbf{Z} - \nabla^2 \mathbf{Z} && ...(c) \\ \mathbf{B} &= \frac{1}{c^2}\frac{(\partial \text{curl} \mathbf{Z})}{\partial t} && ...(d) \end{aligned}\right\} \qquad ...(4)$$

Here we shall not find the solution corresponding to a plane wave for this does not result to any singularity at the origin, rather we attempt to find the solution corresponding to a spherical wave, in which the surfaces of equal phase are spheres. For this purpose we write $\nabla^2 \mathbf{Z}$ in spherical polar coordinates, assume that $\mathbf{Z}$ is a function of r only, so that from equation (3), we obtain

$$\frac{\partial^2 \mathbf{Z}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathbf{Z}}{\partial r} - \frac{1}{c^2}\frac{\partial^2 \mathbf{Z}}{dt^2} = 0. \qquad \text{...(5)}$$

Since $\mathbf{Z}$ is a function of r and t, let us choose a simple periodic variation with time, so that

$$\mathbf{Z} = \mathbf{F}(r)\, e^{i\omega t} \qquad \text{...(6)}$$

hence

$$\frac{\partial \mathbf{Z}}{\partial r} = \frac{\partial F(r)}{\partial r} e^{i\omega t}; \frac{\partial^2 \mathbf{Z}}{\partial r^2} = \frac{\partial^2 F(r)}{\partial r^2} e^{i\omega t}$$

and

$$\frac{\partial^2 \mathbf{Z}}{\partial t^2} = -\,\omega^2 F(\mathbf{r})\, e^{i\omega t}$$

Making above substitutions and dropping out the common factor $e^{i\omega t}$, equation (5) takes the form

$$\frac{\partial^2 \mathbf{F}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathbf{F}}{\partial r} + \frac{\omega^2}{c^2}\mathbf{F} = 0. \qquad \text{...(7)}$$

Equation (7) may be expressed as

$$\frac{\partial^2}{\partial r^2}(r\,\mathbf{F}) + \frac{\omega^2}{c^2}(r\,\mathbf{F}) = 0. \qquad \text{...(8)}$$

The integration of above equation yields

$$r\,\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\mathbf{p}_0\, e^{-i\omega r/c} \qquad \text{...(9a)}$$

i.e.,

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p_0}}{\mathbf{r}} e^{-i\omega r/c} \qquad \text{...(9b)}$$

The negative sign corresponds to a wave spreading outward from the centre $r = 0$. Then we have

$$\begin{aligned} \mathbf{Z} &= \mathbf{F}(r)\, e^{i\omega t} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p_0}}{\mathbf{r}} e^{-i\omega r/c}\, e^{i\omega t} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p_0}}{\mathbf{r}} e^{-i\omega(t - r/c)} && \text{...(10a)} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}(t)}{r} e^{-i\omega r/c} && \text{...(10b)} \end{aligned}$$

where $\mathbf{p}(t)$ replaces $\mathbf{p}_0 e^{i\omega t}$. This spherical wave differs from a plane wave in that the amplitude decreases as $1/r$, corresponding to the spreading of energy over sphere of increasing size. The field vectors **E** and **B** may be obtained from **Z** by simple differentiations. Starting with **B**, the simpler of the two, we have

$$\text{curl}\ \mathbf{Z} = \text{curl}\left(\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}(t)}{r} e^{-i\omega r/c}\right)$$

$$= \frac{1}{4\pi\varepsilon_0} \text{curl}\ \{ f(r)\ \mathbf{p}\ (t)\} \text{ where } f(r) = \frac{e^{-i\omega r/c}}{r}$$

$$= \frac{1}{4\pi\varepsilon_0} [f(r) \text{ curl } \mathbf{p}\ (t) + \text{grad} f(r) \times \mathbf{p}\ (t)]$$

$$= \frac{1}{4\pi\varepsilon_0} \text{grad } f(r) \times \mathbf{p}\ (t) \text{ since curl } \mathbf{p}\ (t) = 0.$$

Now

$$\text{grad} f(r) = \frac{\partial f(r)}{\partial r} \hat{\mathbf{r}} = \frac{\partial}{\partial r}\left(\frac{1}{r} e^{-i\omega r/c}\right)\hat{\mathbf{r}}$$

$$= \left(-\frac{i\omega}{cr} - \frac{1}{r^2}\right) e^{-i\omega r/c}\ \hat{\mathbf{r}} \qquad \text{...(11)}$$

where $\hat{\mathbf{r}}$ is a unit vector along **r**.

Hence

$$\text{curl } \mathbf{Z} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{i\omega}{cr} - \frac{1}{r^2}\right) e^{-i\omega r/c}\ \hat{\mathbf{r}} \times \mathbf{p}\ (t)$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{-\frac{i\omega}{cr} - \frac{1}{r^2}\right\} e^{-i\omega r/c}\ e^{i\omega t}\ (\hat{\mathbf{r}} \times \mathbf{p_0})$$

∴

$$\mathbf{B} = \frac{1}{c^2} \cdot \frac{\partial\ (\text{curl } \mathbf{Z})}{\partial t}$$

$$= \frac{1}{c^2}\left[\frac{1}{4\pi\varepsilon_0}\left(-\frac{i\omega}{cr} - \frac{1}{r^2}\right) e^{-i\omega r/c}\ e^{i\omega t}\ \hat{\mathbf{r}} \times \mathbf{p_0}\right]$$

$$= \frac{i\omega}{4\pi\varepsilon_0 c^2}\left(\frac{1}{r^2} + \frac{i\omega}{cr}\right) e^{i\omega(t - r/c)}\ \mathbf{p_0} \times \hat{\mathbf{r}} \qquad \text{...(12a)}$$

The vector **B** is thus seen to be composed of two parts similarly directed but decreasing according to different powers of (1/r). We can locate the regions where one of the other term predominates by writting

$$\left(\frac{1}{r^2} + \frac{i\omega}{cr}\right) = \frac{1}{r}\left(\frac{1}{r} + \frac{i2\pi\nu}{c}\right) = \left(\frac{1}{r} + \frac{2\pi i}{\lambda}\right) \qquad \text{...(12b)}$$

The scale of comparison is thus given by the wavelength. In the near zone ($r << \lambda$) the first term predominates, while in the distant zone ($r >> \lambda$)–the actual radiation field–second term predominates and hence controls the situation. Between these two regions both the terms are of the same order of magnitude. Let us first investigate the near zone ($r << \lambda$) symbolised by N, where we have

$$\mathbf{B_N} = \frac{i\omega}{4\pi\varepsilon_0 c^2} e^{i\omega(t - r/c)} \frac{1}{r^2} (\mathbf{p_0} \times \hat{\mathbf{r}})$$

$$= \frac{1}{4\pi\varepsilon_0 c^2 r^2} \cdot \left(\frac{\partial \mathbf{p}}{\partial r} \times \hat{\mathbf{r}}\right) e^{-i\omega r/c}$$

$$= \frac{\mu_0}{4\pi} \frac{\left(\frac{\partial \mathbf{p}}{\partial r} \times \hat{\mathbf{r}}\right) e^{-i\omega r/c}}{r^2} \qquad \text{...(13)}$$

$$\left(\text{since } \mu_0\varepsilon_0 = \frac{1}{c^2} \text{ or } \mu_0 = \frac{1}{\varepsilon_0 c^2}\right)$$

If we compare this expression with the Biot-Savart law of the magnetic field due to an element of current carrying conductor viz.

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i\,d\mathbf{l}\times\hat{\mathbf{r}}}{r^2} \qquad ...(14)$$

We see that, apart from factor $e^{-i\omega r/c}$, the field in the local region is that, due to current element

$$i d\mathbf{l} = \frac{\partial \mathbf{p}}{\partial t} = i\omega \mathbf{p}_0\, e^{i\omega t} \qquad ...(15)$$

The factor $e^{-i\omega r/c}$ corresponds to the finite velocity of propagation of the magnetic field, which causes a phase displacement, increasing with distance from the centre, with respect to the instantaneous value of the current (*"retarded field"*). In order to gain a concept of such an alternating current element, let us consider an electric dipole whose length and direction are given by $\mathbf{l}$ and whose ends carry periodically oscillating charge

$$q = \pm q_0\, e^{i\omega t} \qquad ...(16)$$

Then the moment of the dipole will also be a periodic function of time given by

$$\mathbf{p} = q\mathbf{l} = q_0\, e^{i\omega t}\,\mathbf{l}. \qquad ...(17a)$$

so that

$$\frac{\partial \mathbf{p}}{\partial t} = \frac{\partial q}{\partial t}\,\mathbf{l}. \qquad ...(17b)$$

But $\partial q/\partial t$ represents the current flowing as the charge changes.

Our solution thus gives the magnetic field of a dipole whose axis is fixed and whose moment $\mathbf{p}$ is periodic. Such a dipole is called an *electric vibrator* or *Hertzian oscillator.*

Let us now find the electric field in the near zone. We have

$$\mathbf{E} = \text{grad div}\,\mathbf{Z} - \nabla^2 \mathbf{Z} \qquad ...(18a)$$

We know that $\mathbf{Z}$ satisfies that wave equation (3), therefore (18a) may be written as

$$\mathbf{E} = \text{grad div}\,\mathbf{Z} - \frac{1}{c^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2}. \qquad ...(18b)$$

Now from equation (10a)

$$\begin{aligned}
\text{div}\,\mathbf{Z} &= \text{div}\left[\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}_0}{\mathbf{r}}\,e^{i\omega(t-r/c)}\right] \\
&= \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,\text{div}\left\{\mathbf{p}_0\left(\frac{e^{-i\omega r/c}}{r}\right)\right\} \\
&= \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,\text{div}\,(\mathbf{p}_0 f(r) \text{ where } f(r) = \frac{e^{-i\omega r/c}}{r} \\
&= \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,[f(r)\,\text{div}\,\mathbf{p}_0 + \mathbf{p}_0 \cdot \nabla f(r)] \\
&= \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,\mathbf{p}_0 \cdot \nabla f(r) \ \ (\text{since div}\,\mathbf{p}_0 = 0) \\
&= \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,\mathbf{p}_0 \cdot \frac{\partial f(r)}{\partial r}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,\mathbf{p}_0 \cdot \frac{1}{r}\frac{\partial f(r)}{\partial r}\mathbf{r}. \qquad ...(19)
\end{aligned}$$

Hence

$$\begin{aligned}
\text{grad div}\,\mathbf{Z} &= \text{grad}\left[\frac{1}{4\pi\varepsilon_0}e^{i\omega t}\,\mathbf{p}_0 \cdot \frac{1}{r}\frac{\partial f(r)}{\partial r}\mathbf{r}\right] \\
&= \frac{1}{4\pi\varepsilon_0}e^{i\omega t}\frac{\partial}{\partial r}\left[\mathbf{p}_0 \cdot \frac{1}{r}\left(\frac{\partial f(r)}{\partial r}\right)\mathbf{r}\right]\hat{\mathbf{r}}
\end{aligned}$$

$$= \frac{1}{4\pi\varepsilon_0} e^{i\omega t}\left[\mathbf{p}_0\left\{\frac{1}{r}\frac{\partial^2 f(r)}{\partial r^2}\mathbf{r} - \frac{1}{r^2}\frac{\partial f(r)}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f(r)}{\partial r}\hat{\mathbf{r}}\right\}\hat{\mathbf{r}}\right]$$

Writing $\frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$ and $(\hat{\mathbf{r}})^2 = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$, we get

$$\text{grad div } \mathbf{Z} = \frac{1}{4\pi\varepsilon_0} e^{i\omega t}\left[\mathbf{p}_0 . \hat{\mathbf{r}}\left(\frac{\partial^2 f(r)}{\partial r^2} - \frac{1}{r}\frac{\partial f(r)}{\partial r}\right) . \hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial r}\mathbf{p}_0\right]$$

Now

$$f(r) = \frac{e^{-i\omega r/c}}{r}$$

$$\therefore \qquad \frac{\partial f(r)}{\partial r} = \left(\frac{-i\omega}{cr} - \frac{1}{r^2}\right)e^{-i\omega r/c} \qquad \text{...(20)}$$

and

$$\frac{\partial^2 f}{\partial r^2} = \left(\frac{-\omega^2}{c^2 r} + \frac{2i\omega}{cr^2} + \frac{2}{r^3}\right)e^{-i\omega r/c} \qquad \text{...(21)}$$

$$\therefore \quad \text{grad div } \mathbf{Z} = \frac{1}{4\pi\varepsilon_0} e^{i\omega(t - r/c)}\left[\mathbf{p}_0 . \hat{\mathbf{r}}\left\{\left(-\frac{\omega^2}{c^2 r} + \frac{2i\omega}{cr^2} + \frac{2}{r^3}\right) + \left(\frac{i\omega}{cr^2} + \frac{1}{r^3}\right)\right\} . \hat{\mathbf{r}} - \left(\frac{i\omega}{cr^2} + \frac{1}{r^3}\right)\mathbf{p}_0\right] \qquad \text{...(22)}$$

Also

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2} = -\frac{1}{4\pi\varepsilon_0}\frac{\omega^2}{c^2}\frac{1}{r}\mathbf{p}_0 e^{i\omega(t - r/c)} \qquad \text{...(23)}$$

$$\therefore \; \mathbf{E} = \text{grad div } \mathbf{Z} - \nabla^2 \mathbf{Z}$$

$$= \frac{1}{4\pi\varepsilon_0} e^{i\omega(t - r/c)}\left[\mathbf{p}_0 \cdot \hat{\mathbf{r}}\left\{\left(-\frac{\omega^2}{c^2 r} + \frac{2i\omega}{c^2 r} + \frac{2}{r^3}\right) + \left(\frac{i\omega}{cr^2} + \frac{1}{r^3}\right)\right\}\hat{\mathbf{r}} - \left(\frac{i\omega}{cr^2} + \frac{1}{r^3}\right)\mathbf{p}_0 + \frac{\omega^2}{c^2}\frac{1}{r}\mathbf{p}_0\right]. \qquad \text{...(24)}$$

The field in the near zone is governed by highest power of $1/r$ only. Since $\hat{\mathbf{r}}$ is a unit vector; highest power of $1/r$ is $(1/r)^3$ here. We note that highest power of $(1/r)$ does not occur in the term $\frac{1}{c^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2}$; hence we may omit this term while considering the field in the near zone. Consequently from equation (18 b), the field **E** in the near zone may be expressed as

$$\mathbf{E} = \text{grad div } \mathbf{Z} = -\text{grad } \phi_N \qquad \text{...(25)}$$

where

$$\phi_N = -\text{ div } \mathbf{Z} \qquad \text{...(26)}$$

Thus the electric field in the near zone is derivable from a scalar potential ϕ_N given by (26).

Substituting value of div Z from equation (19) in (26) we get

$$\phi_N = -\left[\frac{1}{4\pi\varepsilon_0} e^{i\omega t}\left(\mathbf{p}_0 . \frac{\partial f(r)}{\partial r}\hat{\mathbf{r}}\right)\right]$$

$$= -\frac{1}{4\pi\varepsilon_0} e^{i\omega t}\,\mathbf{p}_0 . \left(-\frac{i\omega}{cr} - \frac{1}{r^2}\right)e^{-i\omega r/c}\,\hat{\mathbf{r}}$$

$$= -\frac{1}{4\pi\varepsilon_0}\mathbf{p}_0 . \hat{\mathbf{r}}\left(\frac{i\omega}{cr} + \frac{1}{c^2}\right)e^{i\omega t}\,e^{-i\omega r/c}$$

Neglecting lower power of $1/r$ in near region, we get

$$\phi_N = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}_0\, e^{i\omega t} . \hat{\mathbf{r}}}{r^2} e^{-i\omega r/c}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}(t) \cdot \mathbf{r}}{r^3} e^{-i\omega r/c} \qquad \text{...(27)}$$

Since $\hat{r} = \frac{\mathbf{r}}{|\mathbf{r}|}$ and $\mathbf{p}(t) = \mathbf{p}_0 e^{i\omega t}$

We note that apart from the factor $e^{-i\omega r/c}$, the potential in the near zone is the same as that of a dipole of oscillating moment $\mathbf{p}(t)$. Consequently the electric field in the near zone corresponds with the electrostatic field of a dipole of periodically oscillating moment $\mathbf{p}(t) = \mathbf{p}_0 e^{i\omega t}$. The term $e^{-i\omega r/c}$ again represents finite velocity of propagation of electric field which causes phase displacement increasing with distance. ϕ_N is so called the *retarded electric potential.*

A dipole of this kind may be represented by a short linear antenna in which high frequency oscillating currents are induced, say by inductively coupling the antenna to an oscillatory circuit.

Electric and Magnetic fields in distant zone (i.e. radiation field)

In contrast with steady currents. where the impossibility of isolating individual conductor elements implies the restriction of Biot-Savart law as a fundamental basis, current elements are experimentally attainable when we deal with oscillations. However the magnetic field of such a segment is given by the Biot-Savart law only in the immediate neighbourhood of the conductor and then only to a first approximation. According to the *theory of action at a distance*, we should use this law for arbitrarily great distance from the conductor, this would yield result which are quite contrary to experience, for at a great distances (*i.e.* in radiation field) the second term of (12a) controls the situation while considering magnetic field. Hence from (12a) the magnetic induction $\mathbf{B}_R$ in radiation field is given by

$$\mathbf{B}_R = -\frac{\omega^2}{4\pi\varepsilon_0 c^2 r} e^{i\omega(t-r/c)} (\mathbf{p}_0 \times \hat{\mathbf{r}})$$

$$= -\frac{\mu_0 \omega^2}{4\pi c r} e^{i\omega t} e^{i\omega(t-r/c)} (\mathbf{p}_0 \times \hat{\mathbf{r}}) \qquad \text{...(25a)}$$

$$= \frac{\mu_0}{4\pi} \frac{\left(\frac{\partial^2 \mathbf{p}}{\partial t^2} \times \hat{\mathbf{r}}\right)}{cr} e^{-i\omega r/c} \qquad \text{...(25b)}$$

The magnetic field which decreases as $1/r$ is no longer proportional to the first derivative of $\mathbf{p}$- which by equation (17b) is proportional to antenna current-but to its second derivative *i.e.* the rate of change of current.

To determine $\mathbf{E}$ in the radiation field we must collect the terms containing powers of $1/r$ only in equation (24).

Since $(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = 1$, we get

$$\mathbf{E}_R = \frac{1}{4\pi\varepsilon_0} \frac{\omega^2}{c^2 r} e^{i\omega(t-r/c)} \left[-\left(\mathbf{p}_0 \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{r}} + \mathbf{p}_0\right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\omega^2}{c^2 r} e^{i\omega(t-r/c)} \left[(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\, \mathbf{p}_0 - \left(\mathbf{p}_0 \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{r}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\omega^2}{c^2 r} e^{i\omega(t-r/c)} \left\{\hat{\mathbf{r}} \times \left(\hat{\mathbf{p}}_0 \times \hat{\mathbf{r}}\right)\right\} \qquad \text{...(26)}$$

From equations (25a) and (26) it is clear that $\mathbf{B}_R$ is along $(\mathbf{p}_0 \times \hat{\mathbf{r}})$ *i.e.* perpendicular to plane containing $(\mathbf{p}_0 \times \hat{\mathbf{r}})$]. while $\mathbf{E}_R$ is perpendicular to plane containing $(\mathbf{p}_0 \times \hat{\mathbf{r}})$. Consequently $\mathbf{B}_R$ and $\mathbf{E}_R$ are perpendicular to each other.

Poynting vector S. The Poynting vector (or energy flux vector) which is in the direction of $\mathbf{r}$ is given by

$$\mathbf{S} = \mathbf{E}_R \times \mathbf{H}_R = \frac{\mathbf{E}_R \times \mathbf{B}_R}{\mu_0} \qquad \text{(Since } \mathbf{B} = \mu_0 \mathbf{H} \text{ for free space).}$$

Here E and B are real parts of complex $\mathbf{E}_R$ and $\mathbf{B}_R$.

∴ Since E and B are mutually perpendicular, we have

$$|\mathbf{S}| = \frac{1}{\mu_0} |\mathbf{E}_R| \, |\mathbf{B}_R|$$

Here
$$|\mathbf{E}_R| = \frac{1}{4\pi\varepsilon_0} \frac{\omega^2}{c^2 r} p_0 \sin\theta \cos\omega\left(t - \frac{r}{c}\right)$$

and
$$|\mathbf{B}_r| = \frac{\mu_0 \omega^2}{4\pi c r} p_0 \sin\theta \cos\omega\left(t - \frac{r}{c}\right)$$

$$|\mathbf{S}| = \frac{1}{\mu_0}\left[\frac{1}{4\pi\varepsilon_0} \cdot \frac{\omega^2}{c^2 r} p_0 \sin\theta \cos\omega\left(t - \frac{r}{c}\right)\right] \times \left[\frac{\mu_0 \omega^2}{4\pi c r} \cdot p_0 \sin\theta \cos\omega\left(t - \frac{r}{c}\right)\right].$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\omega^4}{4\pi c^3 r^2} p_0^2 \sin^2\theta \cos^2\omega\left(t - \frac{r}{c}\right)$$

Since average value of $\cos^2 \omega (t - r/c)$ over a complete cycle is $\frac{1}{2}$, therefore magnitude of time-averaged Poynting vector is given by

$$\langle S \rangle = \frac{1}{4\pi\varepsilon_0} \frac{\omega^4}{8\pi c^3 r^2} p_0^2 \sin^2\theta \qquad \text{...(27)}$$

The *average energy which passes* through a sphere of radius *r* each second is obtained by taking the surface integral.

$$\oint \langle S \rangle \cdot d\mathbf{a} = \frac{1}{4\pi\varepsilon_0} \frac{\omega^4}{8\pi c^3 r^2} p_0^2 \cdot 2\pi r^2 \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\omega^4}{8\pi c^3} p_0^2 \cdot 2\pi\left(\tfrac{4}{3}\right) \left[\text{Since} \int_0^{\pi} \sin^2\theta \, d\theta = \frac{4}{3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\omega^4 p_0^2}{3c^3} \qquad \text{...(28)}$$

Using $\omega = 2\pi\nu = \frac{2\pi c}{\lambda}$, the energy loss by the source of radiation per second may be expressed as

$$\oint \langle S \rangle \cdot d\mathbf{a} = \frac{1}{4\pi\varepsilon_0} \frac{(2\pi c/\lambda)^4 p_0^2}{3c^3} = \frac{1}{4\pi\varepsilon_0} \frac{16\pi^4 c}{3\lambda^4} p_0^2 \qquad \text{...(29)}$$

11.8. Spherical Waves

The scalar equation in a non-conducting homogeneous isotropic medium in absence of charge and current is

$$\nabla^2 \psi(\mathbf{r}, t) - \mu\varepsilon \frac{\partial^2 \psi(\mathbf{r}, t)}{\partial t^2} = 0 \qquad \text{...(1)}$$

where $\psi(\mathbf{r}, t)$ is the scalar field function of space and time coordinates.

Let
$$\psi(\mathbf{r}, t) = u(\mathbf{r}) \, e^{-i\omega t} \qquad \text{...(2)}$$

where u (**r**) represents spatial wave function independent of time.

Using equation (2) and dropping out the common factor $^{-i\omega t}$ equation (1) may be expressed as

$$\nabla^2 u + \mu\varepsilon\omega^2 u = 0 \qquad \text{...(3)}$$

But $\mu\varepsilon = 1/v^2$, where v is phase velocity of the wave; so that

$$\mu\varepsilon\omega^2 = \frac{\omega^2}{v^2} = k^2 \qquad \text{...(4)}$$

Using (4), equation (3) takes the form

$$\nabla^2 u + k^2 u = 0 \qquad \text{...(5)}$$

Now let us solve equation (5) in spherical polar coordinates (r, θ, ϕ) by method of separation of variables. In spherical polar coordinates equation (5) is expressed as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0 \qquad \text{...(6)}$$

Let

$$u\,(r, \text{thcta}, \phi) = R\,(r)\,\Theta(\theta)\,\Phi(\phi) \qquad \text{...(7)}$$

Substituting this value of u from (7) in equation (6) and dividing by $R\Theta\Phi$, we get

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} + k^2 = 0$$

Multiplying throughout by $r^2 \sin^2\theta$, we get

$$\frac{\sin^2\theta}{R}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \theta^2} + k^2 r^2 \sin^2\theta = 0$$

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial \Theta}{\partial \theta}\right] + k^2 r^2 \sin^2\theta = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} \qquad \text{...(8)}$$

In above equation R.H.S. is dependent on ϕ only ; while L.H.S. is independent of ϕ and depends on r and θ only. Therefore if this equation is satisfied, each side must be equal to same scalar constant , m^2 (say); so that

$$-\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = m^2 \text{ or } \frac{\partial^2 \Phi}{\partial \phi^2} + m^2\,\Phi = 0 \qquad \text{...(9)}$$

and

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial \Theta}{\partial \theta}\right] + k^2 r^2 \sin^2\theta = m^2 \qquad \text{...(10)}$$

Dividing equation (10) by $\sin^2\theta$ throughout, we obtain

$$\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] + \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial \Theta}{\partial \theta}\right] + k^2 r^2 = \frac{m^2}{\sin^2\theta}$$

i.e.

$$\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] + k^2 r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Theta}{\partial \theta}\right)$$

In this equation L.H.S. depends only on r; while R.H.S. depends only on θ; therefore if this equation is satisfied, each side must be equal to the same constant ; $l\,(l+1)$ say. Thus

$$\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] + k^2 r^2 = l\,(l+1) \qquad \text{...(11)}$$

and

$$\frac{m^2}{\sin^2\theta}-\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]=l\,(l+1) \quad \text{...(12a)}$$

Above equation may be expressed as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right]+\left[k^2-\frac{l\,(l+1)}{r^2}\right]R=0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]+\left[l\,(l+1)-\frac{1}{\sin^2\theta}\right]\Theta=0. \quad \text{...(12b)}$$

The solutions of equation (7) are well known

$$\phi_m=\frac{1}{\sqrt{2\pi}}e^{im\phi}$$

where

$$m=0,\pm 1,\pm 2 \quad \text{...(13)}$$

and $\frac{1}{\sqrt{2\pi}}$ *is normalisation* constant.

The solution of Θ equation. If we substitute a new variable x such that

$$x=\cos\theta$$

is equation (12b), then we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\Theta}{\partial x}\right\}+\left\{l\,(l+1)-\frac{m^2}{(1-x^2)}\right\}\Theta=0 \quad \text{...(14)}$$

The equation is known as associated *Legendre's equation*. Hence its solutions for arbitrary $m\leq l$ are associated Legendre polynomials $P_l^m\,(x)$ defined by

$$P_l^m\,(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l\,(x) \quad \text{...(15a)}$$

with $x=\cos\theta$. it is clear that $P_l^0\,(x)=P_l\,(x)$, known as Legendre polynomial. For $m\neq 0$ the functions are given in the following table. Taking normalising constant, the solution of Θ– equation may be expressed as

$$\Theta\,(\theta)=\left[\frac{(2l+1)}{2}\frac{(l-m)\,!}{(l+m)\,!}\right]^{1/2}P_l^m\,(\cos\theta) \quad \text{...(15b)}$$

Table 11.1. Associated Legendre Polynomials $x=\cos\theta$.

Polynomials ($x=\cos\theta$)	Function
$P_\theta\,(x)$	$1\ x=\cos\theta$
$P_1\,(x)$	$(1-x^2)^{1/2}=\sin\theta$
$P_1^1\,(x)$	$\frac{1}{2}(3x^2-1)=\frac{1}{4}(3\cos 2\theta+1)$
$P_2^1\,(x)$	$3\,x\,(1+x^2)^{1/2}=\frac{3}{2}(\sin 2\theta)$
$P_2^1\,(x)$	$3\,(1-x^2)=\frac{3}{2}(1-\cos 2\theta)$
$P_2^2\,(x)$	$\frac{1}{2}(5x^3-x)$
$P_3^1\,(x)$	$\frac{3}{2}(1-x^2)^{1/2}(5x^2-1)$
$P_3^2\,(x)$	$15x\,(1-x^2)$
$P_3^3\,(x)$	$15\,(1-x^2)^{3/2}$

Solution of the Radial Equation. Let us substitute $kr = \rho$ in equation (11b), then we obtain

$$\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial R}{\partial \rho}\right) - \left[l(l+1) - \rho^2\right] R = 0$$

Substituting $R = \rho^{-1/2} Z$, we obtain

$$\rho^2 \frac{\partial^2 Z}{\partial p^2} + \rho \frac{\partial Z}{\partial \rho} - \left[\left(l + \frac{1}{2}\right)^2 - \rho^2\right] Z = 0 \qquad \text{...(16)}$$

This is a familiar equation, known as *Bessel's equation*. The solutions of this equation are well known and have been extensively investigated and tabulated. The common solutions are given by *Bessel function* $J_{l+\frac{1}{2}}(kr)$ and *Neumann function* $N_{l+\frac{1}{2}}(kr)$; of order $l + \frac{1}{2}$. For purposes of wave equation it is convenient to define spherical *Bessel's function* as

$$j_l(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr\,;\, n_l(kr) = \sqrt{\frac{\pi}{2kr}}\, N_{l+\frac{1}{2}}(kr) \qquad \text{...(17)}$$

and from these in turn, we obtain

Spherical Henkel function $h_l(kr)$ defined as

$$h_i(kr) = j_i(kr) \pm i n_l(kr) \qquad \text{...(18)}$$

The functions $j_l(kr)$, $n_l(kr)$ and $h_l(kr)$ are all solutions of the radial equation (11). These functions are tabulated for $l = 0, 1, 2$ in table 11.2.

Table 11.2. Spherical Bessel, Neumann and Hankel Functions

$$h_l(\rho) = j_i(\rho) \pm i n_l(\rho)$$

i.e.

$$h_l(+)(\rho) = j_l(\rho) + i n_l(\rho)$$

$$h_l(-)(\rho) = j_l(\rho) - i n_l(\rho)$$

Type	Function
$j_0(\rho)$	$(1/\rho)\sin\rho$
$n_0(\rho)$	$-(i/\rho)\cos\rho$
$n_0^{(+)}(\rho)$	$-(i/\rho)\, e^{i\rho}$
$h_0^{(-)}(\rho)$	$(i/\rho)\, e^{-i\rho}$
$j_1(\rho)$	$(1/\rho)\sin\rho - (1/\rho)\cos\rho$
$n_1(\rho)$	$-(1/\rho)\sin\rho - (1/\rho^2)\cos\rho$
$h_1^{(+)}(\rho)$	$-(1/\rho)\, e^{i\rho}(1 + i/\rho)$
$h_1^{(-)}(\rho)$	$-(1/\rho)\, e^{i\rho}(1 - i/\rho)$
$j_2(\rho)$	$(3/\rho^3 - 1/\rho)\sin\rho - (3/\rho^2)\cos\rho$
$n_2(\rho)$	$-(3/\rho^2)\sin\rho\,(3/\rho^3 - 1/\rho)\cos\rho$
$h_2^{(+)}\rho$	$(i/\rho)\, e^{i\rho}\,.\,(1 + 3i/\rho - 3/\rho^2)$
$h_2^{(-)\rho}$	$-(i/\rho)\, e^{-i\rho}(1 - 3i/\rho - 3/\rho^2)$

The spherical Hankel functions h_l are particularly convenient for radiation problems because for large distances they approach the values

$$h_l(kr) = j(kr) \pm i n_l(kr) \rightarrow \frac{1}{kr} e^{\pm i[kr - (l+1)\pi/2]} \qquad ...(19)$$

thus leading to outgoing and ingoing spherical waves ; since for (+) ve sign this represents the space part of . a wave travelling in along r and for (–) ve sign a wave travelling out, in each case with an amplitude that falls off inversely as the distance, so that the intensity, which is proportional to the square of amplitude, falls off inversely as the square of the distance. For radiation from a source then we choose a negative sign in (19), and thus secure only an outgoing wave without an incoming wave. We note that the Hankel functians, containing the Neumann functions, all have the singularities at the origin. Thus an outgoing wave must either be propagated from a finite charge distribution or the outgoing wave must be propagated by a singularity or field, which can take the form of certain point charges, dipoles and other multipoles.

Thus the general form of u may be written as

$$\begin{aligned} u_{lm} &= R(r)\,\Theta(\theta)\,\Phi(\phi) \\ &= h_l(kr)\left[\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\frac{1}{\sqrt{(2\pi)}}e^{im\phi} \\ &= h_l(kr)\,Y_l^m(\theta,\phi) \end{aligned} \qquad ...(20)$$

where the spherical function

$$Y_l^m(\theta,\phi) = \left[\frac{2l-1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\,e^{im\phi}$$

satisfies the following properties

$$\left.\begin{aligned} Y_l^{-m}(\theta,\phi) &= (-1)^m\,Y_l^{m}{}^*(\theta,\phi) \\ \text{and} \quad \int Y_l^m(\theta,\phi)\,Y_{l'}^{m'}{}^*(\theta,\phi)\,d\Omega &= \delta_{mm'}\,\delta_{ll'} \end{aligned}\right\} \qquad ...(21)$$

where l is a positive integer and $|m| \leq l$,

Since the vector equation is equivalent to three scalar equations for the three components of a vector ; hence if the function u_{lm} satisfies the scalar equation (5), then the gradient of u is a solution of the corresponding vector equation, and so is $\mathbf{r} \times \text{grad}\, u$ if the equation is invariant under an infinitesimal rotation co-ordinates.

Now

$$\mathbf{r} \times \text{grad}\, u = -\nabla \times (\mathbf{r}\, u) \qquad ...(22)$$

which follows from the vector relations

$$\left.\begin{aligned} \text{curl}\,(u\,\mathbf{r}) &= u\,\text{curl}\,\mathbf{r} - \mathbf{r} \times \text{grad}\, u \\ \text{and} \quad \text{curl}\,\mathbf{r} &= 0 \end{aligned}\right\}$$

Since the divergence of any curl is zero, therefore the divergence of vector $\mathbf{r} \times$ grad u (or $\mathbf{r} \times \nabla u$) vanishes, thus it satisfies a necessary condition on **E** or **B** in a uniform source free medium. A vector field having no radial component can therefore be expressed as a sum over l and m of such solutions, with coefficients appropriate to the detailed nature of the problem. If we let

$$\mathbf{B}_1 = \mathbf{r} \times \nabla u_1 \qquad ...(23)$$

where u_l is a solution (or a sum of solutions) of equation (5). Now the corresponding electric field may be determined from the Maxwell's equation $\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ ...(24)

Here $\mathbf{J} = 0$, $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\,\mathbf{E}$; therefore equation (24) becomes

$$\text{curl}\,\mathbf{B} = \mu\varepsilon\frac{\partial \mathbf{E}}{\partial t} \qquad ...(25)$$

$$\left.\begin{aligned}\text{Now assuming } \mathbf{E} &= \mathrm{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\\ \mathbf{B} &= B_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\end{aligned}\right] \quad ...(26)$$

Equation (25) give

$$\text{curl } \mathbf{B} = -\, i\omega\mu\varepsilon\, \mathbf{E}$$

Hence required electric field

$$\mathbf{E}_1 = \frac{i \text{ curl } \mathbf{B}_1}{\mu\varepsilon\omega} \quad ...(27)$$

Since $\mathbf{B}_1$ has no component along $\mathbf{r}$, hence it can not represent a general magnetic field even through the scalar function u_1^m form a complete set. However we may obtain a linearly independent solution of vector equation by interchanging the roles of **E** and **B**

i.e.
$$\mathbf{E}_2 = \mathbf{r}\times\nabla u_2 \quad ...(28)$$

where u_2 is a solution of equation (5), the corresponding magnetic iduction **B** can then, be obtained from the relation

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\ \ i.e.\ \text{curl } \mathbf{E} = i\omega\, \mathbf{B}$$

Hence
$$\mathbf{B}_2 = -\frac{i}{\omega}\,\text{curl } \mathbf{E}_2 \quad ...(29)$$

In general an electromagnetic field can be expressed as sum of these two kinds of partial fields. The two scalar functions u_1 and u_2 are known as Debye potentials and from examination of space dependence of $\mathbf{E}_1$ and $\mathbf{B}_2$ equations (27) and (29) it may be seen that the scalar functions u_1 and u_2 are essentially equivalent to radial components of $\mathbf{E}_1$ and $\mathbf{B}_2$ respectively. It can be readily verified form the wave equation that if

$$\mathbf{B} = \mathbf{r}\times\nabla u_l^m$$

then
$$\begin{aligned} i\mu\varepsilon\omega\, \mathbf{E}_\mathbf{r} &= -\,(\text{curl } \mathbf{B})_\mathbf{r} = -\,[\text{curl}\,(\mathbf{r}\times\nabla u_l^m)]_r\\ &= -\,[\nabla\times(\nabla\times \mathbf{r}u_l^m)]_r = \frac{(l+1)}{r}\,u_l^m \end{aligned} \quad ...(30)$$

Thus all the fields denoted by the subscript 1 may be expressed in terms of E_r. Similarly the fields denoted by subscript 2 may be expressed in terms of B_r. But usually the fields are not expressed in this way, but it is convenient ot characterise them by subscripts E and M rather than by 1 and 2 accordingly $\mathbf{E}_E$, $\mathbf{B}_E$ is a set of fields for which the electric field, but not the magnetic fields has a parallel component, whereas $\mathbf{E}_M$, $\mathbf{B}_M$ represents a set of fields for which only the magnetic field has a radial component. For each value of the separation parameters and m, we may find field of each type.

$$\left.\begin{aligned}\mathbf{E}_E(l, m) &= \nabla\times\nabla\times\mathbf{r}\, u_l^m\\ \mathbf{B}_E(l, m) &= -\,ik\sqrt{\mu\varepsilon}\ \nabla\times\mathbf{r}\, u_l^m\end{aligned}\right\} \quad ...(31)$$

$$\left.\begin{aligned}\mathbf{E}_M(l, m) &= ik\,\nabla\times\mathbf{r}u_l^m\\ \mathbf{B}_M(l, m) &= \sqrt{\mu\varepsilon}\ \nabla\times\nabla\,\mathbf{r}\, u_l^m\end{aligned}\right\} \quad ...(32)$$

where the constants are introduced so that both types of fields $\mathbf{E}_E$ and E_M may have the same dimensions.

Explicitly the components for type **E** are given by

$$\left.\begin{aligned} E_r &= \frac{l(l+1)}{r} h_l(kr)\, Y_l^m(\theta, \phi), \quad B_r = 0 \\ E_\theta &= \frac{1}{r}\frac{\partial}{\partial r}[r h_l(kr)] \frac{\partial}{\partial r}[Y_l^m(\theta, \phi)], \; B_\theta = \sqrt{\mu\varepsilon}\, \frac{km}{\sin\theta} h_l(kr)\, Y_l^m(\theta, \phi) \\ E_\phi &= \frac{im}{r\sin\theta}\frac{\partial}{\partial r}[r h_l(kr)]\, Y_l^m(\theta, \phi), \; B_\phi = ik\sqrt{\mu\varepsilon}\, h_l(kr) \frac{\partial}{\partial \theta}[Y_l^m(\theta, \phi)] \end{aligned}\right\} \quad \text{...(33)}$$

and independently the components for type M are given by

$$\begin{aligned} E_r &= 0, \; B_i = \sqrt{\mu\varepsilon}\, \frac{l(l+i)}{r} h_l(kr)\, Y_l^m(\theta, \phi) \\ E_\theta &= -\frac{km}{\sin\theta} h_l(kr)\, Y_l^m(\theta, \phi), \; B_\theta = \sqrt{\mu\varepsilon}\, \frac{1}{r}\frac{\partial}{\partial r}[r h_l(kr)] \frac{\partial}{\partial \theta}[Y_l^m(\theta, \phi)] \\ E_\phi &= -ik h_l(kr) \frac{\partial}{\partial \theta}[Y_l^m(\theta, \phi)], \; B_\phi = \sqrt{\mu\varepsilon}\, \frac{im}{r\sin\theta}\frac{\partial}{\partial r}[r h_l(kr)]\; Y_l^m(\theta, \phi) \end{aligned} \quad \text{...(34)}$$

It may be seen readily that these fields vanish identically for $l = 0$. According to their spatial symmetry they are classified as multipole fields of order $2l$, those for which the radial component of **E** does not vanish [equations (31) and (33)] are characterised as electric multipoles, while those for which the radial component of **B** does not vanish [equations (32) and (34)] represent magnetic multipole fields.

QUESTIONS AND PROBLEMS

1. Show how the four-Maxwell's equations for the electromagnetic field can be written as a pair of equations in terms of the scalar and vector potenials. Discuss the gauge transformation used. **(Meerut 1996)**
2. Write Maxwell's equations in terms of scalar and vector potentials. Show that these equations are invariant under gauge transformations. Discuss the significance of this invariance. Are these potentials independent of each other and physically measurable. **(Rohilkhand 1993, 89; Meerut 87, 82)**
3. Write down the Maxwell's equations for an electromagnetic field. Show how they can be written as a pair of inhomogeneous wave equations involving scalar and vector potentials. Explain the gauge transformation used. **(Meerut 1971)**
4. Discuss the non-uniqueness of electromagnetic potentials and hence explain the significance of guage transformation. **(Meerut 1971)**
5. What are gauge transformations ? What are Coulomb's and Lorentz gauges ? What are their importance. **(Meerut 1991)**
6. Discuss the importance and utility of Lorentz and Coulomb gauges in dealing with inhomogeneous wave equations.
7. What do you understand by Lorentz gauge and Coulomb's gauge ? Show that coupled in homogeneous Maxwell's equations are uncoupled by a guage transformation. **(Meerut 1989)**
8. What is Hertz vector. Show that a set of four Maxwell's equations can be reduced to a single wave equation in terms of Hertz vector.
9. (a) Derive expression for inhomogeneous wave equations for scalar and vector potentials.
   (b) Using Fourier analysis, derive the solutions of scalar and vector potentials in terms of charge and current distributions respectivley. **(Agra 1975, 73)**
   (c) What are electromagnetic potentials ? Discuss their uniqueness.Obtain Maxwell equations in terms of electromagnetic potentials. **(Meerut 1996)**
10. Comment on the statement "Maxwell's equations may equivalently be expressed in terms of scalar and vector potentials." **(Rohilkhand 1988, 86)**
11. Obtain Maxwell's equations in terms of scalar and vector potentials using Coulomb's gauge for potentials. Discuss the usefulness of this gauge. **(Meerut 1983)**
12. Starting with the expression for the force per unit volume on a region of *free space* containing charges and currents.

$$\mathbf{F}_v = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

and using Maxwell's equations and vector indentity

$$\mathbf{B} \times (\text{curl } \mathbf{B}) = \text{grad } (\tfrac{1}{2} B^2) - (\mathbf{B} \cdot \text{grad})\, \mathbf{B}$$

show that

$$\mathbf{F}_v = -\varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) + \varepsilon_0 \mathbf{E} \text{ div } \mathbf{E} - \frac{1}{2} \varepsilon_0 \text{ grad } (\mathbf{E}) + \varepsilon_0 (\mathbf{E} \cdot \text{grad})\, \mathbf{E}$$

$$+ \frac{\mathbf{B}}{\mu_0} \text{ div } \mathbf{B} - \frac{1}{2\mu_0} \text{ grad } (B^2) + \frac{1}{\mu_0} (\mathbf{B} \cdot \text{grad})\, \mathbf{B}$$

(The quantity $\varepsilon_0 \mathbf{E} \times \mathbf{B}$ is sometimes referred to as the momentum density of the electromagnetic field).

13. A medium is characterised by $\rho = 0$, $\mathbf{J} = 0$, $\mu = \mu_0$, but where the polarisation **P** is a given function of position and time, $\mathbf{P} = \mathbf{P}(\mathbf{r}, t)$. Show that the Maxwell's equations are correctly obtained from a single vector function **Z** (the Hertz vector), where **Z** satisfies the equation

$$\nabla^2 Z - \frac{1}{c^2} \frac{\partial^2 Z}{\partial t^2} = -\frac{\mathbf{P}}{\varepsilon_0}$$

and

$$\mathbf{E} = \text{curl curl } Z - \frac{1}{\varepsilon_0} \mathbf{P}\,; \quad \mathbf{B} = \frac{1}{c^2} \text{ curl } \frac{\partial \mathbf{Z}}{\partial t}.$$

14. Give Hertz solution of wave equation to obtain the electric and magnetic fields from an oscillating electric dipole and calculate the time averaged power radiated per unit solid angle and the total power radiated by it. **(Meerut 1972)**
15. Write a note on spherical waves from a point charge. **(Meerut 1988)**
16. Write short notes on

    (i) Coulomb gauze transformation

    (ii) Lorentz gauze. **(Meerut 1998)**

13

Propagation of Radio Waves

13.1. Introduction.

The electromagnetic radio frequency waves containing intelligence (audio signal) are launched from the transmitting antenna. These waves are radiated in all possible directions with speed of light, $c = 3 \times 10^8$ m/s. These waves carry oscillating electric and magnetic fields mutually perpendicular and also perpendicular to direction of wave propagation. That is the electromagnetic waves are transverse in nature. These waves do not require a material medium for their propagation and travel best in free space or vacuum. But the electromagnetic waves starting from transmitter antenna reach the receiving antenna through atmosphere whose density varies with altitude. Therefore the waves are disturbed by the surrounding atmosphere, disturbed weather like rain and snow.

According to density and temperature variation, the atomsphere is divided into three regions.

1. Troposphere. This is region of earth's atmosphere extending from earth's surface upto a height of nearly 12 km. In this region the density of air molecules is sufficient to cause convective currents to lift the warm air from earth's surface and carry it away. In this region temperature and density both decrease with increase of altitude.

2. Stratosphere. Above the troposphere the region of earth's atmosphere extending from 12 km to 50 km is called the *stratosphere*. In this region too the density falls but the temperature rises with increasing altitude.

3. Ionosphere. The region of earth's atmosphere above 50 km to 350 km. contains ionised particles and hence is called the *Ionosphere*. This region plays an inportant part for the propagation of *radio waves*, because it acts as a *radio mirror* to bounce the radio waves back to earth. The layers of ionised particles are created by various strong radiations from the sun and are therefore subject to disturbances as well as atmospheric conditions.

In this chapter we shall study the propagation to radio waves through the atmospheric medium, considering the effect of medium and several electronic devices.

13.2. Types of Wave Propagation.

There are a number of methods, depending primarily on frequency, by which radio waves may travel from transmitting to receiving antenna. The most important of these are :

1. Ground or Surface Waves. These are the waves radiated by antenna that travel at zero or lower angles with respect to earth's surface.

2. Space Waves. These are the waves that travel more or less directly through space between transmitting and reciving antenna. The space waves are within the troposphere region of atmosphere and have two components.

(i) Direct or line of sight waves. The space waves that travel directly from transmitting to receiving antenna are called the *direct* or *line of sight waves.*

(ii) Ground Reflected Waves. The space waves that travel from transmitting antenna to earth and then reflected to receiving antenna are called *ground reflected waves.*

3. Sky Waves. These are the waves radiated by antenna at high angles that travel upward and reach the receiving antenna after suffering reflection from ionosphere.

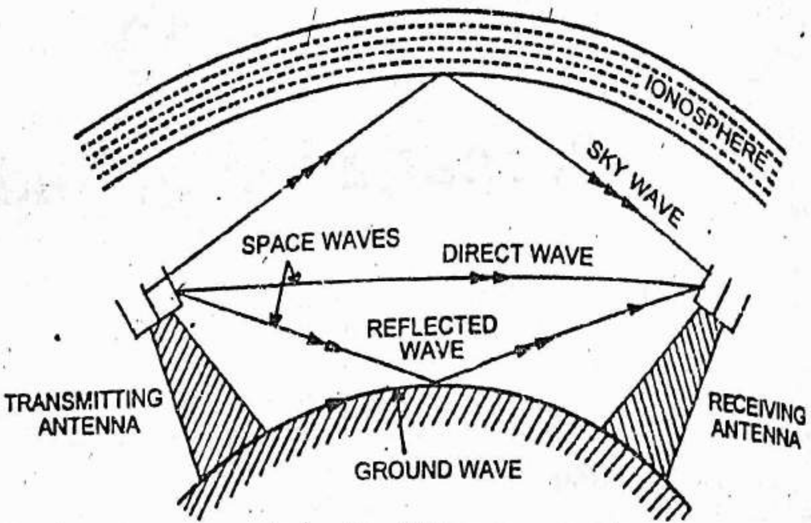


Fig. 13.1

13.3. Propagation of Ground or Surface Waves.

In ground waves propagation, the radio waves are guided along the earth's surface because of electrical discontinuity existing between the earth and the atmosphere. The discontinuity produces a diffraction effect that causes the wave to follow the curvature of earth. The electric vector of ground wave is vertical. If the horizontal component of electric vector exists, it is short-circuited by the earth. During the passage the ground wave induces charges on the earth which travel with the wave and constitute a current. In carrying the induced currents, the earth behaves as a leaking capacitor. Therefore it may be represented by a resistance shunted by a capacitance. Due to this fact the electrical characteristics of earth may be described by conductivity σ and dielectric constant K.

As ground wave passes over the earth's surface, the energy is dissipated due to flow of induced current through earth's resistance, therefore the wave is weakened. The weakening of the wave depends upon two factors.

1. The resistance (or conductivity) **of the surface at the side of the wave.** The conductivity of moist soil or sea water is much greater than that of dry soil. Therefore the waves are strongly absorbed by solid ground but much less absorbed by salt water.

2. The frequency of the travelling wave. The absorption of the wave increases considerably with increasing frequency; so that ground wave propagation is generally useful at low frequency. At frequencies above $2MHz$. the absorption is so high that the transmission range becomes essentially negligible.

For frequency below 500 kHz, the reliable communication can be obtained over distances upto 1500 km, by the ground wave alone. Amplitude modulated radio broadcasts in the medium frequency band (550-1600 kHz) are transmitted primarily via the ground waves. But at higher frequencies used by frequency modulation and television the ground wave is increasing absorbed and becomes insignificant beyond a distance of few kilometres from the transmitter.

However at frequencies, where surface wave transmission can be effectively used, it represents a reliable propagation mode that is not subject to seasons or atmospheric conditions.

According to Sommerfield prediction the surface wave field intensity at a distance d from the transmitting antenna, assuming the earth as flat, is given by the relation

$$E = \frac{AE_0}{d}$$

where A is called the sommerfeld reduction factor which depends upon the conductivity and dielectric constant of the earth and E_0 is the field strength of the wave at a unit distance from the transmitting antenna.

13.4. Propagation of Space Waves

This type of propagation is also called the *tropospheric propagation.* It is used at frequencies above about 30 Mc/s, where neither surface wave nor sky wave propagation is possible. In this frequency range, the surface wave amplitude is reduced in strength by absorption as to become virtually useless within a few hundred feet, because due to short wavelength more charges are induced in the earth per unit length and their flow causes heavy dissipation of energy. The sky wave is also not reflected back to earth by the ionosphere at such a high frequency.

In this frequency range, useful propagation can be achieved by space wave. Fig. 13.2 illustrates the manner of space wave propagation, assuming the earth as flat *i.e.*, neglecting the earth curvature. It may be seen from Fig. 13.2 that energy reaches the receiver in two ways ;

(i) By a ray travelling directly between transmitting and receiving antenna over a path TR.

(ii) By a ray travelling over path TOR which is reflected from the earth at O.

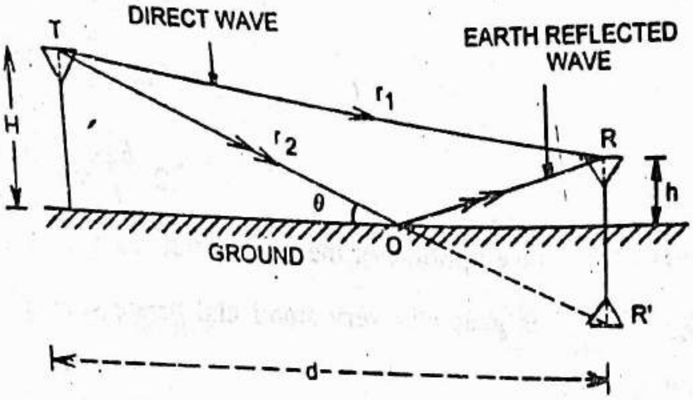


Fig. 13.2

The field strength at the receiving antenna R is the vector sum of fields represented by these rays. As both the rays travel through space, they suffer negligible attenuation and their amplitdues can be supposed to be inversely proportional to the distance from the transmitter. Also, if the distance d between the transmitting antenna T and receiving antenna R is considerably greater than the antenna heights H and h, the angle θ which the ray TO makes with the surface of the earth will be very small and the reflection occuring at O will simply cause a reversal of phase and no change in magnitude. The waves *i.e.*, the direct and earth reflected waves, will, therefore, arrive at the receiving antenna with equal amplitudes but differing in phase.

The resultant field strength at the receiving antenna R may be calculated as follows :

Distance $(TR)^2 = r_1^2 = d^2 + (H-h)^2$

or
$$r_1 = d\left[1 + \frac{(H-h)^2}{d^2}\right]^{1/2} = d + \frac{(H-h)^2}{2d} \qquad \text{...(i)}$$

Similarly, distance
$$r_2 = d + \frac{(H+h)^2}{2d} \qquad \text{...(ii)}$$

Consequently, the path difference is
$$r_2 - r_1 = \frac{(H+h)^2}{2d} - \frac{(H-h)^2}{2d} = \frac{2Hh}{d}$$

As
$$\text{phase difference} = \frac{2\pi}{\lambda} \times \text{path difference}$$

$\therefore$
$$\text{Phase difference} = \frac{2\pi}{\lambda}\cdot\frac{2Hh}{d} = \frac{4\pi Hh}{\lambda d}$$

Due to reflection at the denser medium (at point O), an additional phase difference of π takes place. Hence

$$\text{Total phase difference} = \pi + \frac{4\pi Hh}{\lambda d} \qquad \text{...(iii)}$$

If E represents the common amplitude of the two waves, then the resultant amplitude E_R is given by

$$E_R^2 = E^2 + E^2 + 2E.E\cos\left(\pi + \frac{4\pi Hh}{\lambda d}\right)$$

$$= 2E^2\left[1 + \cos\left(\pi + \frac{4\pi Hh}{\lambda d}\right)\right]$$

$$= 2E^2\left[2\cos^2\left\{\frac{\left(\pi + \frac{4\pi Hh}{\lambda d}\right)}{2}\right\}\right]$$

$$E_R = 2E\cos\left[\frac{\pi}{2} + \frac{2\pi Hh}{\lambda d}\right]$$

$$= -2E\sin\frac{2\pi Hh}{\lambda d}$$

$$= -2\frac{E_0}{d}\sin\frac{2\pi Hh}{\lambda d} \qquad \text{...(4)}$$

where E_0 is the amplitude of the wave produced at unit distance by the transmitting antenna. Since d is very large, $\frac{2\pi Hh}{\lambda d}$ is generally very small and hence sine of the angle is eq. (4) can be replaced by the angle giving

$$E_R = -\frac{2E_0}{d}\cdot\frac{2\pi Hh}{\lambda d} = -\frac{4\pi Hh\,E_0}{\lambda d^2} \qquad \text{...(5)}$$

This equation depicts the importance of antenna heights. If the height of the transmitting antenna is doubled, the field strength at the receiving point will be doubled but if the transmitted power is doubled, the field strength increases only $\sqrt{2}$ times because field strength is proportional to E_0 which is turn is proportional to the square root of the transmitted power.

Typical curves showing the variation of field strength with distance from the transmitter, on the basis of eq. (5), are given in Fig. 13.3. The dotted line shows the *free space wave* of strength $\frac{E_0}{d}$. It is due to the direct wave TR between transmitter and receiver. Due to earth reflected wave TOR, the resultant field strength oscillates about the value $\frac{E_0}{d}$ till at larger distances from the transmitter, the field strength at the receiving antenna becomes less than the free space value as shown in Fig. 13.3, because for larger distances the field intensity decreases as $\frac{1}{d^2}$ and hence drops off rapidly.

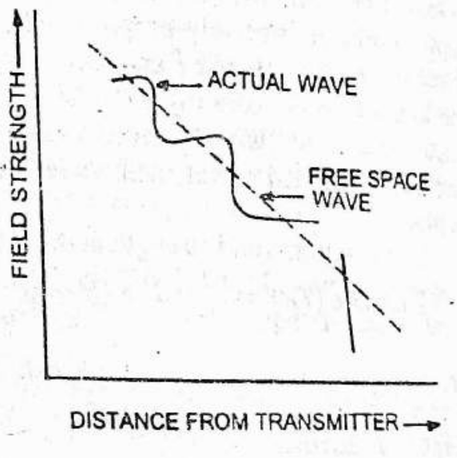


Fig. 13.3

13.5. Effect of Earth's Curvature on Space Wave Propagation

When the distance between transmitting and receiving antennas is considerably large, it becomes necessary to take into account the curvature of the earth.

When the receiving and transmitting antennas are above the horizon of the curved earth, the space wave propagation will be in the manner shown in Fig. 13.4. The field strength at the receiving antenna is still the vector sum of direct and earth reflected waves but now the effective antenna heights H' and h', above the tangent line, $t\,t'$, should be used in eq. (4) of section 13.4 for field strength in place of H and h.

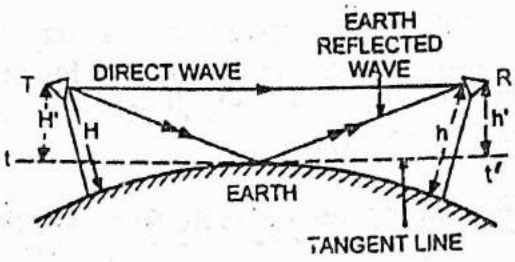


Fig. 13.4

The curvature of the earth also causes the earth reflected wave to be diverging rather than plane wave. Consequenctly, the earth reflected wave at the receiver is weaker than if the reflection were from a flat surface. This effect is negligible when the angle of incidence of the incident wave at the earth's surface is moderately large, it increases as the angle of incidence becomes less.

Another consequence of the earth's curvature is that if the receiving antenna R of moderate height is at a considerably greater distance from the transmitting antenna T, then it is below the radio horizon and hence will not be able to receive the wave by either means *i.e.*, either direct wave or earth reflected wave. This situation has been shown in Fig. 13.5.

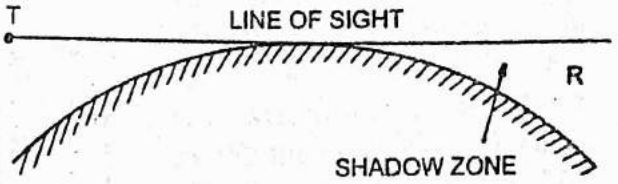


Fig. 13.5

There are, however, certain methods by which transmitted energy can be received beyond the direct line of sight, known as *shadow zone*. These methods have been discussed in the next section.

13.6. Atmospheric Effects on Space Wave Propagation

The atmosphere, through which the space wave propagation takes place, influences the transmission to a significant degree. Some distance beyond the direct line of sight can be covered due to refraction and reflection of radio waves by the earth's atmosphere (troposphere) and diffraction by the roughness of the earths surface.

Near the surface of the earth, the dielectric constant and hence the refractive index of the atmosphere usually decreases linearly with height. It is called the *standard atmosphere*. If, any how, the moisture content of the air at the earth surface is very high but decreases sharply with increasing height, then the refractive index of the air decreases with height much more rapidly than normal. It is called the *region of the duct*. It causes the bending of radio waves moving through the atmosphere towards the earth's surface in accordance with Snell's law. Hence the rays initially moving parallel to the earth's surface get refracted in the atmosphere and follow a path having slightly downward curvature. This downward curvature of the path permits the direct ray to reach slightly beyond the horizon (shadow zone). The radio waves may then be reflected back from the earth to follow a series of hops, as shown in Fig. 13.6. The phenomenon is known as *duct propagation*. Such ducts are generally found over the ocean and less frequently over lands.

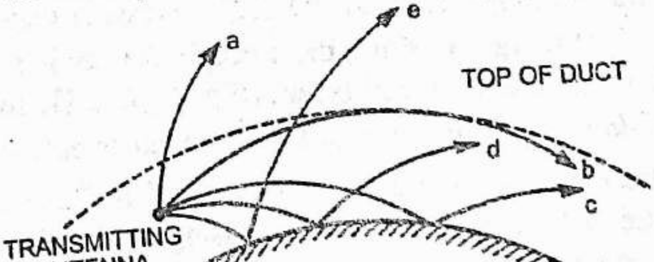


Fig. 13.6

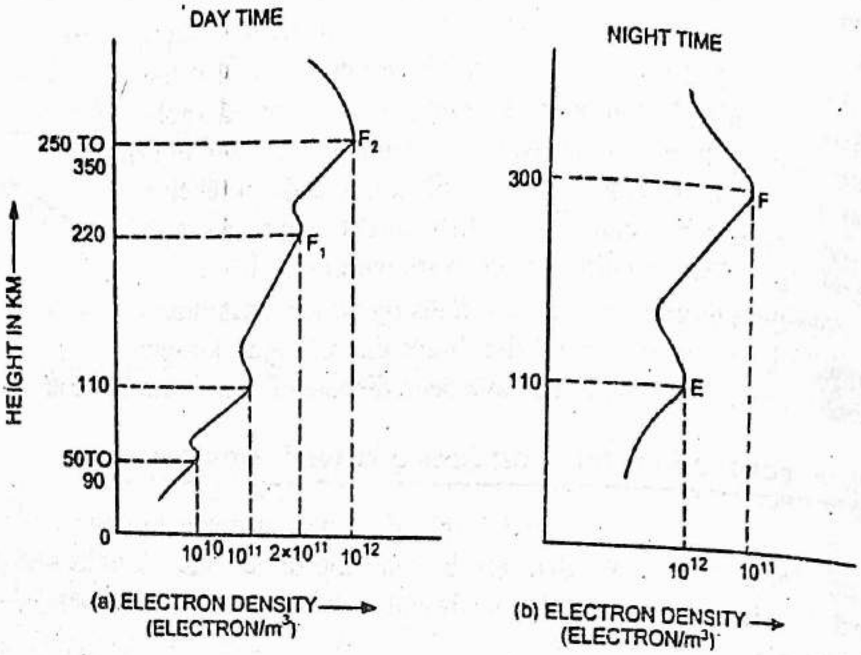
In addition to the refraction of waves occuring in the troposphere, there also occurs reflection at places of abrupt changes in the dielectric constant. The effect of tropospheric reflection is to extend the range of propagation by hundreds of kilometers. In addition to it, the inhomogeneties in the atmosphere give rise to a scattering of radio signals. By using highly directional high gain antennas and large transmitted power, *tropospheric scatter propagation* can be used to establish a link well beyond the radio horizon.

Space wave propagation is affected by atmospheric conditions, more seriously at frequencies in the microwave region. Rain, cloud and fog result in serious attenuation of waves at frequencies above about 30 GHz. Hail has a very little effect except at frequencies above about 100 GHz, while snow has a negligible effect at all frequencies.

13.7. Ionosphere and Its Stratification.

The upper part of the atmosphere continually absorbs large quantities of radiant energy from the sun and consequently gets heated and ionized *i.e.*, electrons are detatched from atmospheric gas atoms. This *portion of the upper atmosphere, where the ionisation is considerable is known as the ionosphere.* It is located between about 50 to 350 km. above the earth's surface.

Due to the difference in the chemical composition and physical properties of the atmosphere at different heights and also due to the presence of different kinds of radiations (*e.g.* X-rays, cosmic rays, ultraviolet radiations etc.) the ionisation in the ionosphere is not uniform throughout but is stratified *i.e.*, in the form of layers. The region or stratum of the ionosphere where the electron density is a maximum and falls off on either side is called a layer. The different ionospheic layers that generally exist during day and night are shown in Figs. 13.7 (a) and 13.7 (b) respectively.



(a) ELECTRON DENSITY→ (ELECTRON/m^3)

(b) ELECTRON DENSITY→ (ELECTRON/m^3)

Fig. 13.7

During day time, the electron density is found to be maximum at four different heights *i.e.*, there are four different ionized layers (Fig. 13.7a). These layers are designated as D, E, F_1 and F_2 respectively. The *D-layer* is the lowermost layer and exists only during the day time. It extends from 50 to 90 km. This layer has the maximum electron density of 10^{10} electron/m^3 and an average thickness of 10 km. It offers a high attenuation to radio waves passing through it. It can reflect low frequency waves and absorb medium frequency waves to a certain extent. It is believed to be formed due to the ionization of O_2 at its first ionization potential.

Above D layer, lies the *E-layer,* when is also called the Kennely-Heavside layer after its discoverer. Its height and maximum electron density are 100 km. and 10^{11} electron/m^3 respectively, which remain fairly constant throughout the day and from season to season. Its average thickness is 25 km. and is governed by the amount of ultraviolet light received from the sun. Nicolet assigned its existence to the pre-ionisation of O_2 in transition region.

The F_1 *layer* exists at a height of 220 km. and its electron density is twice that of the *E–layers.* Its thickness is about 20 km. It undergoes inappreciable diurnal (with time of the day) and seasonal (with month of the year) variations. Bhar attributed its formation to the ionization of N_2 at its second ionization potential but the accepted view is that it is produced by ionization of atomic oxygen.

F_2 layer is the uppermost layer. Its height ranges from 250 to 350 km. with a maximum electron density of 10^{12} electron/m^3. It is subjected to considerable diurnal and seasonal variations. It is the most important reflecting medium for high frequency radio waves. Bohr attributed its production to the ionization of oxygen at its first ionization potential but later works support its production by a sort of bifurcation of F_1 layer.

At night time, Fig. 13.7 (b) *the D-layer is practically absent.* Since D-layer merely attenuates the radio waves passing through it, its absorption during night vanishes and hence the *reception of the sky wave is better at night.* The maximum electron density of *E*-layer decays uniformly with time and at night decreases by a factor of 10 from it day time value. Further F_1 and F_2 layers merge at night to form a single layer, called the *F*-layer. In the *F*-layer ionized density is very high but decreases by a factor of 10 from the value of day time F_2 layer. The air density in *F* layer is very low and hence molecules ionized during day time have little chance to deionize themselves by recombination due to large mean free path and less rate of collision.

In addition to these layers, an anomalous ionization, termed *sporadic E,* often appears in the *E*-region anywhere in the height range from 90 to 130 km. It often appears in the form of clouds of various sizes. It may be observed both at day and night but its occurence is quite unpredictable and its cause is still uncertain.

13.8. Effect on Ionosphere on the Propagation of Radio Waves-Eccles Larmor Theory

The presence of charged particles (*i.e.* electrons and ions) of ionosphere affect the path of the radiowaves. The theory was given by Eccles and Larmor by assuming that *there is no effect of earth's magnetic field and the collisions between* electrons and gas atoms.

When the radiowaves pass through the ionosphere, the electric field of waves exert a force on the existing electrons and ions. Due to large inertia the ions do not vibrate with the field, but the electrons are found to execute oscillatory motion at the wave frequency.

Consider a region of ionosphere containing *N* free electrons per unit volume. Let the electric field of radio wave passing through this region be represented as

$$E = E_0 \sin \omega t$$

where E_0 is the amplitude of electric field vector and $f = \frac{\omega}{2\pi}$ its frequency.

Due to this field, the electric force on electron.

$$\mathbf{F} = -e\,\mathbf{E} = -e\,\mathbf{E}_0 \sin \omega t$$

where $(-e)$ is charge on electron; e being positive quantity $= 1{\cdot}6 \times 10^{-19}$ C.

If v is the instantaneous velocity of electron, then from Newton's II law,

$$\mathbf{F} = m\,\mathbf{a} = m\,\frac{d\,\mathbf{v}}{dt}, \text{ we have}$$

$$m\,\frac{d\,\mathbf{v}}{dt} = -e\,\mathbf{E}_0 \sin \omega t$$

Integrating we get

$$\mathbf{v} = \frac{e\,\mathbf{E}_0}{m\,\omega} \cos \omega t + \mathbf{A}$$

A being constant of integration.

At $t = 0$, $\mathbf{v} = 0$, this gives $A = 0$.

$$\therefore \qquad v = \frac{e\,E_0}{m\,\omega} \cos \omega t \qquad ...(1)$$

The motion of electrons causes *current density,* given by

$$J_e = -Ne\,v$$

$$= -\frac{Ne^2\,E_0}{m\,\omega} \cos \omega t \qquad ...(2a)$$

$$= \frac{Ne^2\,E_0}{m\,\omega} \sin\left(\omega t - \frac{\pi}{2}\right) \qquad ...(2b)$$

Obviously the current lags behind the electric field by an angle $\frac{\pi}{2}$. This shows that the *current induced is inductive.*

But the *displacement current density* is $J_d = \frac{\partial D}{\partial t}$ where $D = \varepsilon E$

For *free space* $\varepsilon = \varepsilon_0$

$$\therefore \qquad J_d = \frac{\partial(\varepsilon_0 E)}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}(E_0 \sin \omega t)$$

$$= \omega\,\varepsilon_0\,E_0 \cos \omega t$$

$\therefore$ *Total current density,* $J = J_e + J_d$ ...(3)

$$= -\frac{N e^2\,E_0}{m\,\omega} \cos \omega t + \omega\,\varepsilon_0\,E_0 \cos \omega t$$

i.e.

$$J = \omega\left[\varepsilon_0 - \frac{Ne^2}{m\,\omega^2}\right] E_0 \cos \omega t \qquad ...(4)$$

This equation shows that *effective permittivity of ionised space* containing free electrons is reduced to

$$\varepsilon = \varepsilon_0 - \frac{Ne^2}{m\,\omega^2} = \varepsilon_0\left[1 - \frac{Ne^2}{m\,\omega^2\,\varepsilon_0}\right]. \qquad ...(5)$$

The *equivalent dielectric constant,*

$$K = \frac{\varepsilon}{\varepsilon_0} = 1 - \frac{Ne^2}{m\,\omega^2\,\varepsilon_0} \qquad ...(6)$$

Substituting

$$e = 1{\cdot}6 \times 10^{-19}\ \text{C},\ m = 9{\cdot}1 \times 10^{-31}\ \text{kg}.$$

and

$$\varepsilon_0 = 8{\cdot}86 \times 10^{-12}\ F/m, \text{ and } \omega = 2\pi f$$

$$K = \left(1 - \frac{81\,N}{f^2}\right) \qquad ...(7)$$

Thus the effeect of *existence of electrons in the ionosphere is to reduce the dielectric constant of space.* From equations (7), it is clear that the equivalent dielectric constant K depends upon the electron density n and the frequency of passing wave.

The refractive index of the ionised layers is

$$\mu = \sqrt{K} = \sqrt{\left(1 - \frac{81\,N}{f^2}\right)} \qquad ...(8)$$

This shows that *the real value of refractive index of ionosphere is always less than unity.*

The *phase velocity of wave in ionosphere* is therefore, given by

$$v_p = \frac{c}{\mu} = \frac{c}{\sqrt{\left(1 - \frac{81\,N}{f^2}\right)}} \qquad ...(9)$$

Thus shows that the *phase velocity of wave in ionosphere is greater than the speed of light.*

The Actual Mechanism of Propagation of sky Waves through Ionoshere.

The bending of radio waves in the ionosphere may be explained on the basis of variation of refractive index of ionosphere with ion density. Let a radio wave be incident on the ionosphere from the underlying unionised medium. The refractive index of ionosphere is less than 1, therefore the ionosphere acts as a rarer medium. Therefore the ray moves away from the normal.

If ϕ_i is the angle of incidence and ϕ_r that of refraction at the surface of E-layer, then from Shell's law

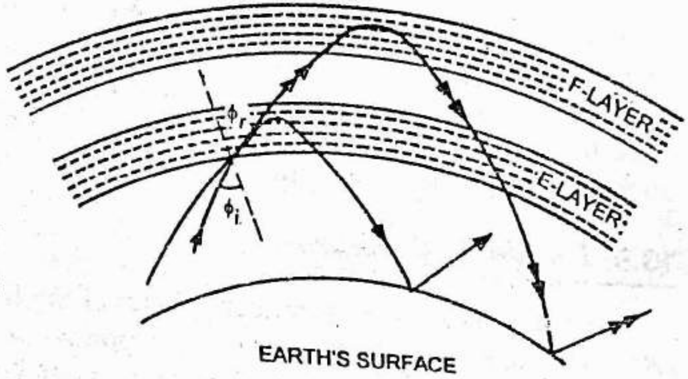


Fig. 13.10

$$\mu = \frac{\sin \phi_i}{\sin \phi_r} \quad i.e. \quad \frac{\sin \phi_i}{\sin \phi_r} = \sqrt{\left(1 - \frac{81\,N}{f^2}\right)} \qquad ...(10)$$

As the wave penetrates E layer, the ion density increases, therefore the angle of refraction ϕ_r increases continuously, hence the wave follows a curved path. At a particular stage when ϕ_r is 90°, the wave travels parallel to earth's surface. If the refractive index at this point is denoted by μ_1, and ion-density by N_1, then (10) gives

$$\mu_1 = \frac{\sin \phi_i}{\sin 90^\circ} = \sqrt{\left(1 - \frac{81\,N_1}{f^2}\right)} \quad i.e. \quad \sin \phi_i = \sqrt{\left(1 - \frac{81\,N_1}{f^2}\right)} \qquad ...(11)$$

Equation (11) shows that if the ion-density is greater than or equal the value N_1, the wave will return to the earth. If the maximum electron density is less than N_1, the wave will pass through the layer. If the maximum ion density for a given layer is fixed at $N_{\max}$ ions/m^3, the critical limiting frequency f_c for the wave to return to earth depends on angle of incidence to the layer. *For vertically directed beam,* $\phi_i = 0$ therefore (11) gives

$$0 = \sqrt{\left[\left\{1 - \frac{81\,N_{\max}}{f_c^2}\right\}\right]}$$

or

$$f_c = \sqrt{(81\,N_{\max})} = 9\sqrt{(N_{\max})}\,. \qquad ...(12)$$

For any other angle of incidence the critical frequency is higher and may be evaluated as follows from (11)

$$\sin \phi_i = \sqrt{\left(1 - \frac{81\,N_1}{f^2}\right)}$$

This gives $f = 9\sqrt{N_1}\,\sec \phi_i$

∴ *Critical frequency*

$$f_c = 9\,(\sqrt{N_{\max}})\,\sec \phi_i \qquad ...(13)$$

This shows that the highest frequency that can be reflected from the earth depends on the angle of incidence. The maximum frequency which will undergo reflection for a particular distance of transmission is known as the *maximum usuable frequency* for the distance.

Thus it follows that the degree of bending depends upon (*i*) angle of incidence (*ii*) ion-density and (*iii*) frequency (or wavelength) of incident radio wave. For constant values of angle of incidence and ion-density, the bending increases with decrease of frequency. The radiowaves incident normally have greater probability of penetration of the layer to outerspace. Even so all waves of wavelengths greater than 400 m (or frequency less than 0.75 MHz), even close to normal incidence are reflected from to earth's surface. Shorter waves than these incident near the normal, will cross to *E* layer to reach *F* layer. The *F*-layers reflect back to earth the waves of wavelength greater than 10 *m*, though the longer wavelengths than this will cross both layers if incident below a certain critical angle. Ultra high frequency (*U.H.F.*) waves of wavelengths below 10 m will not, except in unusual ionospherical circumstances, be reflected back to earth. Thus the ultra-short waves are only suitable for direct radio communication over optical distances upto 50 km in practice.

13.9. The Veritcal Height.

An actual sky wave of frequency *f* travelling from transmitter *T* to receiver *R* will follows a path such as *TBR*. The apex *B* of the path corresponds to a point in the ionosphere where the electron density is sufficient to reflect the wave. Consider now the triangular path *TAR*, shown in fig. 13.11, by extending the straight line portions of the actual path *TBR*. The time required for a signal to travel with the group velocity over the actual path *TBR* is the same as that taken by a signal to travel over the equivalent path *TAR* with a velocity equal to that of light. At the lowre edge of the ionosphere, the horizontal component of the velocity is $c \sin \phi_0$ where c is the velocity of light *i.e.*, the velocity of the wave before entering the ionosphere. At some point in the ionosphere, the horizontal component of the group velocity $c\mu$ is $c\mu \sin \phi$. But from Snell's law

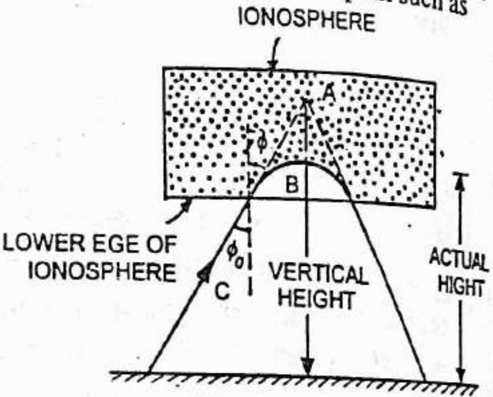


Fig. 13.11

$$\mu = \frac{\sin \phi_0}{\sin \phi}$$

or

$$\sin \phi_0 = \mu \sin \phi$$

or

$$c \sin \phi_0 = c\mu \sin \phi$$

which means that the horizontal component of velocity remains unchanged or is independent of the refractive index μ. Hence the wave would arrive at a point *B* of the actual path at the same instant as a signal travelling at the velocity of light would arrive at a point *A* of the equivalent triangular path. Thus the ionosphere could be replaced by a perfect reflector located at the height of point *A* without affecting the conditions of propagation. The height of point *A* is known as the *virtual height* and is *defined as the height from which a wave sent up at an angle appears to be reflected.*

Virtual height varies with the angle of incidence and is an important characteristic of any given layer. It is always greater than the actual (or true) height.

13.10. Skip Distance and Maximum Usable Frequency (MUF).

Sky wave is the primary means of short wave communication around the world. It is possible due to the fact that ionospheric layers act as a 'radio mirror' to bound the sky waves back to earth. The maximum frequency of the radio waves which is returned from a layer at vertical incidence is known as the *critical* frequency f_c. When the frequency of a wave is greater than the critical frequency of the layer ($f > f_c$), the effect of the ionosphere on the path of propagation depends upon the angle of incidence ϕ_0.

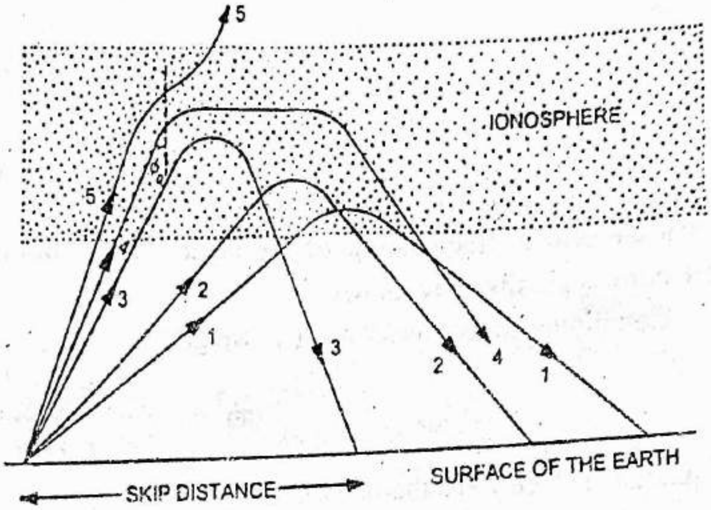


Fig. 13.12

Let us consider Fig. 13.12 where ray path have been drawn for different angle of incidence ϕ_0. With a large value of angle of incidence ϕ_0 (ray 1) the condition $\mu_m = \sin \phi_0$ is satisfied when refractive index μ_m is only slightly less than unity. Under this condition the wave returns to earth after slight penetration of the ionized layers. As the angle of incidence ϕ_0 is decreased more and more (rays 2, 3 nd 4), refractive index required for returning the wave earthward progressively decreases. Consequenctly the penetration of the ray into the layers increases. With further decrease in the value of ϕ_0, the maximum electron density will not be able to reflect the wave and the wave penetrates through the layers, as shown for Ray 5 in Fig. 13.12, and is never returned to ground.

It may be seen from Fig. 13.12 that as the angle of incidence ϕ_0 decreases progressively, the distance from the transmitter to the point, where a ray returns to the ground, first decreases (rays 1, 2) attains a minimum value (ray 3) and then increases again (ray 4). Ray 4 seems to be a victim of indecision and travels a longer horizontal distance in the layer compared with the other rays due to some complicated reason. With still smaller angle of incidence the ray penetrates the layer (ray 5).

The minimum distance from the transmitter at which a sky wave of given frequency ($f > f_c$) is returned to earth by the ionosphere is called the skip distance. The skip distance varies directly with (i) the frequency of transmission, and (ii) the height of the layer. The skip distance is larger for a wave of higher frequency. For a given frequency, the skip distance becomes greater as the reflecting layer moves up. Since the height of an ionospheric layer is a minimum at noon, the *skip distance will also be minimum at noon* and increases during pre-noon or after-noon hours.

The frequency which makes a given distance between the transmitter and receiver equal to the skip distance for that frequency is known as the maximum usable frequency (M.U.F.). An ionospheric layer will, therefore, be unable to return a sky wave if the frequency of transmission exceeds the maximum usable frequency for that receiving point. The M.U.F. is also maximum at noon and gradually decreases on either side of the noon hour.

Expression for skip distance and M.U.F.

To find the skip distance and the M.U.F., let us assume that the earth is flat and let the average height of the ionospheric layer above the ground be h. Then we see from Fig. 13.13, that skip distance

$$D_{skip} = 2h \tan \phi_0 \qquad \text{...(1)}$$

But from eq. (11) section 13.8

$$\sin^2 \phi_0 = 1 - \frac{81\, N_m}{f^2}$$

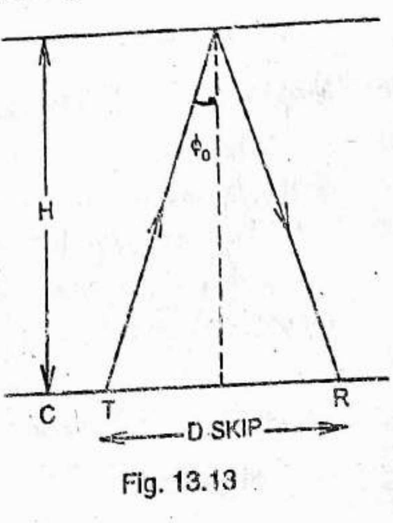


Fig. 13.13

$$= 1 - \frac{f_c^2}{f^2} \qquad ...(2)$$

[on using eq. (12), section 13.8]

$$\therefore \qquad \cos^2\phi_0 = 1 - \sin^2\phi_0 = f_c^2/f^2 \qquad ...(3)$$

where f_c is the critical frequency for the layer under consideration. Therefore from eq. (2)

$$f = f_c \sec\phi_0 \qquad ...(4)$$

This equation is known as the *Secant Law* and gives the maximum usable frequency between two stations when the critical frequency f_c of the layer and the limiting minimum angle of incidence ϕ_0, for which reflection is possible, are known.

Combining equations (2) and (3), we get

$$\tan^2\phi_0 = \frac{1-(f_c^2/f^2)}{(f_c^2/f^2)} = \frac{f^2}{f_c^2} - 1$$

Equation (1) then gives the *skip distance*

$$D_{skip} = 2h\tan\phi_0 = 2h\sqrt{\left\{\left[\left(\frac{f^2}{f_c^2}\right) - 1\right]\right\}} \qquad ...(5)$$

The circular area of radius D_{skip} around the transmmiter in which no sky wave of given frequency is possible, is known as the *skip zone.*

If the distance between the transmitter and receiver is given, say D, then to find the maximum usable frequency for this distance, we use equation (5), When the frequency of transmission is equal to M.U.F. ($f_{\max}$) then the distance D corresponds to D_{skip}. Thus from eq. (5), we have

$$D^2 = 4h^2\left(\frac{f_{\max}^2}{f_c^2} - 1\right)$$

or

$$\frac{f_{\max}^2}{f_c^2} = \frac{D^2}{4h^2} + 1$$

or

$$f_{\max}^2 = f_c^2\left(\frac{D^2}{4h^2} + 1\right)$$

or, *Maximum usable frequency*

$$f_{\max} = f_c\sqrt{\left[\left[1 + \frac{D^2}{4h^2}\right]\right]} \qquad ...(6)$$

Maximum usable frequency for large distance.

When the distances are large (7500 km.), the earth can no longer be assumed as flat and earth curvature should be taken into account.

The maximum distances between transmitter T_r and receiver R_e will be obtained when the rays leave the earth surface tangentially and reach the receiver in the same way. If ϕ_0 is the angle of incidence at the ionosphere, then from Fig. 13.14, we have

$$\sin\phi_0 = \frac{R}{R+h} \qquad ...(7)$$

where R is the radius of the earth and h is the height of the layer above the ground.

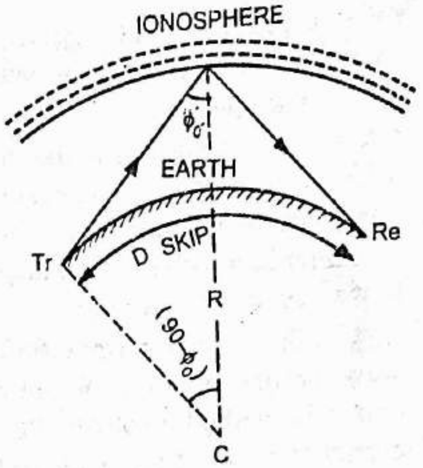


Fig. 13.14

From eq. (2), we get for the maximum usable frequency f_{max}

$$\sin^2 \phi_0 = 1 - \frac{f_c^2}{f_{max}^2}$$

or $$\frac{f_c^2}{f_{max}^2} = 1 - \sin^2 \phi_0 = \cos^2 \phi_0 \qquad \text{...(8)}$$

or $$\frac{f_c}{f_{max}} = \cos \phi_0$$

or $$f_{max} = f_c \sec \phi_0 \qquad \text{...(9)}$$

Equation (9) is known as the *secant law* and gives the maximum usable frequency between two stations taking into account the effect of earth's curvature.

Now from eq. (8), we get

$$f_{max} = \frac{f_c}{\sqrt{(1-\sin^2 \phi_0)}} = \frac{f_c}{\sqrt{[1-(R/R+h)^2]}} \qquad \text{using (7)}$$

The *skip distance*, as seen from fig. 13.14, is given by

$$\text{Angle } (90-\phi_0) = \frac{\text{Arc}\left(\dfrac{D_{skip}}{2}\right)}{\text{radius } (R)}$$

or $$D_{skip} = 2R\,(90-\phi_0)$$
$$= 2R \sin (90-\phi_0)$$

because when angle ϕ_0 is large, $(90-\phi_0)$ is small.

Hence $$D_{skip} = 2R \cos \phi_0$$
$$= 2R \sqrt{\left\{\left[1-\left(\frac{R}{R+h}\right)^2\right]\right\}} \qquad \text{...(10)}$$

13.11. Fading

Fading of radio waves is the name given to undesirable changes in the intensity or loudness of the waves at the receiving point. It is the fluctuation in signal intensity coming after reflection from the ionosphere. It may be due to several reasons including the change in electron density at or below the reflection point, roughness and movement of the reflecting layer and also due to ionization blobs moving within the layer itself below the reflection point. Some times the ionospheric signals may suddenly disappear altogether. The phenomenon is then called *Dellinger effect* and is caused by sudden burst of ionizing radiations from the sun. It causes the ionization density in the *D*-layer to rise suddently, causing heavy absorption of radio waves of all frequencies. This phenomenon occurs simultaneously throughout the part of the globe illuminated by the sun but obviously does not occur at night.

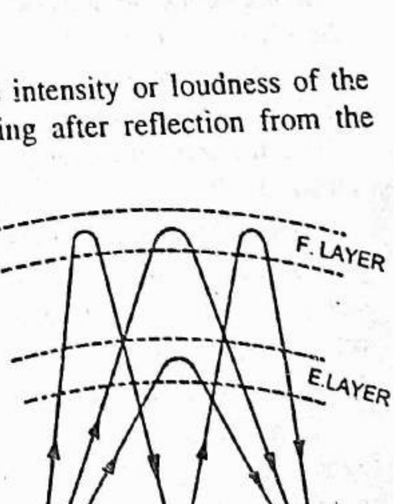


Fig. 13.15

One of the most serious types of fading, known as *interference fading*, is produced by the arrival of energy at the receiver via a number of paths as shown in Fig. 13.15. The received signal strength then depends upon the relative phases of the various arriving components and thus may produce alternate reinforcements and cancellations resulting in a fluctuation in the signal strength.

If the signal contains various frequency components, such as a modulated wave contains different frequency components in the side bands, then the different frequency components of the wave will combine differently causing a frequency distortion. Since the variation in the length of the ray path through the ionosphere is a function of frequency, the carrier wave and the side band components will fade differently in a manner in which each is independent of the other and will produce a serious signal distortion. The effect is knwon as *selective fading*.

Fading is very harmful for any communication system and may be considered as one of the vagaries of wave propagation via ionosphere Several methods are, therefore, used for minimizing fading. The most common method employed in radio receivers is the use of *automatic volume control (AVC)*, in which the carrier strength of the signal received is used to control the volume of the receiver so that the output is kept fairly constant inspite of the variations in the signal strength, However, sometimes the fading is so severe that the signal strength drops below the noise level and no amount of amplification in the receiver will make the signal audible. *A.V.C.* is then not the solution of fading. One of the best method of reducing fading is known as *diversity reception*. In telegraph transmitters, sometimes frequency diversity systems are employed which are based on the fact that signals differing in frequencies as small as 500 c/s fade independently.

13.12. Effect of Earth's Magnetic Field on Radio Wave Propagation :

Appleton Hartree Formula.

When the electromagnetic waves pass through the ionized medium in the presence of external magnetic field, the electrons experience a force *Hev* in a direction perpendicular to both, the velocity v of electron and the earth's magnetic field H. Consequently the path of the electron is changed. Since the instantaneous velocity of electrons is inversely proportional to frequency, at high radio frequencies, the instantaneous velocity of the electrons is small and hence the magnetic field makes the vibrating electrons to follow elliptical paths as shown in Fig. 13.16. Here the arrow E dipicts the direction of motion of the electron and the dot represents the direction of the magnetic field which is perpendicular to E *i.e.*, perpendicular to the plane of the paper. The force on electron is exerted perpendicular to both and causes the electron to follow an elliptical path. As the frequency is decreased, the instantanteous velocity of electrons increases and hence the effect of force also increases. It increases the ratio of minor axis to major axis until at a frequency of about 1400 kc/s, the electron follows a spiral path. This frequency is called the *gyro frequency*. At still lower frequencies, the electron vibrates in loops, as shown in Fig. 13.16.

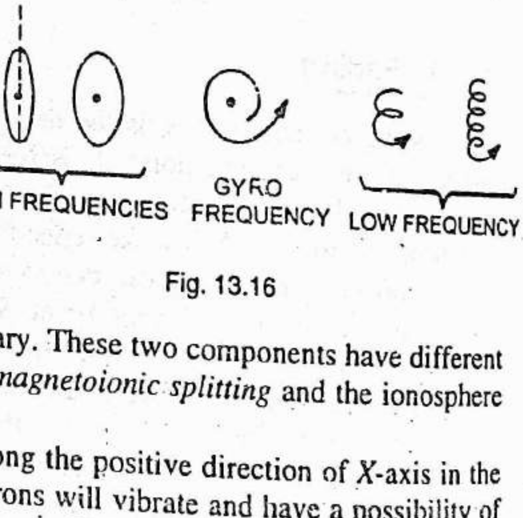


Fig. 13.16

The presence of magnetic field not only changes the polarization of the wave from plane, to elliptical but also causes it to be split into two components-ordinary and extraordinary. These two components have different velocities and travel on different paths. This action is known as *magnetoionic splitting* and the ionosphere thus acts as a doubly refracting crystal.

Consider a plane polarized electromagnetic wave moving along the positive direction of X-axis in the ionosphere. Under the effect of the electrostatic field E, the electrons will vibrate and have a possibility of collision with other electrons and gaseous atoms. Consequently a type of frictional force is developed which can be represented by the product of the coefficient of friction β and the velocity v of electrons. Due to earth's magnetic field H acting in Z–X plane, a force $e\,(\vec{v} \times \vec{H})$ acts an the electrons which is perpendicular to both $\vec{v}$ and $\vec{H}$. If e is the electronic charge and m is its mass, then the equations of motions of electrons in the field of advancing electromagnetic wave is S.I. system are given by

$$m\frac{d^2x}{dt^2} = e\,E_x - \beta\frac{dx}{dt} + e\,(v \times H)_x \qquad \text{...(1)}$$

$$m\frac{d^2y}{dt^2} = e\,E_y - \beta\frac{dy}{dt} + e\,(v \times H)_y \qquad \text{...(2)}$$

$$m\frac{d^2z}{dt^2} = e\,E_z - \beta\frac{dz}{dt} + e\,(v \times H)_z \qquad \text{...(3)}$$

But $(\vec{v} \times \vec{H}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ H_x & 0 & H_z \end{vmatrix}$

so that

$$(\vec{v} \times \vec{H})_x = H_z\frac{dy}{dt}$$

$$(\vec{v} \times \vec{H})_y = H_x\frac{dz}{dt} - H_z\frac{dx}{dt}$$

$$(\vec{v} \times \vec{H})_z = -\,H_x\frac{dy}{dt}$$

Considering eq. (1) and substituting the value of $(\vec{v} \times \vec{H})$ in it, we get

$$m\frac{d^2x}{dt^2} = e\,E_x - \beta\frac{dx}{dt} + e\,H_z\frac{dy}{dt} \qquad \text{...(4)}$$

If ω is the angular frequency of the wave, then the displacement of electron from the mean position along the axis of X at any instant is given by

$$x = A\,e^{j\omega t} \qquad \text{where } j = \sqrt{-1}$$

Then

$$\frac{dx}{dt} = j\,\omega A\,e^{j\omega t} = j\,\omega\,x$$

and

$$\frac{d^2x}{dt^2} = -\,\omega^2 x.$$

Similar expressions can be obtained for Y and Z components. Then equation (4), on these substitutions; gives,

$$-m\omega^2 x = e\,E_x - j\omega x\,\beta + e\,H_z\,j\omega y \qquad \text{...(5)}$$

If we represent by N the number of electrons per unit volume and divide eq. (5) by $\frac{Ne^2}{\varepsilon_0}$, than we get

$$-\frac{\varepsilon_0\,m\omega^2 x}{Ne^2} = \frac{\varepsilon_0\,eE_x}{Ne^2} - \frac{\varepsilon_0\,j\omega x\beta}{Ne^2} + e\,\frac{\varepsilon_0\,H_z j\omega\,y}{Ne^2}$$

Putting

$$-\frac{\varepsilon_0\,m\omega^2}{Ne^2} = A, \qquad \frac{\varepsilon_0\,\omega\beta}{Ne^2} = B$$

and

$$\frac{\varepsilon_0\,\omega\,e\,H_z}{Ne^2} = C_z,$$

we get,

$$\left.\begin{aligned} Ax &= \frac{\varepsilon_0 E_x}{Ne} - jBx + jy\,C_z \\ \text{or} \qquad E_x &= (A+jB)\frac{Ne\,x}{\varepsilon_0}\frac{-j\,C_z\,y\,Ne}{\varepsilon_0} \\ \text{similarly} \qquad E_y &= (A+jB)\frac{Ne\,y}{\varepsilon_0} + (j\,C_z\,x - jC_x\,z)\frac{Ne}{\varepsilon_0} \\ \text{and} \qquad E_z &= (A+jB)\frac{Ne}{\varepsilon_0}z + j\,C_x\,y\frac{Ne}{\varepsilon_0} \end{aligned}\right\} \quad \text{...(6)}$$

Now, due to the presence of electrons, ionosphere behaves like a conducting medium for which Maxwell's electromagnetic equation can be written as

$$\text{Curl } H = \varepsilon\frac{\partial \vec{E}}{\partial t} + \vec{J}.$$

Since the dielectric constant of the medium (air) $K = 1$ so $\varepsilon = \varepsilon_0$ and the current density $J = \sigma E = Nev$, therefore

$$\text{Curl } H = \varepsilon_0\frac{\partial E}{\partial t} + Nev \qquad \text{...(7)}$$

But

$$\text{Curl } H = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Since the wave is travelling in positive X-direction, therefore derivatives with respect to y and z are zero *i.e.*,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \qquad \text{But } \frac{\partial}{\partial x} \neq 0$$

Therefore, the components of curl H in different directions are given by

$$[\text{curl } H]_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = 0$$

$$[\text{curl } H]_y = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = -\frac{\partial H_z}{\partial x}$$

$$[\text{curl } H]_z = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \frac{\partial H_y}{\partial x}$$

Accordingly eq. (7) gives

$$\left.\begin{aligned} \varepsilon_0\left[\frac{\partial E_x}{\partial t} + Ne\frac{dx}{dt}\right] &= 0 \\ \varepsilon_0\left[\frac{\partial E_y}{\partial t} + Ne\frac{dy}{dt}\right] &= -\frac{\partial H_z}{\partial x} \\ \text{and} \qquad \varepsilon_0\left[\frac{\partial E_z}{\partial t} + Ne\frac{dz}{dt}\right] &= \frac{\partial H_y}{\partial x} \end{aligned}\right\} \quad \text{...(8)}$$

Now the equation of a plane wave travelling along the axis of X can be taken as

$$\psi = \psi_0\, e^{j\omega\left(t - \frac{x}{v}\right)} = \psi_0\, e^{j\omega\left(t - \frac{x\sqrt{k'}}{c}\right)}$$

where the velocity $v = \frac{c}{\sqrt{k'}}$, k' being the effective dielectric constant of the ionosphere modified by the presence of electrons and is not equal to 1 like that of air.

Differentiating the above equation w.r.t t and x, we get

$$\frac{\partial \psi}{\partial t} = j\omega\, \psi_0\, e^{j\omega\left(t - \frac{x\sqrt{K'}}{c}\right)} = j\omega\psi$$

$$\frac{\partial \psi}{\partial x} = -j\omega\, \frac{\sqrt{K'}}{c} \cdot \psi_0\, e^{j\omega\left(t - \frac{x\sqrt{K'}}{c}\right)} = -\frac{j\omega\sqrt{K'}}{c}\,\psi$$

Thus the operator $\frac{\partial}{\partial t} = j\omega$ and $\frac{\partial}{\partial x} = -j\omega\,\frac{\sqrt{K'}}{c}$, substituted in eq. (8) gives

$$\left.\begin{aligned} \varepsilon_0 j\omega E_x + Ne\, j\omega x &= 0 \\ \varepsilon_0 j\omega E_y + Ne\, j\omega y &= j\omega\,\frac{\sqrt{K'}}{c}\,H_z \\ \text{and} \qquad \varepsilon_0 j\omega E_z + Ne\, j\omega z &= -\,j\omega\,\frac{\sqrt{K'}}{c}\,H_y \end{aligned}\right\} \qquad \text{...(9)}$$

Above equations, on rearranging, become

$$\left.\begin{aligned} E_x &= -\frac{Nex}{\varepsilon_0} \\ E_y &= -\frac{Ney}{\varepsilon_0} + \sqrt{K'}\,H_z \\ \text{and} \qquad E_z &= -\frac{Nez}{\varepsilon_0} - \sqrt{K'}\,H_y \end{aligned}\right\} \qquad \text{...(10)}$$

Now the other field equation of Maxwell is

$$\text{curl } E = -\frac{\partial H}{\partial t} \qquad \text{...(11)}$$

But

$$\text{curl } E = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

But in our case $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$ because the plane wave is travelling in positive X-direction and the function will vary only w.r.t. x. Hence

$$\therefore \qquad [\text{curl } E]_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

$$[\text{curl } E]_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial E_z}{\partial x}$$

$$\text{and} \qquad [\text{curl } E]_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$$

Hence eqn. (11) gives

$$\left.\begin{aligned} -\frac{\partial H_x}{dt} &= 0 \\ -\frac{\partial H_y}{dt} &= -\frac{\partial E_z}{\partial x} \\ \text{and} \qquad -\frac{\partial H_z}{dt} &= \frac{\partial E_y}{\partial x} \end{aligned}\right\} \qquad \text{...(12)}$$

But for a plane wave we have proved that operator, $\frac{\partial}{\partial t} = j\omega$ and $\frac{\partial}{\partial x} = -j\omega\,\frac{\sqrt{k'}}{c}$. Therefore, eq. (12) becomes

$$\left.\begin{aligned} -j\omega H_x &= 0, && \text{or} \quad H_x = 0 \\ -j\omega H_y &= j\omega\,\frac{\sqrt{k'}}{c}\,E_z && \text{or} \quad H_y = -\frac{\sqrt{k'}}{c}\,E_z \\ \text{and} \qquad -j\omega H_z &= j\omega\,\frac{\sqrt{k'}}{c}\,E_y && \text{or} \quad H_z = \frac{\sqrt{k'}}{c}\,E_y \end{aligned}\right\} \qquad \text{...(13)}$$

Substituting these values in eq. (10), we get

$$\left.\begin{aligned} E_x &= -\frac{Nex}{\varepsilon_0} \\ E_y &= -\frac{Ney}{\varepsilon_0} + k'\,E_y \\ \text{or} \quad (k'-1)E_y &= \frac{Ney}{\varepsilon_0} \\ \text{and} \qquad E_z &= -\frac{Nez}{\varepsilon_0} + k'\,E_z \\ \text{or} \quad (k'-1)\,E_z &= \frac{Nez}{\varepsilon_0} \end{aligned}\right\} \qquad \text{...(14)}$$

Putting the value of E_x from eq. (14) into 6(a), we have

$$-\frac{Nex}{\varepsilon_0} = (A+jB)\,\frac{Nex}{\varepsilon_0} - \frac{jC_z\,Ney}{\varepsilon_0}$$

or

$$\frac{Nex}{\varepsilon_0} = \frac{jC_z\,Ney}{(1+A+jB)} \qquad \text{...(15)}$$

Also, substituting the value of E_y from (14) in eq. 6(b), we get

$$\frac{Ney}{k'-1} = (A+jB)\,Ney + jC_z\,Nex - jC_x\,Nez$$

$$= (A+jB)\,Ney + jC_z \cdot \frac{jC_z\,Ney}{(1+A+jB)} - jC_x\,Nez \qquad \text{[using eq. 15]}$$

or

$$Ney\left[\frac{1}{k'-1} - (A+jB) + \frac{C_z^2}{(1+A+jB)}\right] = -\,jC_x\,Nez \qquad \text{...(16)}$$

Now putting the value of E_z from (14) in 6(c), we get

$$\frac{Nez}{k'-1} = (A+jB)\,Nez + jC_x\,Ney$$

or

$$Nez\left[\frac{1}{k'-1} - (A+jB)\right] = jC_x\,Ney \qquad \text{...(17)}$$

Multiplying eq. (16) and (17), we get

$$\left[\frac{1}{k'-1} - (A+jB)\right]\left[\frac{1}{k'-1} - (A+jB) + \frac{C_z^2}{1+A+jB}\right] = C_x^2$$

or

$$\left[\frac{1}{K'-1} - (A+jB)\right]^2 + \frac{C_z^2}{1+A+jB}\left[\frac{1}{K'-1} - (A+jB)\right] - C_x^2 = 0 \qquad \text{...(18)}$$

Equation (18) is quadratic in $\left[\dfrac{1}{k'-1}-(A+jB)\right]$ and gives on being solved

$$\left[\frac{1}{k'-1}-(A+jB)\right]=\frac{-\dfrac{C_z^2}{1+A+jB}\pm\sqrt{\left[\left(\dfrac{C_z^4}{(1+A+jB)^2}+4C_x^2\right)\right]}}{2}$$

or

$$\frac{1}{k'-1}=(A+jB)-\frac{C_z^2}{2\,(1+A+jB)}\pm\sqrt{\left[\left(\frac{C_z^4}{4\,(1+A+jB)^2}+C_x^2\right)\right]}$$

or

$$k'=1+\frac{1}{(A+jB)-\dfrac{C_z^2}{2\,(1+A+jB)}\pm\sqrt{\left[\left(\dfrac{C_z^4}{4\,(1+A+jB)^2}+C_x^2\right)\right]}} \qquad \text{...(19)}$$

This is *Appleton Hartree formula* for the dielectric constant of the ionosphere in the presence of magnetic field. Evidently the effective dielectric constant k' has got two values corresponding to the positive and negative signs involved in the formula. Thus the incident wave is split up into two waves—ordinary and extraordinary and the *ionosphere acts as a doubly refracting crystal.*

Splitting of radio waves in ionosphere.

The splitting of radio waves, when they travel upwards in the ionoshphere, can be explained on the basis of Appleton Hartree formula given by eq. (19). If μ is the refractive index of the ionosphere, then putting $k'=\mu^2$ in eq. (19), we get

$$\mu^2=k'=1+\frac{1}{(A+jB)-\dfrac{C_z^2}{2\,(1+A+jB)}\pm\sqrt{\left[\left(\dfrac{C_z^4}{4\,(1+A+jB)^2}+C_x^2\right)\right]}}$$

In order to simplify matters, if we neglect collisions, then co-efficient of friction $\beta=0$ and hence

$$B^2=\frac{\varepsilon_0\omega\beta}{Ne^2}=0$$

Then,

$$\mu^2=1+\frac{1}{A-\dfrac{C_z^2}{2\,(1+A)}\pm\sqrt{\left[\left(\dfrac{C_z^4}{4\,(1+A)^2}+C_x^2\right)\right]}} \qquad \text{...(20)}$$

If the wave is sent upwards normally, then the angle of incidence on the surface of ionosphere is zero and hence according to Snell's law, the refractive index of the ionosphere should be reduced to zero to get the wave reflected. Hence putting $\mu=0$ in eq. (20) for the condition of vertical reflection, we have

$$A-\frac{C_z^2}{2\,(1+A)}\pm\sqrt{\left[\left(\frac{C_z^4}{4\,(1+A)^2}+C_x^2\right)\right]}=-1$$

or

$$(1+A)^2-\frac{C_z^2}{2}=\pm\sqrt{\left[\left(\frac{C_z^4}{4}+C_x^2\,(1+A^2)\right)\right]}$$

Squaring both the sides, we get

$$(1+A)^4 + \frac{C_z^4}{4} - 2(1+A)^2 \cdot \frac{C_z^2}{2} = \frac{C_z^4}{4} + C_x^2 (1+A)^2$$

or

$$(1+A)^2 [(1+A)^2 - (C_z^2 - C_x^2)] = 0$$

Therefore, either

$$(1+A)^2 = 0$$

i.e.,

$$1 + A = 0 \qquad \ldots(21)$$

or

$$(1+A^2) - (C_z^2 - C_x^2) = 0$$

or

$$(1+A)^2 = (C_z^2 - C_x^2) = r^2$$

∴

$$(1+A) = \pm r \qquad \ldots(22)$$

Thus a radio wave is splitted up into two parts :

(*i*) *Ordinary ray,* which follows the condition of reflection $1+A=0$ given by eq. (21), which is the same as that for the case of no magnetic field.

(*ii*) *Extra ordinary ray,* which follows the condition of reflection $1+A = \pm r$ given by eq. (22). Its condition of reflection is affected by the presence of magnetic field.

13.13. Elementary Idea of Micro-Wave and Optical Communications : The Communication Satellite

The micro waves are high frequency electromagnetic waves; their wavelength being of the order of 1 cm – 1 m. The optical waves have still lower wavelengths.

In the preceding section we have seen that the radio waves of wavelength less than 10 m penetrate the ionosphere and can not be reflected back. Therefore the communication of micro-waves and optical waves in not possible by sky waves. Their communication may be possible by space waves. *A micro-wave transmission, therefore, works between two points at a short distance of direct sight only.* To send the signals at large distances the *repeater stations* should be formed which may receive and transmit the information to the next relay terminal. The signal can thus be made to bend around the earth on land but it causes difficulty where repeater towers can not be installed such as at wide oceans and unconquerred mountains etc.

The difficulty is solved by **communication satellite,** which has exactly the same motion as the earth, so that such a satellite appears stationary from the earth.

A satellite is a celestial body, natural or artificial which revolves around the earth in circular orbit.

The necessary centripetal force for circular motion is provided by the gravitational attraction of earth. If m is the mass of satellite, M_e the mass of earth and r the radius of satellite's orbit around the earth, then we have

$$m\,\omega^2 r = \frac{G M_e m}{r^2}$$

ω being angular speed of satellite

$$i.e.\ \omega = \sqrt{\left(\frac{G M_e}{r^3}\right)}$$

If T is the period of revolution of the satellite relative to earth, then $T = \frac{2\pi}{\omega}$

$$\therefore \qquad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{(G\,M_e/r^3)}} = 2\pi\sqrt{\left(\frac{r^3}{G\,M_e}\right)} \qquad \text{...(1)}$$

The period of rotation of earth about its own axis is 24 hours.

If we put $T = 24$ hours $= 24 \times 60 \times 60$ sec.

Mass of earth, $M_e = 5{\cdot}98 \times 10^{24}$ kg, $G = 6{\cdot}67 \times 10^{-11}$ mks units, then from (1), we get

$$24 \times 60 \times 60 = 2\pi\sqrt{\left(\frac{r^3}{6{\cdot}67 \times 10^{11} \times 5{\cdot}98 \times 10^{24}}\right)}$$

radius of satellite orbit, $r = 4{\cdot}2 \times 10^7$ m

Height of such a satellite from earth's surface

$$h = r - R_e \quad \text{(radius of earth)}$$

$$= 4{\cdot}2 \times 10^7\,\text{m} - 6{\cdot}4 \times 10^6\,\text{m} = (4{\cdot}2 - 0{\cdot}64) \times 10^7\,\text{m}$$

$$= 3{\cdot}56 \times 10^7\,\text{m} = \mathbf{35600\ km}$$

The speed of such a satellite will be

$$v = \frac{2\pi r}{T} = \frac{2\pi \times 4{\cdot}2 \times 10^7}{24 \times 60 \times 60} = 3{\cdot}2 \times 10^3\,\text{m/s} = \mathbf{3{\cdot}2\ km/s}$$

If such a satellite orbits the earth from west to east with a period of 24 hours, as does the earth, then the satellite will appear stationary to an observer fixed on the earth.

As the angular speed of such a satellite is synchronous with that of earth, it is called the *synchronous satellite* or *communication satellite*.

The communication satellite provides a link betwen distant points on the earth. Radio or T.V. signals are directed to the communication satellite from one point on earth and the reflected signals are received at another distant points on earth. The schematic operation of communication satellite is shown in fig. 13.17.

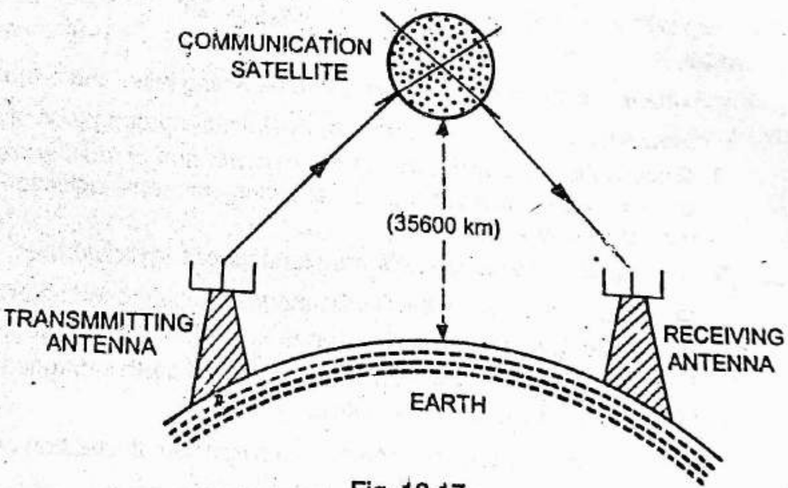


Fig. 13.17

The first communication satellite in synchronous orbit called syncom I was launched in 1963.

The important communication satellite called *Intelsat* (The International Tele Communication Satellite Organisation) was established in orbit in 1975. Most of the nations are members of this organisation and use it for communications of important TV and radio programmes. A powerful communication satellite. *ATS–F* was launched by America. A multipurpose satellite of India *INSAT–2D* is being used as communication satellite. It has made possible the relay of T.V. programmes all over the country. The life span of a communication satellite is limited mainly by solar cells which get damaged by radiations from space and also by the fuel needed for *course correction and spinning* of the **satellite.**

Questions and Problems

1. Discuss the ground, space and sky wave propagations with particular reference to frequency and range.
2. Show how the ionized layers of the upper atmosphere help in the propagation of radio waves.
3. Give the theory of propagation of electromagnetic waves through ionized atmosphere and derive the expression for the critical frequency. Describe a method for determining the equivalent height of the regions of ionosphere responsible for reflecting the electromagnetic waves.
4. Derive expressions for the effective dielectric constant and loss for the electromagnetic wave propagating in ionosphere by considering the effect of collisions only and neglecting earth's magnetic field.
5. How does the presence of electrons affect the dielectric constant of the ionosphere ? Derive and explain the necessary formulae.
6. Explain what is meant by the terms : ground waves, sky waves and space waves in relation to the propagation of radio waves.

   Discuss the reflection and refraction of sky waves by the ionosphere and show that the ionosphere behaves like a medium of refractive index

$$\mu = \sqrt{\left(1 - \frac{81\,N}{f^2}\right)}$$

   where N is the electron density of the layer and f is the frequency in kc/s.

7. Describe the nature of ionosphere and is stratification. Discuss its role in the propagation of radio waves.

   Show how the skip distance for a given frequency is related to the ionospheric height and the critical frequency.
8. Explain the terms : skip distance, maximum usable frequency and critical frequency.

   Show that on the flat earth, the skip distance for a given frequency is given by

$$D_{skip} = 2h\sqrt{\left[\left(\frac{f^2}{f_c^2} - 1\right)\right]}$$

   where h is the average height of the reflecting layer and f_c the critical frequency of the layer.

   How does the curvature of the earth effect the propagation of radio waves ?
9. Discuss theoritically the method of transmission of radio waves through the ionosphere at oblique incidence and show how the maximum usable frequency for communication between two distant stations can be calculated for thin ionized layers.
10. (a) Explain how are the different ionospheric layers formed. What is their importance ?
    (b) Explain how does the earths magnetic field affect the propagation of radio waves.
11. Obtain the Appleton-Hartree magneto-ionic formula. Discuss its significance.
12. Show that the ionosphere under the action of earth's magnetic field behaves like a birefringent crystal.
13. Give your comments on the following :
    (i) The fading of radio signals results from the fluctuation of power at the transmitter and can not be eliminated
    (ii) The range of medium wave transmission increases during night time in comparison to that during the day.
    (iii) Radio waves of very high frequencies do not propagate as sky waves and surface waves.
    (iv) The critical frequency of the propagating electromagnetic wave is independent of the electron density of the ionosphere and is an exclusive function of the earths magnetic field.
14. Write short notes on :
    (i) Nature of ionosphere and is stratification.
    (ii) Method for investigating the ionosphere.
    (iii) Skip zone.
    (iv) Maximum usable frequency.
    (v) Microwave communication.

14

Plasma Physics

14.1. PLASMA—An Introduction

The three states of matter–solid, liquid and gas are well known. In recent years the word *plasma* has been used to represent the *fourth state of matter*. As we know that a gas is formed of particles which move and collide with one another randomly. When the temperature of a gas is increased above a certain value, the kinetic energy of the gas particles increases to such an extent that the gas atoms ionise *i.e.* they may be splitted into electrons and positive ions as a result of their mutual collisions. Generally a *plasma can be regarded to be an ionised gas with a sufficiently high density so that there exist finite forces of interaction between gas particles.* The interaction forces are basically the coulomb electrostatic forces between charged particles of plasma. It is obvious that a plasma is a *mixture of free electrons, positive ions and neutral particles with a high density.*

Plasma is the most widely spread in the universe. The ionosphere consisting of ionised layers form plasma envelope surrounding the earth's atmosphere. The so called great radiation belts (the Van-Allen belts) outside the ionosphere contain plasma formations. The sun and stars may be regarded as the lumps of hot plasma. In laboratory plasma occurs in discharge of electricity through gases.

In this chapter we shall study the various processes occurring in the plasma.

14.2. Conditions for Plasma Existence

In order to produce plasma it is necessary to free electrons that are normally bound into atoms. The binding energies for many elements run from several eV to a few tens of eV. These energies may be added to an atomic system to cause ionisation by a number of means. A convenient method is to impart the energy by collision events of one sort or another.

On addition of the ionisation energy to a fraction of atoms of a neutral gas, *an ionised gas* is formed. If sufficient total energy can be added, the gas may be completely ionised with no neutral particles existing.

One of the most powerful influences upon plasma behaviour is the electromagnetic interaction of the charged particles. Even a partially ionised gas can be affected by externally applied electric and magnetic fields and can conduct electric current, the ions and electrons serving as the charge carriers. Since the coulombs electric fields of charged particles fall off only as the inverse of square of distance, the electrostatic forces are of long range and can act upon a considerable number of other particles. this interaction of substantial number of particles causes them to react in a collective manner the other forces and the collective behaviour constitutes the prime characteristic of a plasma.

Quasi-neutrality of plasma :

Debye shilding distance : The tendency of plasma is to remain quasi neutral (*i.e.* to balance positive and negative space charges in each macroscopic volume element). Whatever be the method of producing a plasma, it is evident that initially the number of positive and negative charges must be the same. Since the velocities of electrons and ions are different; the velocity of electrons being more than that of positive ions; it is likely to happen that more electrons leave the plasma boundary (e.g. discharge tube) than positive ions. As a result of the loss of electrons there will be excess charge of opposite sign and this will tend to equalise the current of electrons and positive ions to reduce the concentration of particles of opposite sign.

Suppose the excess positive charge within the plasma at any instant is represented by Q. Let us assume a large number of neighbouring particles so that the electric field can be taken as the continuous function of distance. Then the electric field satisfies the Poisson equation

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad ...(1)$$

where ρ is the charge density.

If ϕ is the potential, then

$$\mathbf{E} = -\operatorname{grad} \phi \qquad ...(2)$$

If N_i and N_e are the number of ions and electrons per unit volume respectively, then

$$\rho = (N_i e - N_e e) \qquad ...(3)$$

e being electronic charge. In view of (2) and (3), equation (1) becomes

$$\operatorname{div}(-\operatorname{grad} \phi) = \frac{N_i e - N_e e}{\varepsilon_0}$$

or

$$\nabla^2 \phi = -\frac{(N_i - N_e)\, e}{\varepsilon_0} \qquad ...(4)$$

If N_0 is the mean density of charged particles in the *quasineutral plasma,* then from Boltzmann distribution

$$N_i = N_0 e^{-e\phi/kT} \quad \text{and} \quad N_e = N_0 e^{+e\phi/kT}$$

where k is Boltzmann's constant and T is the *absolute temperature of the plasma.* Substituting (5) in (4), we get

$$\nabla^2 \phi = -\frac{N_0 e}{\varepsilon_0}\left[e^{-e\phi/kT} - e^{e\phi/kT}\right]$$

$$= \frac{2N_0 e}{\varepsilon_0} \sinh\left(\frac{e\phi}{kT}\right)$$

Assuming $\frac{e\phi}{kT} \ll 1$, we write $\sinh \frac{e\phi}{kT} \approx \frac{e\phi}{kT}$ so that

$$\nabla^2 \phi = \frac{2N_0 e}{\varepsilon_0} \cdot \frac{e\phi}{kT}$$

$$= \frac{2N_0 e^2 \phi}{\varepsilon_0 kT} \qquad ...(6)$$

If charge distribution is spherically symmetric, it is independent of angular coordinates (θ and ϕ' of spherical coordinate system), hence $\nabla^2 \phi$ may be expressed as

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$$

and so equation (6) takes the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{2N_0 e^2\phi}{\varepsilon_0 kT} \quad ...(7)$$

This differential equation has infinite number of solutions. The one of our interest is that which satisfies the boundary conditions,

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad \text{at } r \to 0 = \phi_0 \text{ (say)}$$

and $\phi \to 0$ at $r \to$ infinity (∞)

Then the solution may be expressed as

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}e^{-r/D} = \phi_0 e^{-r/D} \quad ...(8)$$

where

$$D = \left(\frac{\varepsilon_0 kT}{2N_0 e^2}\right)^{1/2} \quad ...(9)$$

where D is called the **Debye length.**

Accordingly at $r = D$, the effect of charge becomes $\frac{1}{e}$ times of its original value. From equation (9) it therefore follows that if the linear dimensions of the region occupied by the ionised gas of given electron concentration is much greater than D, then inside the region $N_e \neq N_i$ and under this condition a considerable departure of N_e from N_i will give rise to an electric field which will eject particles of opposite sign. This mechanism will automatically maintain the neutrality and will cease to be effective when $r << D$. This concept automatically gives the definition of plasma.

Langmuir, the pioneer of plasma physics, defined the plasma as an ionised gas in which the Debye length is small compared to the linear dimension occupied by the gas, *i.e.*

for existence of plasma $\qquad D \ll L \qquad$ ...(10)

The assumption of a large number of particles is a further criterion for the existence of a plasma. The large number of particles provides the smooth decrease in the field required for the deduction of Debye length D. The requirement is insured if the number of particles within a sphere of radius equal to Debye length D must be greater than 1, *i.e.*

$$\left(\tfrac{4}{3}\pi D^3\right)\cdot n \gg 1 \quad ...(11)$$

where n is number of particles per unit volume.

The third, but not independent, criterion for the existence of a plasma is its *quasi-neutrality, i.e.* $n_i \approx n_e$.

Plasma Oscillations. The above conditions hold for the steady-state plasma. However, these steady state conditions are not sufficient to insure that collective plasma motions can occur. One of the most important aspect of the collective motions is the bulk oscillations of the plasma electrons with respect to the ions. The plasma electrons may be expected to oscillate about the much more massive ions under the collective restoring force provided by the ion-electron coulomb-attraction.

Fig. 14.1 represents the plasma containing electrons and ions such that the electrons and ions have been displaced slightly but collectively relative to each other through a distance $\delta \ll L$, when L is plasma thickness. The electric field E produced in the plasma interior by the displaced charge is

$$E = \frac{\sigma}{\varepsilon_0}$$

where $\sigma = ne\delta =$ charge per unit area of surface.

The force per unit area acting on the electron in the plasma interior is the charge density multiplied by electric field strength E, *i.e.*

$$f = (Lne)\,E = \frac{Lne\;ne\delta}{\varepsilon_0} = \frac{n^2e^2 L\delta}{\varepsilon_0}.$$

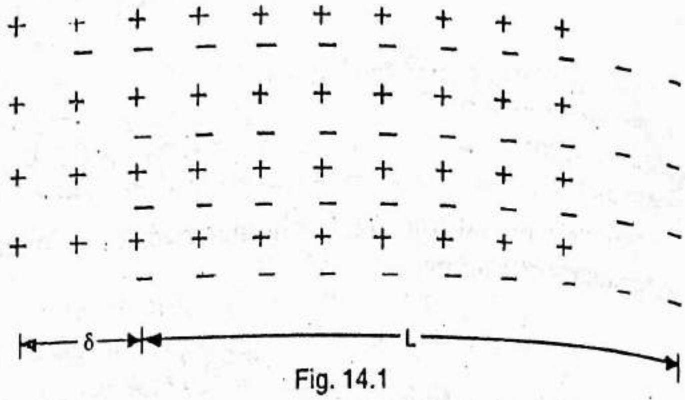


Fig. 14.1

If m is mass of electron, the mass per unit area under the influence of force $= mnL$.

∴ From Newton's II law, the equation of motion is

$$(Lmn)\frac{d^2\delta}{dt^2} = -\frac{n^2e^2L\delta}{\varepsilon_0}$$

i.e. acceleration

$$\frac{d^2\delta}{dt^2} = -\frac{ne^2\delta}{\varepsilon_0 m}.$$

This represents the differential equation of oscillatory motion of angular frequency ω given by

$$\omega^2 = \frac{ne^2}{\varepsilon_0 m} \quad i.e. \quad \omega = \left(\frac{ne^2}{\varepsilon_0 m}\right)^{1/2}$$

∴ The frequency ν_p called the ***plasma frequency*** is given by

$$\nu_p = \frac{\omega}{2\pi} = \frac{1}{2\pi}\left(\frac{ne^2}{\varepsilon_0 m}\right)^{1/2} = 9 \times 10^3\, n^{1/2} \text{ cycles/sec.}$$

The collective oscillations are damped due to collisions between electrons and ions. In order for the oscillations to be slightly damped, the collision frequency ν_c $\nu_c \ll \nu_p$

These are the conditions for a collection of charge particles to exhibit plasma like behaviour. These conditions may be summarised as :

(i) The Debye length must be small compared to the linear extent of the plasma *i.e.* $D \ll L$.

(ii) There must be a large number of electrons in the Debye sphere *i.e.*

$$\frac{4}{3}\pi D^2 n_e \gg 1$$

(iii) The plasma is approximately neutral *i.e.* $n_i \approx n_e$.

(iv) The plasma oscillations are slightly damped *i.e.* plasma frequency $\nu_p \ll$ collision frequency ν_c.

14.3. Occurence of Plasma

There are two categories of plasmas as regards their occurence.

1. Laboratory Plasma. 2. Space Plasma.

1. Laboratory Plasma. The gas discharge is the most popular technique for producing plasma under laboratory conditions. The gas discharge is used for exciting most gas lasers ; gas discharge as well as radiation sources and lamps in a wide wave-length range are the basis of plasmatrons. The density of charged particles (electrons and ions) in the gas discharge is much lower than the density of the neutral particles. This plasma is called *weakly ionised* or *low temperature plasma* since the mean energy of electrons or ions in it is much lower than the ionisation potential for the gas particles.

Another type of laboratory plasma is the *thermo-nuclear plasma* also called the hot *plasma.* This is the plasma which is created for the course of the controlled thermo-nuclear reaction. In this plasma the mean energy of ions is much higher than the ionisation potential of the gas particles. The most practicable thermonuclear fusion reaction involves deuterium or tritium nuclei. To make possible this reaction the deuterium or tritium ions must be abie to enter the reaction during the time of plasma confinement.

In the existing laboratory installations, this condition is satisfied when the ion-temperature exceeds $\approx 10^8$ degrees and when the *product of density (N) of charged particles and the plasma confinement time* (τ) *exceeds* 10^{22} s–m^{-3}. For the plasma containing deuterium or tritium nuclei $N.\tau$ exceeds 10^{20}. s-m^{-3}, this criterion is called the *Lewson criterion.* If this condition is met, the self-maintaining thermo-nuclear reaction can occur provided the heat released by the reaction maintains the temperature of the particles required for the continuation of reaction.

The energy radiated by the sun or stars also is produced in thermonuclear fusion reaction of hydrogen nuclei or protons. This reaction is less efficient than the reaction with deutrium or tritium ions but since the reaction volume in stars is very large, the temperaure proves to be $\approx 10^7$ *K*, which is considerably lower than that of laboratory thermonuclear plasma for a more favourable reaction.

Space-Plasma. The space plasma involves the plasma in atmospheres of the earth and the planets, in the star including the sun and in the outer space. Each of above plasma type exists under rather special conditions. For example the plasma of earth's atmosphere (hundreds of kilometers above earth's surface) is created by the *ultraviolet solar radiation.* This plasma's parameters are affected by certain processes occuring on solar surface and to parameters of atmosphere itself. The plasma of stars differ greatly in their parameters. For example in the interior of sun where the thermonuclear fusion occurs, the temperature is 16×10^6 K. The surface region of sun about 1000 km thick which radiates most of the solar energy called the photosphere is about 6000 K and its distance from the centre of sun is $\approx 7 \times 10^5$ km. The region closer to sun's centre is called the convective region since energy is transferred there by convection of the solar plasma in strong magnetic fields. Such movement of the solar plasma gives rise to the granular structure of the photosphere, development of sun's spots and other non-stationary phenomena on sun's surface.

Over the sun's surface there is a low-density high-temperature (order 10^6 K) region called *solar corona* ; the powerful source of ultraviolet radiation. The sun emits plasma from its surface. The stationary proton flux emitted by solar corona is referred as *solar wind.* The plasma flow from the sun's surface gives rise to interplanetary plasma. The electrons in this plasma are captured by the earth's magnetic field and gives rise to radiation belts around the earth. The high energy electrons and protons produce auroras-like effects in earth's atmosphere.

The interplanetary and inter-steller plasma are of low density and low temperature. The energy exchange between the particles of the plasma proceeds in a peculiar way via the interaction with electromagnetic radiation field.

14.4. Charged Particle in Electric and Magnetic Fields

1. Charged particle in uniform constant Electric Field.

Let **E** be the strength of uniform constant electric field and let it act on a charged particle of charge *q* and mass *m*. The electrical force $\mathbf{F}_e$ on the particle is given by

$$\mathbf{F}_e = q\mathbf{E}.$$

If **a** is the acceleration produced in the particle due to this force, then

$$\mathbf{F}_e = m\mathbf{a} \quad \text{or} \quad m\mathbf{a} = q\mathbf{E},$$

i.e. acceleration
$$\mathbf{a} = \frac{q\mathbf{E}}{m}, \qquad \text{...(1)}$$

i.e., the acceleration a of the charged particle in the uniform constant electric field is constant as shown in fig. 14·2.

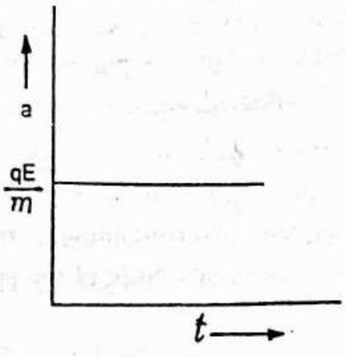


Fig. 14.2

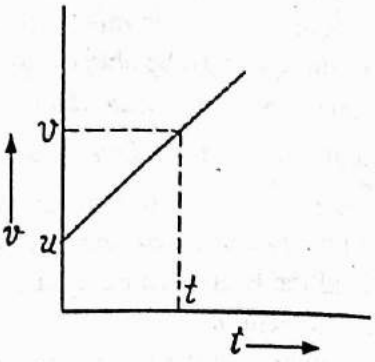


Fig. 14.3

If **u** is the initial velocity ($t = 0$), **v** the velocity of the charged particle after time t, then; we have from (1)

$$(\mathbf{a} =) \frac{d\mathbf{v}}{dt} = \frac{q\mathbf{E}}{m}$$

Integrating with respect to t, the velocity $\mathbf{v} = \frac{q\mathbf{E}}{m} t + C_1$, C_1 being constant of integration.

At $t = 0$, $\mathbf{v} = \mathbf{u}$, this gives $C_1 = \mathbf{u}$.

$$\therefore \qquad \mathbf{v} = \mathbf{u} + \frac{q\mathbf{E}}{m} t. \qquad \text{...(2)}$$

That is the velocity of the charged particle is the linear function of time t as shown in the fig. 14·3. If **r** is the position vector of particle, then velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, therefore (2) gives

$$\frac{d\mathbf{r}}{dt} = \mathbf{u} + \frac{q\mathbf{E}}{m} t.$$

Integrating with respect to time again

$$\mathbf{r} = \mathbf{u}t + \frac{q\mathbf{E}}{m} \frac{t^2}{2} + C_2,$$

C_2 being constant of integration.

Applying initial condition that $\mathbf{r} = \mathbf{r}_0$ at $t = 0$, we get

$$C_2 = \mathbf{r}_0.$$

$\therefore$ We have
$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2} \frac{q\mathbf{E}}{m} t^2 \qquad \text{...(3a)}$$

or *displacement*,
$$\mathbf{s} = \mathbf{r} - \mathbf{r}_0 = \mathbf{u}t + \frac{1}{2} \frac{q\mathbf{E}}{m} t^2. \qquad \text{...(3b)}$$

This indicates that the displacement varies in a quadratic manner with time as shown in fig. 14·4.

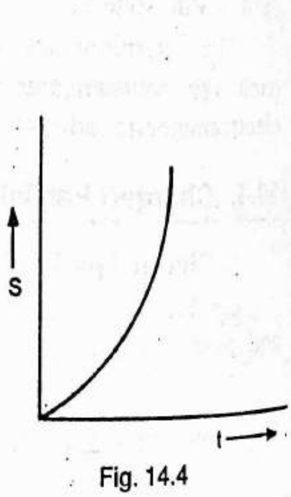


Fig. 14.4

Now let us consider two particular cases :

Case (i) Field applied along the direction of motion of the particle (Longitudinal Electric Field) : Let electric field **E** act along *X*-axis along the initial direction of motion of the charged particle, the initial velocity *u* being along *X*-axis *i.e.* at $t = 0$, we have :

$$u_x = u$$

As the electric force $\mathbf{F} = q\mathbf{E}$ acts along *X* -axis only, the acceleration will also be in *X*-direction only. That is

$$\left.\begin{aligned} a_x &\left\{ = \frac{qE_x}{m} \right\} = \frac{qE}{m} & \quad ...(a) \\ a_y &\left\{ = \frac{qE_y}{m} \right\} = 0 & \quad ...(b) \\ a_z &\left\{ = \frac{qE_z}{m} \right\} = 0 & \quad ...(c) \end{aligned}\right\} \qquad ...(4)$$

Integrating equation (4), we get

$$v_x = \frac{qE}{m} t + C_1 .$$

$$v_y = C_2 \text{ and } v_z = C_3 .$$

C_1, C_2, C_3 being constants of integration.

Now applying initial condition that at $t = 0$

$$(v_x)_{t=0} = u_x = u$$
$$(v_y)_{t=0} = u_y = 0$$
$$(v_z)_{t=0} = u_z = 0.$$

we get $C_1 = u, C_2 = 0$ and $C_3 = 0$.

$$\therefore \qquad v_x = u + \frac{qE}{m} t , v_y = 0 \text{ and } v_z = 0. \qquad ...(5)$$

Integrating again, we get the displacement components

$$x = ut + \frac{qE}{m}\frac{t^2}{2} + C_1' , y = C_2' , z = C_3'.$$

Applying initial condition that the particle is at the origin at $t = 0$ (*i.e.* $x = y = z = 0$ at $t = 0$), we have

$$C_1' = C_2' = C_3' = 0.$$

∴ The displacement of the particle is given by

$$x = ut + \frac{1}{2}\frac{qE}{m} t^2 , y = z = 0. \qquad ...(6)$$

Case (ii). Field applied perpendicular to the direction of motion of the particles (Transverse Electric Field).

Let the charged particle be moving with an initial uniform velocity *u* along *X*-direction and a uniform constant electric field *E* be applied along *Y*-axis. Then at time $t = 0$.

$$(v_x)_{t=0} = u_x = u , (v_y)_{t=0} = (v_z)_{t=0} = 0.$$

Also $E_x = E_z = 0$ and $E_y = E$.

The components of force acting on the particle are given by

$$F_x = qE_x = 0, F_y = qE_y = qE \text{ and } F_z = qE_z = 0.$$

∴ Components of acceleration vector **a** are

$$a_x = 0 , a_y = \frac{qE}{m} , a_z = 0. \qquad ...(7)$$

If v_x, v_y and v_z are cartesian components of velocity **v** , then

$$a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt} \text{ and } a_z = \frac{dv_z}{dt}.$$

$$\therefore \qquad \frac{dv_x}{dt} = 0, \frac{dv_y}{dt} = \frac{qE}{m} \text{ and } \frac{dv_z}{dt} = 0.$$

Integrating, we get velocity components

$$v_x = C_1; v_y = \frac{qE}{m}t + C_2, v_z = C_3.$$

Applying initial conditions $v_x = u, v_y = v_z = 0$ at $t = 0$, we get

$$C_1 = u, C_2 = C_3 = 0.$$

$$\therefore \qquad v_x = u, v_y = \frac{qE}{m}t, v_z = 0. \qquad \text{...(8)}$$

If particle is initially at origin and (x, y, z) are components of displacement vector $\mathbf{r}$, then we have

$$\frac{dx}{dt} = u, \frac{dy}{dt} = \frac{qE}{m}t \text{ and } \frac{dz}{dt} = 0.$$

Integrating with respect to time, we get

$$x = ut + C_1',$$

$$y = \frac{qE}{m}\cdot\frac{t^2}{2} + C_2'$$

and

$$z = C_3'.$$

Applying boundary conditions that $x = y = z = 0$ at $t = 0$, we get

$$C_1' = C_2' = C_3' = 0.$$

$$\therefore \qquad x = ut, y = \frac{1}{2}\frac{qE}{m}t^2 \text{ and } z = 0. \qquad \text{...(9)}$$

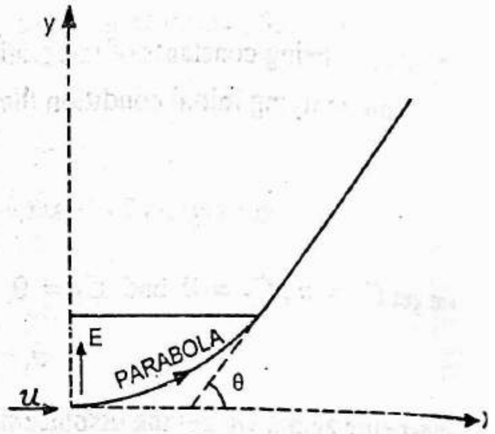


Fig. 14.5

Eliminating t between first two of above equations, we get

$$y = \frac{1}{2}\frac{qE}{m}\frac{x^2}{u^2}$$

i.e.,

$$x^2 = \frac{2mu^2}{qE}y.$$

This equation represents a parabola. Thus *the path traversed by a charged particle moving with a finite initial velocity and subjected to a constant transverse electric field is a parabola* as shown in fig. 14·5.

2. Charged Particle in Homogeneous Magnetic Field

Let a particle of mass m, charge q be shot into a magnetic field of induction $\mathbf{B}$ with velocity $\mathbf{v}$; then the *magnetic force on charged particle*

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$$

The magnitude of this force is $qvB\sin\theta$ (θ being angle between $\mathbf{v}$ and $\mathbf{B}$) and the direction of force is normal to the plane containing $\mathbf{v}$ and $\mathbf{B}$.

Case (i) : ***When the particle enters parallel to the magnetic field*** *i.e.* $\theta = 0°$; then magnetic force $F_m = qvB\sin 0° = 0$

Therefore *the path of the charged particle is the undeflected straight line.*

Case (ii) : ***When the charged particle enters perpendicular to the magnetic field i.e.***

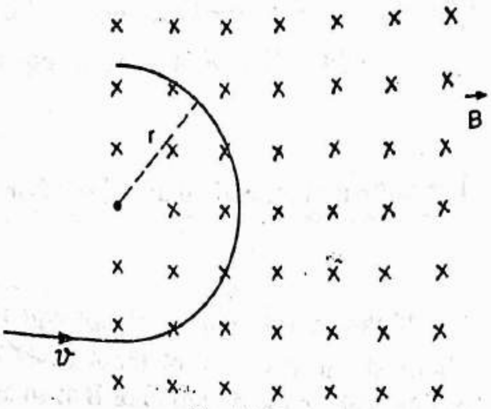


Fig. 14.6

$$\theta = 90^\circ$$

then the magnetic force acting on the particle, $F_m = qvB$ is always perpendicular to **v** as well as **B** and hence causes the particle to travel along a circular path,

The magnetic force qvB provides the necessary centripetal force for circular path *i.e.*,

$$qvB = \frac{mv^2}{r}$$

Radius of circular orbit $r = \frac{mv}{qB}$.

This radius is called *Larmor radius (or cyclotron radius or gyro radius) of the orbit.*

The *angular frequency* ω_c is given by

$$\omega_c = \frac{v}{r} = \frac{qB}{m}.$$

the *linear frequency*

$$\nu_c = \frac{\omega_c}{2\pi} = \frac{qB}{2\pi m}.$$

This is known as *cyclotron (or gyro) frequency* and is independent of speed v of particle and radius r of circular path.

Case (ii). When the charged particle enters the magnetic field at any angle θ such that $\theta \neq 0^\circ$ or $\pi/2$; then the velocity v may be resolved into two components :

$v_2 = v\cos\theta$ parallel to the field **B** and $v_\perp = v\sin\theta$ in the plane perpendicular to **B**. The path of the particle due to $v_\|$ is the straight line while that due to $v_\perp$ is a circle of radius r given by

$$r = \frac{mv_\perp}{qB}$$

the resulting path will be the superposition of the straight line and circular paths *i.e.* helix or spiral as shown in fig. 14·7. In other words the particle gyrates about the magnetic field direction or it is locked to the field line.

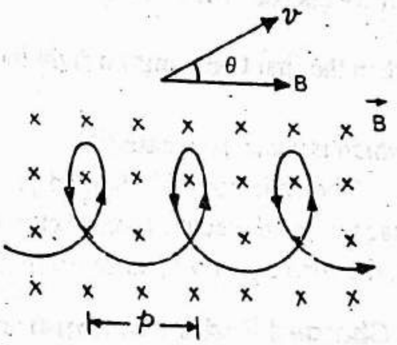


Fig. 14.7

3. Charged Particle in Simultaneous Electric and Magnetic Fields

If a charged particle of charge q enters simultaneous electric and magnetic field **E** and **B** respectively, then the total force on the particle is

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \text{...(1)}$$

v being the velocity of the particle.

If electric and magnetic fields are mutually perpendicular and the velocity **v** is perpendicular to both **E** and **B**. The force exerted by electric field **F** is along **E** and that exerted by magnetic field is perpendicular to **B** and hence parallel or antiparallel to **E**. When the magnitudes and directions of field vectors **E** and **B** are so adjusted that the electric and magnetic forces are equal and opposite, the particle will proceed undeflected.

For example it **E** is along *Y* axis and **B** along *Z*-axis with velocity vector **v** along *X*-axis, then electric force $\mathbf{F}_e = q\mathbf{E} = q\,E\mathbf{j}$ and magnetic force, $\mathbf{F}_m = q\,(\mathbf{v}\times\mathbf{B}) = q\,(v\mathbf{i}\times B\mathbf{k}) = -qvB\mathbf{j}$.

If magnitude of $\mathbf{F}_e$ and $\mathbf{F}_B$ are equal, then $qE = qvB$, *i.e.*

$$v = \frac{E}{B}$$

The velocity perpendicular to both **E** and **B** is called the *drift velocity* and is symbolised by v_0, *i.e.*

$$v_0 = \frac{E}{B} \qquad \text{...(2)}$$

If the initial velocity is not equal to v_0 or is not perpendicular to **B**, more complicated results are obtained. However all of these cases may be described by particle gyration about the direction of **B** at an angular frequency

$\omega_c = \dfrac{qB}{m}$, superimposed upon a linear drift motion of velocity

$v_0 = \dfrac{E}{B}$.

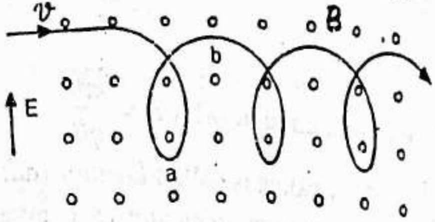


Fig. 14.8

For example, if a positively charged particle of charge q moves with velocity $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, then the force

$$\begin{aligned} F &= q\,(\mathbf{E}+\mathbf{v}+\mathbf{B}) \\ F &= q\,[\mathbf{E}+(\mathbf{v}_0+\mathbf{v}')\times\mathbf{B}] \\ &= q\,(\mathbf{E}+\mathbf{v}_0\times\mathbf{B}+\mathbf{v}'\times\mathbf{B}) \end{aligned} \qquad \text{...(3)}$$

If we choose drift velocity
$$\mathbf{v}_0 = \frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}^2}, \qquad \text{...(4)}$$

then the first two terms on right hand side of (3) cancel each other and force is given by

$$\mathbf{F} = q\mathbf{v}'\times\mathbf{B}$$

which is same as in case (i).

The trajectory of charged particle is shown in fig. 14·8. From the figure we note that at point *a* the magnetic and electric forces act in the same sense and so the radius of curvature is smaller while at the point *b*, the forces partly balance each after and therefore the radius of curvature is larger.

4. Charged Particle in non-homogeneous magnetic field

Let a charged particle of charge $+q$ enter a non-homogeneous magnetic field as shown in fig. 14·9. The result of non-homogeneity of magnetic field is the variation of radius of curvature.

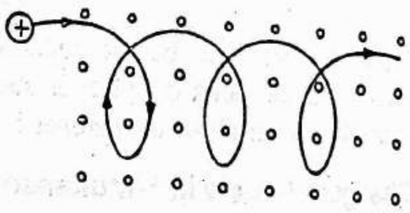


Fig. 14.9

This also induces a particle's drift. The gyrating charged particle gives rise to a magnetic moment defined as

$\mu = IA$ = current due particles gyration × area enclosed by that current

$$= \frac{qv_\perp}{2\pi r}\times\pi\,r^2 = \frac{qv_\perp r}{2} \qquad \text{...(5)}$$

But
$$B = \frac{m.v_\perp}{qr} \text{ i.e } v_\perp = \frac{qrB}{m} \qquad \text{...(6)}$$

$$\therefore \qquad \mu = \tfrac{1}{2} q \,.\, qrB = \left(\frac{q^2}{2\pi m}\right)(\pi r^2 B)$$

$$= \left(\frac{q^2}{2\pi m}\right)\phi \qquad \text{...(7)}$$

where ϕ — magnetic flux $= BA = B\pi r^2$

This can be shown that the magnetic moment remains constant if the variation of B is slow either in space or time.

The induced e.m.f. ε around a closed path in accordance with Faraday's law is given by

$$\varepsilon = -\frac{d\phi}{dt} = -\pi r^2 \frac{dB}{dt} = \pi r^2 \frac{dB}{dt} \text{ (numerically)} \qquad \text{...(8)}$$

The work done upon a particle of charge q in one orbit is

$$W = q\varepsilon$$

But W is also equal to $\frac{dW}{dt}$ × time for one orbit

i.e.

$$q\varepsilon = \frac{dW}{dt} \times \left(\frac{2\pi r}{v}\right) = \frac{2\pi r}{v} \cdot \frac{dB}{dt}$$

Using (8), we get

$$q\,\pi r^2 \frac{dB}{dt} = \frac{2\pi r}{v}\frac{dW}{dB} \cdot \frac{dB}{dt}$$

$$\therefore \qquad \frac{dW}{dB} = \frac{qrv}{2} = \mu \qquad \text{...(9)}$$

Also

$$W = \tfrac{1}{2} m v^2 = \mu B \qquad \text{...(10)}$$

$$\frac{dW}{dB} = \frac{d}{dB}(\mu B) = B\frac{d\mu}{dB} + \mu \qquad \text{...(11)}$$

Comparing (9) and (11), we get

$$\frac{d\mu}{dB} = 0 \;\; i.e. \; \mu \text{ is independent of } B.$$

For the magnetic moment μ to be constant, equation (7) implies that the flux ϕ within particle's orbit must remain constant. Thus in a magnetic field that varies slowly with position in the particle's orbit is compressed as the magnetic field increases in such a way to maintain the magnetic flux ϕ constant (fig. 14.10).

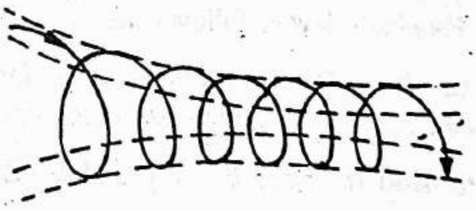


Fig. 14.10

14.5. Magneto-hydrodynamics

The properties of a plasma, in principle, may be accounted for by taking into account the individual contributions of all the particles and their interactions. But this procedure is difficult to realise in practice and therefore a macroscopic approach is convenient.

In the macroscopic treatment the plasma is considered to be made of small volume elements, each of which is large compared to the average spacings of the individual particles it contains, yet small compared to any distance over which the macroscopic plasma properties vary appreciably. Now one such volume element is considered and the average values of velocity, magnetic field, density, temperature, pressure, conductivity etc. are associated with this volume element. Then the behaviour of the plasma as a whole is deduced in terms of the actions of all such volume elements of the plasma. This technique is generally used in the treatment of electromagnetic and hydrodynamic phenomena. In this section we shall treat a conducting fluid in particular plasma-subject simultaneously to the laws of electromagnetics and hydrodynamics. The treatment is known as *magnetohydrodynamics or hydromagnetics.*

Let us consider an electrically neutral, conducting non-permeable fluid in electromagnetic fields, characterised by a matter density $\rho(\mathbf{r}, t)$, a velocity $\mathbf{v}(\mathbf{r}, t)$, a pressure $p(\mathbf{r}, t)$ taken to be a scalar and a real conductivity σ.

The hydrodynamic equations are :

The continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{...(1)}$$

and the force equation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + (\mathbf{J} \times \mathbf{B}) + \mathbf{F}_v + \rho \mathbf{g} \qquad \text{...(2)}$$

In this equation $-\nabla \rho$ and $\mathbf{J} \times \mathbf{B}$ are the pressure and magnetic force terms respectively. $\mathbf{F}_v$ is the viscous force term and $\rho\mathbf{g}$ is the gravitational force term.

For an *incompressible fluid* the viscous force is

$$\mathbf{F}_v = \eta \nabla^2 \mathbf{v} \qquad \text{...(3)}$$

When η is coefficient of viscosity of the plasma and the time derivative of velocity is the *connective derviative* given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} . \nabla . \qquad \text{...(4)}$$

This derivative gives the total time rate of change of a quantity moving instantaneously with velocity v.

The *electromagnetic fields* in the fluid, with the neglect of diplacement current, are described by .

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{...(5)}$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{...(6)}$$

The condition $\nabla \cdot \mathbf{J} = 0$, equivalent to neglect of displacement current follows from equation (6) and from Faraday's law it follows that $\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$ and the requirement $\nabla \cdot \mathbf{B} = 0$ can be imposed as an initial condition. Coulomb's law may be ignored if the displacement current is neglected. The reason is that the electric field is completely determined by curl equations and Ohm's law. If the displacement current is retained in Ampere's law and $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ (ρ being electric charge density) is taken into account, the corrections of only of the order of $\frac{v^2}{c^2}$ result and these are completely negligible for normal magneto-hydrodynamic problems.

To complete the specification of dynamical equations it is essential to specify the relation between current density **J** and the fields **E** and **B**. For a simple conducting medium of conductivity σ, Ohm's law is applicable and the current density is

$$\mathbf{J}' = \sigma \mathbf{E}' \qquad \text{...(7)}$$

$\mathbf{J}'$ and $\mathbf{E}'$ being measured in the rest frame of medium. For a medium moving relative to laboratory the *non-relativistic transformation* of field E and current density J are

$$\mathbf{E} = \mathbf{E}' + \mathbf{v} \times \mathbf{B}' \qquad \text{...(8)}$$

and

$$\mathbf{J} = \mathbf{J}' + \rho \mathbf{v} \qquad \text{...(9)}$$

For a one-component conducting fluid $\rho = 0$ consequently Ohm's law takes the form

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad ...(10)$$

In case if the conductivity of the fluid is effectively infinite, then the condition

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \qquad ...(11)$$

is satisfied.

Equations (10, (2), (5), (6) and (10), supplemented by an equation of state for the fluid form the equations of magnetohydrodynamics.

Electromagnetic and mechanical behaviour of fluid

The behaviour of afluid in the electromagnetic fields is governed mainly by its electrical conductivity. The effects are both electromagnetic and mechanical.

From equations (5), the *time dependence of magnetic field is* given by

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad ...(12)$$

From (10)

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} - \mathbf{v} \times \mathbf{B} \qquad ...(13)$$

Substituting **E** from (13) in (12), we get

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left(\frac{\mathbf{J}}{\sigma} - \mathbf{v} \times \mathbf{B}\right)$$

$$= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left(\frac{\mathbf{J}}{\sigma}\right)$$

Using (6) and assuming σ to be constant in space, we get

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times \mathbf{B})$$

$$= \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu_0 \sigma}\left[\text{grad div } \mathbf{B} - \nabla^2 \mathbf{B}\right]$$

As div $\mathbf{B} = 0$, we get

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \qquad ...(14)$$

For a fluid at rest (*i.e.* $\mathbf{v} = 0$), then equation, (14) reduces to *diffusion equation*

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \qquad ...(15)$$

On the other hand when the conductivity is large enough, so that the second term in (14) can be neglected, then the temporal behaviour of the magnetic field is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \qquad ...(16)$$

This equation effectively indicates that magnetic lines of force also move with velocity **v** with the fluid, so that the magnetic flux through any circuit moving with local fluid velocity is constant in time. As conductivity is assumed to be sufficiently large ; equation (11) is applicable.

From vector analysis the component of v perpendicular to **B**, symbolised by **w**, is given by

$$\mathbf{w} = \frac{(\mathbf{B} \times \mathbf{v}) \times \mathbf{B}}{B^2} \qquad ...(17)$$

$$= \frac{\mathbf{E} \times \mathbf{B}}{B^2} \text{ [using (11)]} \qquad ...(18)$$

The so called "$\mathbf{E} \times \mathbf{B}$ drift" of both fluid and lines of force can be understood in terms of individual particle orbits of electrons and ions in the crossed electric and magnetic fields.

The *mechanical behaviour of the system* can be studied with the form of equation (2)

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{F} - (\mathbf{J} \times \mathbf{B}) \qquad ...(19)$$

when $\mathbf{F} = -\nabla p + \mathbf{F}_v + \rho \mathbf{g} =$ sum of all the non-electromagnetic forces.

Substituting $\mathbf{J}$ from (10), we get

$$\begin{aligned} \rho \frac{d\mathbf{v}}{dt} &= \mathbf{F} - \sigma \{\mathbf{E} + (\mathbf{v} \times \mathbf{B})\} \times \mathbf{B} \\ &= \mathbf{F} - \sigma \mathbf{E} \times \mathbf{B} - \sigma (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \\ &= \mathbf{F} - \sigma B^2 (\mathbf{v}_\perp - \mathbf{w}) \qquad ...(20) \end{aligned}$$

where $\mathbf{v}_\perp = \dfrac{\mathbf{E} \times \mathbf{B}}{\mathbf{B}^2}$ and $\mathbf{w} = \dfrac{(\mathbf{B} \times \mathbf{v}) \times \mathbf{B}}{B^2}$

For fluids of infinite conductivity

$$\mathbf{v}_\perp = \mathbf{w} \text{ i.e. } \rho \frac{d\mathbf{v}}{dt} = \mathbf{F}.$$

Thus it is obvious that the flow parallel to $\mathbf{B}$ is governed by non-electromagnetic forces alone.

In the limit of large conductivity it is convenient to relate the current density $\mathbf{J}$ in force equation to the magnetic induction $\mathbf{B}$ via Ampere's law and to use infinite conductivity condition (11) to eliminate $\mathbf{E}$ from Faraday's law to yield equation (16).

The *magnetic force term in force equation* (2), using

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ can be written as

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \mathbf{B}) \qquad ...(21) \end{aligned}$$

using vector identity

$$\nabla (\mathbf{B} \cdot \mathbf{B}) = 2 (\mathbf{B} \cdot \nabla) \mathbf{B} + 2\mathbf{B} \times (\nabla \times \mathbf{B})$$

i.e.

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \tfrac{1}{2} \nabla (\mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla) \mathbf{B}$$

then (21) transforms to

$$\begin{aligned} \therefore \qquad \mathbf{J} \times \mathbf{B} &= -\frac{1}{\mu_0} \left[\tfrac{1}{2} \nabla (\mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla) \mathbf{B} \right] \\ &= -\nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} \qquad ...(22) \end{aligned}$$

This equation shows that the magnetic force is equivalent to a magnetic *hydrostatic pressure*

$$p_m = \frac{B^2}{2\mu_0} \qquad ...(23)$$

plus a term which can be thought as an additional tension along the *magnetic field direction* :

If we neglect the viscous effects and assume that the gravitational force is derivable from a potential ψ, such that $\mathbf{g} = -\nabla \psi$, then the force equation (2) takes the form

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla (p + p_m + \rho \psi) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} \qquad ...(24)$$

In some simple geometrical situations, such as **B** having only one-component, the last representing additional tension, vanishes; so we have

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla (p + p_m + p\psi) \qquad ...(25)$$

Then the static properties of the liquid are described by

$$p + p_m + \rho\psi = \text{constant} \qquad ...(26)$$

This shows that apart, from gravitational effects, *any change in mechanical pressure is balanced by an opposite change in magnetic pressure.* If the fluid is to be confined within a certain region so that the mechanical pressure p falls rapidly to zero outside the region, the magnetic pressure must rise equally rapidly in order to confine fluid.

14.6. Magnetic Confinement—Pinch Effect

The cofinement of a plasma or conducting fluid by self magnetic fields is of considerable interest in thermo-nuclear research.

If we neglect the viscous and gravitational effects, the condition of equilibrium is $\mathbf{F} = 0$, *i.e.*

$$\nabla p - \mathbf{J} \times \mathbf{B} = 0 \quad \text{or} \quad \nabla p = \mathbf{J} \times \mathbf{B} \qquad ..(1)$$

The scalar product with **B** and **J** gives

$$\mathbf{B} \cdot \nabla p = \mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \qquad ...(2)$$

$$\mathbf{J} \cdot \nabla p = \mathbf{J} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \qquad ...(3)$$

These equation indicate that both **B** and **J** are perpendicular to the pressure gradient *i.e.* they lie on surfaces of constant potential.. If these surfaces are closed, then keeping in view equation (2) and (3) according to which no **B**-lines or **J** lines can cross them, they may be viewed as made up from a winding of **B** lines and of **J** lines as shown in fig. 14·11.

In these isobaric surfaces, the pressure increases from outside towards the axis and the force $\mathbf{J} \times \mathbf{B}$ also points towards the axis. This means that the plasma is eontained by the electromagnetic force $\mathbf{J} \times \mathbf{B}$. This is known as the *magnetic confinement.*

The phenomenon of conflnement of a plasma or conducting fluid by self magnetic fields is called the **Pinch effect.**

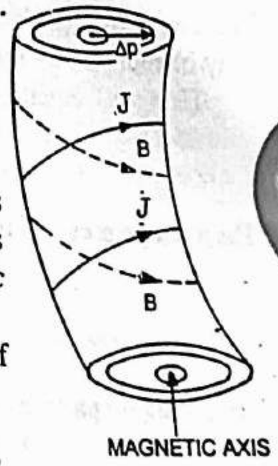


Fig. 14.11

Average pressure for a cylindrical plasma confined by its own magnetic field.

Consider an infinite cylindrical plasma or conducting fluid with an axial current density $J_z = J(r)$ and a resulting azimuthal magnetic induction $B_\phi = B(r)$. For simplicity the magnetic field, the current density, pressure etc. are assumed to depend on distance r from the cylindrical axis and the viscous and gravitational effects are neglected. We first consider whether a steady state condition can exist in which the material is mainly confined within a certain radius $r = R$ by the action of self magnetic field. For a steady state $\mathbf{v} = 0$, therefore equation of motion for the fluid, *i.e.*

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla (p + p_m + \rho\psi) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B}$$

takes the form

$$0 = -\frac{dp}{dr} - \frac{d}{dr}\left(\frac{B^2}{2\mu^2}\right) - \frac{B^2}{\mu_0 r} \qquad ...(4)$$

Ampere's law in integral form relating to $B(r)$ to the current enclosed is

$$B(r) = \frac{\mu_0}{r}\int_0^r r\,J(r)\,dr. \qquad ...(5)$$

If the fluid lies almost entirely inside $r = R$, then equation (5) for the magnetic induction inside the fluid yields

$$B(r) = \frac{\mu_0 I}{2\pi r}. \qquad ...(6)$$

Where $I = \int_0^R 2\pi r\,J(r)\,dt$ is the total current flowing in the cylinder.

Equation (4) can be written as

$$\frac{dp}{dr} = -\frac{1}{2\mu_0 r^2}\frac{d}{dr}(r^2B^2). \qquad ...(7)$$

Integration of above euqation yields

$$p(r) = p_0 - \frac{1}{2\mu_0}\int_0^r \frac{1}{r^2}\frac{d}{dr}(r^2B^2)\,dr, \qquad ...(8)$$

where p_0 is the pressure of fluid at $r = 0$. If the matter is confined to $r \le R$, the pressure drops to zero at $r = R$. Consequently the axial pressure p_0 is given by

$$p_0 = \frac{1}{2\mu_0}\int_0^R \frac{1}{r^2}\frac{d}{dr}(r^2B^2)\,dr. \qquad ...(9)$$

From equation (6) it is obvious that above expression vanishes for $r \ge R$; therefore the upper point in (9) may be replaced by infinity.

Using (9), equation (8) can be written as

$$p(r) = \frac{1}{2\mu_0}\int_0^R \frac{1}{r^2}\frac{d}{dr}(r^2R^2)\,dr. \qquad ...(10)$$

The average pressure inside the cylinder can be expressed as

$$<p> = \frac{2\pi\int_0^R r\,p(r)\,dr}{\pi R^2},$$

Integrating by parts, using equations (10) and (6), we get

$$<p> = \frac{2I^2}{\mu_0 R^2}.$$

This gives the relation between average pressure, total current I and the radius of the cylindrical plasma confined by its own magnetic field. It may be noted that the average pressure of matter is equal to the magnetic pressure $\left[\frac{B^2}{2\mu_0}\right]$ at the surfae of the cylinder.

In thermonuclear work the temperature of (hot) plasma is of the order of 10^8 K ($k\tau \approx 80$ k eV) and density $\approx 10^{21}$ particles/m^3. These condition correspond to a pressure of $\approx 1{\cdot}4 \times 10^6$ N/m^2 or 14 atmosphere. A magnetic induction $B \approx 1{\cdot}9$ Tesla at the surface, corresponding to a current of $9 \times 10^2 R$ Amperes is necessary for confinement. This shows that the extremely high currents are needed to cofine very hot plasma.

14.9. Instabilities in a Pinched Plasma Column

The self-plasma is subject to numerous instabilities. The two important instabilities are :

(i) magneto-hydro-dynamic instabilities that can be explained in terms of magnetic pressures and stresses.

(ii) instabilities associated with particle motion.

Let us consider a uniform cylindrical column of plasma with a surrounding magnetic field. The first type of distortion known as the **kink instability** is shown in fig. 14.12 (a) and occurs when a linear pinch is bent due to some perturbation. The lines of azimuthal induction near the column are bunched together above and separated below, the column by the distortion downwards. Thus the magnetic pressure changes are in such a direction so as to increase the distortion, Such a distortion is unstable.

The second type of distortion known as sausage or neck instability is shown in fig. 14.12 (b) and occurs if the plasma column is pinched by chance. In the neighbourhood of constriction the azimuthal induction increases (as $B \propto 1/r$) causing a greater inwards pressure at the neck than elsewhere. This enhances the existing distortion.

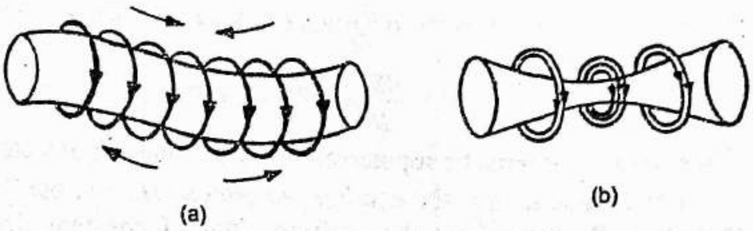


Fig. 14.12

Apart from these instabilities several other kinetic instabilities (*e.g.* flute instability.) exist which are either by chance or by artificial deviation of particles from Maxwell's distribution. These instabilities are hindered by providing a 'frozen in' axial magnetic field along the pinch. This axial magnetic field tends to slow the growth of kink instability by requiring that they expend energy in stretching the field. Sausage instabilities are also to be inhibited because by the compression of axial "frozen in" field required by their development. Nevertheless it has not been possible in practice to prevent either form of pinch instability by the use of such fields.

14.10. Plasma-Waves

In a plasma there can develop and propagate a variety of waves in a very wide frequency range.

It has already been shown that the stress due to a magnetic field is equivalent to an isotropic magnetic field pressure $\frac{B^2}{2\mu_0}$ and a tension $\frac{B^2}{\mu_0}$ acting along the direction of magnetic field (fig. 14·13).

The tension in the lines of force tends to restore them to straight line form, thereby causing a **transverse oscillation.** The situation is analogous to that of a stretched elastic string. The transverse waves in a stretched string can be set up by plucking the string. Similarly it may be thought that the transverse waves propagating along the magnetic field can be produced by plucking a **tube of force.** The existence of such waves was first recognized by Hannes Alfven in 1942 and hence these waves are known as **Alfven waves.**

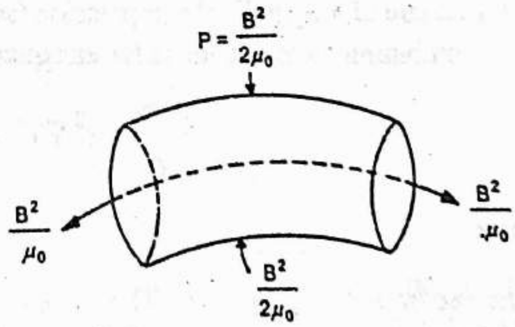


Fig. 14.13

The velocity of transverse waves in a stretched string is

$$v = \sqrt{\left[\frac{T}{\rho}\right]}$$

where T is tension and ρ is mass per unit length of string.

In magneto-hydrodynamics, the tension T is replaced by magnetic tension $\frac{B^2}{\mu_0}$ and ρ becomes the plasma mass density. Thus for hydro-dynamic waves, the velocity of Alfven waves is given by

$$v_A = \sqrt{\left\{\frac{(B^2/\mu_0)}{\rho}\right\}} = \sqrt{\left\{\frac{B^2}{\mu_0\rho}\right\}} \qquad \text{...(1)}$$

To examine the wave motion of a conducting fluid in the presence of a magnetic field, we consider a compressible, non-viscous, perfectly conducting fluid in the magnetic field in the absence of gravitational forces. The equations of motion are

$$\left.\begin{aligned} &\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 && \text{...(a)}\\ &\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p - \frac{\mathbf{B}\times(\nabla\times\mathbf{B})}{\mu_0} && \text{...(b)}\\ &\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) && \text{...(e)} \end{aligned}\right\} \qquad \text{...(2)}$$

These equations must be supplemented by an equation of state relating the pressure to the density.

Let us assume that the *equilibrium velocity is zero,* but there exists a spatially uniform, static magnetic induction $\mathbf{B}_0$ throughout the uniform fluid of constant density ρ_0. Then we imagine small-amplitude departures from the equilibrium values

$$\left.\begin{aligned} \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r}, t) && \text{...(a)}\\ \rho &= \rho_0 + \rho_1(\mathbf{r}, t) && \text{...(b)}\\ \mathbf{v} &= \mathbf{v}_l(\mathbf{r}, t) && \text{...(c)} \end{aligned}\right\} \qquad \text{...(3)}$$

If equations (2) are linearised in small quantities, then they take the form

$$\left.\begin{aligned} &\frac{\partial\rho_1}{\partial t} + \rho_0\,\nabla\cdot\mathbf{v}_1 = 0 && \text{...(a)}\\ &\rho_0\frac{\partial\mathbf{v}_1}{\partial t} + s^2\,\nabla\rho_1 + \frac{\mathbf{B}_0}{\mu}\times(\nabla\times\mathbf{B}_1) = 0 && \text{...(b)}\\ &\frac{\partial\mathbf{B}_1}{\partial t} - \nabla\times(\mathbf{v}_1\times\mathbf{B}_0) = 0 && \text{...(c)} \end{aligned}\right\} \qquad \text{...(4)}$$

where
$$s^2 = \left\{\frac{\partial x}{\partial\rho}\right\}_0 \qquad \text{...(6)}$$

is the square of longitudinal compression (sound) wave velocity.

Combining these equations for an equation to yield $\mathbf{v}_1$, alone, we get

$$\frac{\partial^2\mathbf{v}_1}{\partial t^2} - s^2\,\nabla(\nabla\cdot\mathbf{v}_1) + \mathbf{v}_A\times\nabla\times[\nabla\times(\mathbf{v}_1\times\mathbf{v}_A)] = 0 \qquad \text{...(6)}$$

where
$$\mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{\mu_0\rho_0}} \qquad \text{...(7)}$$

is the vectorial *Alfven velocity.* The wave equation (6) allows simple solutions for wave propagating parallel or perpendicular to the magnetic field direction.

For a plane wave with wave vector $\mathbf{k}$ and frequency ω

$$\mathbf{v}_1(\mathbf{r}, t) = \mathbf{v}_1\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad \text{...(8)}$$

with this equations (6) becomes

$$-\omega^2 \mathbf{v}_1 + (s + \mathbf{v}_A^2)(\mathbf{k}\cdot\mathbf{v}_1)\,\mathbf{k} + \mathbf{v}_A\cdot\mathbf{k}\,[(\mathbf{v}_A\cdot\mathbf{k})\,\mathbf{v}_1 - (\mathbf{v}_A\cdot\mathbf{v}_1)\,\mathbf{k} - (\mathbf{k}\cdot\mathbf{v}_1)\,\mathbf{v}_A] = 0 \qquad ...(9)$$

If propagation wave vector **k** *is perpendicular to* $\mathbf{v}_A$, then the last term in (9) vanishes (since $\mathbf{v}_A \cdot \mathbf{k} = 0$) and the solution for $\mathbf{v}_1$ is a longitudinal *magnetosonic wave* with a *phase velocity*

$$u_l = \sqrt{(s^2 + \mathbf{v}_A^2)}$$

Obviously this wave propagates with a velocity determined by the sum of hydrostatic and magnetic pressures apart from the order of unity.

If *the wave propagation vector* **k** *is parallel to* $\mathbf{v}_A$, then equation (9) reduces to

$$(k\mathbf{v}_A^2 - \omega^2)\,\mathbf{v}_1 + \left(\frac{s^2}{v_A^2} - 1\right) k^2 (\mathbf{v}_A\cdot\mathbf{v}_1)\,\mathbf{v}_A = 0 \qquad ...(10)$$

In this case there are two types of possible wave motion.

1. An ordinary longitudinal wave $\mathbf{v}_1$ (parallel to **k** and $\mathbf{v}_A$) propagating with phase velocity equal to sound velocity s.

2. A transverse wave $(\mathbf{v}_1 \cdot \mathbf{v}_A) = 0$ propagating with a phase velocity equal to Alfven-velocity v_A.

This Alfven wave is purely a magentic hydro-dynamic phenomenon which depends only on magnetic field (tensor) and the density (inertia). At all laboratory field strengths the Alfven velocity is much smaller than the speed of sound ; but in astrophysical problems the Alfven velocity can become very large because of the much smaller densities.

The magnetic fields of these different waves can be found from equation (4c).

$$\mathbf{B}_1 = \begin{cases} \dfrac{k}{\omega} v_1\,\mathbf{B}_0 \text{ for } \mathbf{k} \perp \mathbf{B}_0 \\ 0 \text{ for longitudinal } \mathbf{k} \parallel \mathbf{B}_0 \\ -\dfrac{k}{\omega}\mathbf{B}_0\,\mathbf{v}_1 \text{ for transverse } \mathbf{k} \parallel \mathbf{B}_0 \end{cases} \qquad ...(11)$$

The magnetic wave propagating perpendicular to $\mathbf{B}_0$ causes compressions and rarefactions in the magnetic lines of force without changing their direction (fig. 14.14a). The Alfven wave parallel to $\mathbf{B}_0$ causes the magnetic lines of force to oscillate back and forth laterally (fig. 14.14b), In either case the lines of force are frozen in and move with the field.

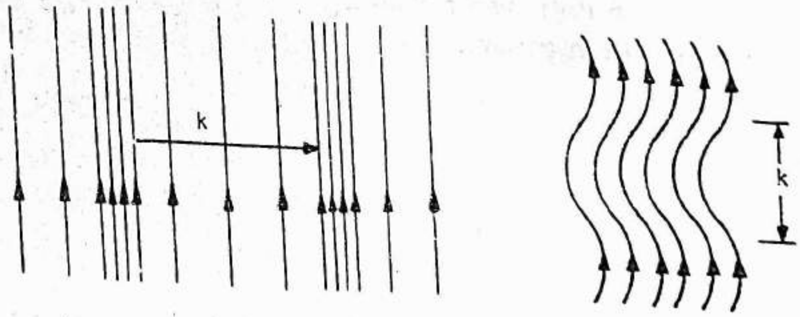


Fig. 14.14. Magnetic hydro-dynamic waves

QUESTIONS AND PROBLEMS

1. What do you mean by a plasma. Discuss the conditions for the existence of a plasma.
2. Discuss the motion of a charged particle in :
   (i) a uniform electric field (ii) a uniform magnetic field (iii) crossed electric and magnetic fields.
3. What do you mean by plasma in general ? Derive the magneto-hydrodynamical equations.
4. Write down the magnetic-hydrodynamic equations and hence show that a magnetic force is equivalent to magnetic hydrostatic pressure $P_m = B^2/2\mu_0$ plus an additional tension along the magnetic lines of force. **(Meerut 1992)**
5. What is Pinch effect in a plasma ? Give a brief account of the dynamical model of Pinch effect. **(Rohilkhand 1993)**
6. What is pinch effect ? Calculate the average pressure for a cylindrical plasma confined by its own magnetic field. Discuss in stabilities in a pinched plasma column. **(Meerut 1998)**

7. Derive the general equation for the velocity of magnetic hydrodynamic waves and find the solution when (i) propagation vector is perpendicular to Alfven velocity.

   (ii) propagation vector is parallel to Alfven velocity.

8. (a) Derive kink, sausage and flute instabilities in a plasma. How can the problem of plasma confinement between magnetic mirror due to these instabilities be overcome ?

   (b) What do you mean by hydromagnotic waves ? show that the phase velocity for plasma waves approximates to Alfven velocity. **(Rohilkhand 1995)**

9. Derive the hydromagnetic equations for electrically neutral plasma in electromagnetic fields. What is meant by pinch effect. **(Meerut 1997)**

10. In the sun's photosphere there are 6×10^{16} hydrogen atoms per cm^3. Assuming the field in photosphere to be 1000 gauss, calculate the phase velocity of Alfven waves.

    [Ans. $8{\cdot}9 \times 10^3$ **m/sec]**

11. A current flows axially in a cylindrical type of plasma. The plasma pressure vanishes at $r = R$ and $p = N(r)\,kT$. Show that

$$2\mu kT = \frac{\pi}{2\mu_0\,(rB)^2_{r=R}} \quad \text{where} \quad \mu = \int_0^R N(r)\,2\pi r\,dr$$

12. An incompressible, non-viscous, perfectly conducting fluid with constant density ρ_0 is acted upon by the gravitational potential ρ and a uniform, static magnetic induction B_0.

    (a) Show that the magneto-hydrodynamic waves of arbitrary amplitude and form $\mathbf{B}_1(r, t), \mathbf{v}(r, t)$ exist, described by equations

$$(\mathbf{B_0} \cdot \nabla)\,B_1 = \pm\sqrt{(\mu_0\,\rho_0)}\,\frac{\partial B_1}{\partial \tau}, \qquad B_1 = \pm\sqrt{(\mu_0\rho_0)}\,v$$

$$p + \rho_0\,\psi + \frac{(B_0 + B_1)}{2\mu_0} = \text{constant}$$

    (b) Suppose that $t = 0$, certain disturbance $B_1(r, 0)$ exists in the fluid such that it satisfies the above equations with the upper sign. What is the behaviour of the disturbance at later times ?

13. Write short notes on :

    (a) Kink, sausage and flute instability in a plasma. **(Rohilkhand 1999)**

    (b) Pinch effect in Plasma. **(Rohilkhand 1996)**

    (c) Alfven waves **(Meerut 1997)**

15

Relativistic Electrodynamics

15.1. Introduction

It is well known that Newton's laws of motion remain unaffected by a *Galilean transformation i.e.* a coordinate transformation between two inertial frames in relative motion. For example let S be a coordinate system at rest and S' be another coordinate system moving with uniform velocity **v** relative to S along positive direction of X-axis. Then the Galilean transformations for space and time are

$$x' = x - vt\,,\; y' = y\,,\; z' = z\,,\; t' = t \qquad \text{...(1)}$$

where (x, y, z, t) and (x', y', z', t') are space and time coordinates of a point P as observed from systems S and S' respectivley.

It was concluded by a large number of carefully performed experiments *e.g.* the aberration of star light, measurement of velocity of light in moving fluids and Michelson Morley experiment, that the laws of Galilean transformations failed to explain the laws of electrodynamics. Among the several possibilities suggesed to remove this conflict, it was concluded that it were the laws of Galilean transformations and not electrodynamics which needed modifications. The modifications in the laws of Galilean transformations originated the theory of relativity according to which the space and time are not asbolute but relative.

The theory of relativity having new concepts of space and time is applicable not only to mechanical phenomena : but also to all optical and electromagnetic phenomena and is divided into two parts.

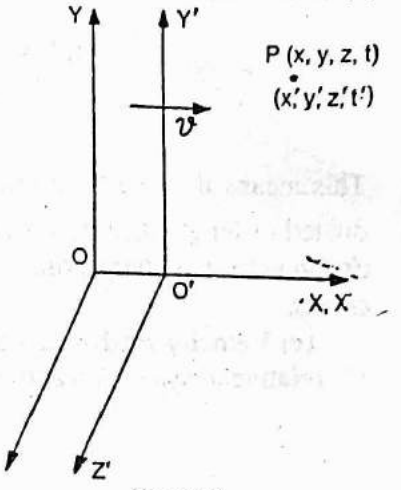


Fig. 15.1

(i) Special Theory of Relativity. It deals with systems known as inertial or unaccelrated systems *i.e.* the systems which move in uniform rectilinear motion relative to one another.

(ii) General Theory of Relativity. It deals with non-inertial or accelerated systems and is applicable to the law of gravitation and explains it in a more refined manner than given by Newton.

Postulates of special theory of relativity. There are two basic postulates of special theory of relativity.

(i) The fundamental laws of physics have the same form for all the inertial systems.

(ii) The velocity of light in vacuum is independent of the relative motion of the source and the observer.

Lorentz Transformations

Based on the new concepts of space and time, Lorentz introduced new transformation equations under which velocity of light in vacuum remains constant independent of the relative motion of the source and observer. These transforamtion equations relate the observations of position and time in different inertial frames and are known as Lorentz transformations.

If S and S' are two inertial frames, S' having uniform veloicity v relative to S along positive direction of X-axis, then Lorentz transforamtions are

$$x' = \frac{x - vt}{\sqrt{(1 - v/c^2)}}; \; y' = y; \; z' = z \quad \text{and} \quad t' = \frac{(t - vx/c^2)}{\sqrt{(1 - v^2/c^2)}} \quad ...(3)$$

where (x, y, z, t) and (x', y', z', t') are the space and time coordinates of an event in systems S and S' respectively.

Consequence of Lorentz Transformation Equations. Following are the important consequences of Lorentz transformation equations.

(a) Length Contraction. If l_0 and l are the proper length (*i.e.* length at rest) and the length in motion of an object respectively, then in the direction of motion we have

$$l = l_0 \sqrt{\left\{1 - \frac{v^2}{c^2}\right\}} = l_0 \sqrt{(1 - \beta^2)} \quad \text{where } \beta = \frac{v}{c} \qquad ...(3)$$

This means that the length of the object is contracted in the direction of motion by the factor $\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ where v is the speed of object relative to observer. The length perpendicular to direction of motion remains unchanged. The phenomenon of contraction of length along the direction of motion is known as *Lorentz-Fitzgerald contraction.*

(b) Time-Dilation. If Δt is the proper time interval and $\Delta t'$ is the corresponding time interval in motion, then

$$\Delta t' = \frac{\Delta t}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{\Delta t}{\sqrt{(1 - \beta^2)}} \quad \text{where } \beta = \frac{v}{c} \; i.e. \; \Delta t' > \Delta t \qquad ...(4)$$

This means that the time interval between two signals appears to the moving observer (velocity v) to be dilated or lengthened by a factor $1/\sqrt{\{1 - (v^2/c^2)\}}$. We may state that ***"A clock will be found to run slow due to velocity between the clock and the observer. This phenomenon is known as apparent retardation of clocks.***

(c) Velocity Addition Theorem. If u and u' are velocity of a particle (along the direction of motion of S') relative to systems S and S', the latter moving with velocity v relative to former with velocity v, then

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}. \qquad ...(5)$$

Accordingly if $u' = c$, then $u = c$ and also if $u' = v = c$, then $u = c$ ***i.e. the speed of light is same for all inertial frames whatever their relative speed may be and no particle can attain a speed greater than the speed of light.***

(d) Variation of mass with velocity. If m_0 is the rest mass and m is the corresponding mass in motion; then

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{m_0}{\sqrt{(1 - \beta^2)}} \quad \text{where } \beta = \frac{v}{c} \qquad ...(6)$$

This indicates that the ***mass of a body increases with increase of velocity.***

(a) Mass-Energy relation. According to Einstein's theory the mass and energy are equivalent *i.e.* mass can converted into energy and vice-versa. The relation is

$$E = mc^2$$

If the laws of electrodynamics are unchanged by the introduction of relativity, then Maxwell's equations should be invariant under Lorentz transformations.

We shall see in this chapter how Maxwell's field equation can be expressed in Lorentz covariant form. The covariant formulation of laws of electrodynamics suggests that the Lorentz force equation is a natural consequence and the splitting of the electromagnetic field is not a fundamental one, rather it depends on the state of motion of the observer. Similarly field invariants like $c^2B^2 - E^2$ and $\mathbf{E} \cdot \mathbf{B}$ follow automatically.

Ex. 1. *Show by direct application of Lorentz transformations that* $x^2 + y^2 + z^2 - c^2t^2$ *is invariant.*

(Rohilkhand 1992; Agra 1997)

Solution. We have only to prove by the help of Lorentz transformations.

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

where (x, y, z, t) and (x', y', z', t') are the co-ordinates of the same event observed by two observers is system S and S' while S' is moving with a velocity v relative to S.

Let us consider the expression

$$x'^2 + y'^2 + z'^2 - c^2t'^2$$

$$= \frac{(x - vt)^2}{1 - \frac{v^2}{c^2}} + y^2 + z^2 - c^2 \frac{\left[t - \frac{vx}{c^2}\right]^2}{1 - \frac{v^2}{c^2}}$$

[Putting values of x', y', z' and t' from Lorentz transformations]

$$\therefore \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = \frac{c^2}{c^2 - v^2}(x - vt)^2 + y^2 + z^2 - \frac{c^2}{c^2 - v^2}\left\{t - \frac{vx}{c^2}\right\}^2$$

$$= \frac{c^2}{c^2 - v^2}\left[(x - vt)^2 - c^2\left\{t - \frac{vx}{c^2}\right\}^2\right] + y^2 + z^2$$

$$= \frac{c^2}{c^2 - v^2}\left[x^2 - 2vxt + v^2t^2 - c^2\left\{t^2 - \frac{2vxt}{c^2} + v^2\frac{x^2}{c^4}\right\}\right] + y^2 + z^2$$

$$= \frac{c^2}{c^2 - v^2}\left[x^2 - 2vxt + v^2t^2 - c^2t^2 + 2vxt - \frac{v^2x^2}{c^2}\right] + y^2 + z^2$$

$$= \frac{c^2}{c^2 - v^2}\left[\frac{c^2x^2 + c^2v^2t^2 - c^4t^2 - v^2x^2}{c^2}\right] + y^2 + z^2$$

$$= \frac{1}{c^2 - v^2}[c^2(x^2 - c^2t^2) - v^2(x^2 - c^2t^2)] + y^2 + z^2$$

$$= \frac{1}{c^2 - v^2}[(c^2 - v^2)(x^2 - c^2t^2))] + y^2 + z^2$$

$$= x^2 - c^2t^2 + y^2 + z^2 = x^2 + y^2 + z^2 - c^2t^2.$$

Thus we have proved that

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2.$$

That is, expression $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformations.

Ex. 2. *Prove that the three dimensional volume element 'dx dy dz' is not invariant, but four dimensional volume element is invariant under Lorentz transformations.* **(Agra 1995, 86)**

Solution. According to Lorentz Fitzgerald contraction; we have

$$dx' = dx\sqrt{(1-\beta^2)} \qquad \text{where } \beta = v/c$$
$$dy' = dy$$
$$dz' = dz.$$

∴ Three dimensional volume element in system S'

$$= dx'\,dy'\,dz' = dx\sqrt{(1-\beta^2)}\,dy\,dz.$$
$$= dx\,dy\,dz\sqrt{(1-\beta^2)}$$
$$\neq dx\,dy\,dz \text{ (i.e. three dimensional volume element in system } S)$$

Thus three dimensional volume element is not invariant under Lorentz transformations.

According to time dilation.

$$dt' = \frac{dt}{\sqrt{(1-\beta^2)}}$$

∴ Four dimensional volume element is system S'

$$= dx'\,dy'\,dz'\,dt'$$
$$= dx\sqrt{(1-\beta^2)}\,dy\,dz\frac{dt}{\sqrt{(1-\beta)}}$$
$$= dx\,dy\,dz\,dt \text{ (i.e. four dimensional volume element in system } S').$$

Hence four dimensional volume element is invariant under Lorentz transformations.

15.2. Transformation of Differential Operators

Consider two systems S and S', S' moving with velocity v relative to S along (+) ve direction of x-axis.

If a wave is travelling in a space with velocity v in system S, then the propagation equation for such a wave is of the form

$$\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\psi = 0$$

Hence ψ is known as *wave function* and differential operator $\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}$ is called *D' Alembert's operator* and is denoted by $\Box^2$. Here ψ is function of x, y, z and t, and thus it may be written as $\psi(x, y, z, t)$.

Now in frame S' which is moving relative to S, the propagation equation of same wave is given by

$$\left\{\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right\}\psi = 0$$

where c is not primed, because according to the principle of relativity c is always constant. Thus ψ may be written as

$$\psi(x', y', z', t).$$

We thus have,

$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x'}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial x'}+\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial x'}+\frac{\partial\psi}{\partial t}\frac{\partial t}{\partial x'} \qquad ...(1)$$

According to Lorentz inverse transformations, we also have

$$\left.\begin{aligned} x &= \frac{x'+vt'}{\sqrt{(1-\beta)^2}},\ y=y',\ z=z'. \\ \text{and} \qquad t &= \frac{t'+\dfrac{vx'}{c^2}}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad \text{...(2)}$$

From equation (2), we have

$$\left.\begin{aligned} \frac{\partial x}{\partial x'} &= \frac{1}{\sqrt{(1-\beta^2)}},\ \frac{\partial x}{\partial y'}=0,\ \frac{\partial x}{\partial z'}=0. \\ \text{and} \qquad \frac{\partial x}{\partial t'} &= \frac{v}{\sqrt{(1-\beta^2)}}, \\ \frac{\partial t}{\partial x'} &= \frac{v/c^2}{\sqrt{(1-\beta^2)}},\ \frac{\partial y}{\partial y'}=1,\ \frac{\partial z}{\partial y'}=0,\ \frac{\partial t}{\partial y'}=0. \end{aligned}\right\} \qquad \text{...(3)}$$

Thus from (1), we have

$$\frac{\partial\psi}{\partial x'} = \frac{1}{\sqrt{(1-\beta^2)}}\frac{\partial\psi}{\partial x} + \frac{v}{c^2\sqrt{(1-\beta^2)}}\frac{\partial\psi}{\partial t}$$

i.e.

$$\frac{\partial\psi}{\partial x'} = \frac{1}{\sqrt{(1-\beta^2)}}\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

or

$$\frac{\partial}{\partial x'} = \frac{1}{\sqrt{(1-\beta^2)}}\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right) \qquad \text{...(4)}$$

Also

$$\frac{\partial\psi}{\partial y'} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial y'}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial y'}+\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y'}+\frac{\partial\psi}{\partial t}\frac{\partial t}{\partial y'}.$$

Putting values from (3) this gives

$$\frac{\partial\psi}{\partial y'} = \frac{\partial\psi}{\partial y}$$

∴

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}. \qquad \text{...(5)}$$

Similarly

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}. \qquad \text{...(6)}$$

Again, we have

$$\begin{aligned} \frac{\partial\psi}{\partial t'} &= \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial t'}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial t'}+\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial t'}+\frac{\partial\psi}{\partial t}\frac{\partial t}{\partial i'} \\ &= \frac{\partial\psi}{\partial x}\frac{v}{\sqrt{(1-\beta^2)}}+\frac{\partial\psi}{\partial t}\frac{1}{\sqrt{(1-\beta^2)}} \end{aligned}$$

$$\left[\text{since } \frac{\partial y}{\partial t'} = \frac{\partial z}{\partial t'} = 0,\ \frac{\partial t}{\partial t'} = \frac{1}{\sqrt{(1-\beta^2)}},\ \frac{\partial x}{\partial t'} = \frac{v}{\sqrt{(1-\beta^2)}}\right]$$

Therefore

$$\frac{\partial\psi}{\partial t'} = \frac{1}{\sqrt{(1-\beta^2)}}\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\psi$$

∴

$$\frac{\partial}{\partial t'} = \frac{1}{\sqrt{(1-\beta^2)}}\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right) \qquad \text{...(7)}$$

From (4), (5), (6) and (7) for transformation equations of differential operators, we have

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= \frac{1}{\sqrt{(1-\beta^2)}}\left\{\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right\} \\ \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z'} &= \frac{\partial}{\partial z} \\ \text{and} \qquad \frac{\partial}{\partial t'} &= \frac{1}{\sqrt{(1-\beta^2)}}\left\{v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right\} \end{aligned}\right\} \quad \text{...(8)}$$

Similarly inverse transformations may be evaluated to be

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{\sqrt{(1-\beta^2)}}\left\{\frac{\partial}{\partial x'}-\frac{v}{c^2}\frac{\partial}{\partial t'}\right\} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'},\ \frac{\partial}{\partial z}=\frac{\partial}{\partial z'}, \\ \frac{\partial}{\partial t} &= \frac{1}{\sqrt{(1-\beta^2)}}\left\{\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right\} \end{aligned}\right\} \quad \text{...(9)}$$

Equations (8) and (9) involve required transformations of differential operators.

15.3. Invariance of D' Alembertian Operator $\Box^2$.

Consider two systems S and S', S' moving with velocity v relative to S along (+) ve direction of x-axis. If $(x', y'; z', t')$ and (x, y, z, t) are co-ordinates of any event in S' and S respectively, then

D' Alembertian in system S is

$$\Box^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

and D' Alembertian in system S' is

$$\Box'^2 = \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}.$$

According to transformation of Differentiatl operators; we have

$$\begin{aligned} \frac{\partial}{\partial x'} &= \frac{1}{\sqrt{(1-\beta^2)}}\left\{\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right\} \\ \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z'} &= \frac{\partial}{\partial z} \\ \text{and} \qquad \frac{\partial}{\partial t'} &= \frac{1}{\sqrt{(1-\beta^2)}}\left\{v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right\} \end{aligned}$$

which give

$$\begin{aligned} \frac{\partial^2}{\partial x'^2} &= \frac{1}{1-\beta^2}\left\{\frac{\partial^2}{\partial x^2}+\frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}+\frac{2v}{c^2}\frac{\partial^2}{\partial x\,\partial t}\right\} \\ \frac{\partial^2}{\partial y'^2} &= \frac{\partial^2}{\partial y^2} \text{ and } \frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z^2} \end{aligned}$$

and
$$\frac{\partial^2}{\partial t'^2} = \frac{1}{1-\beta^2}\left\{v^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x\,\partial t}\right\}$$

Therefore

$$\begin{aligned}
\Box'^2 &= \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} \\
&= \frac{1}{1-\beta^2}\left[\frac{\partial^2}{\partial x^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\,\partial t}\right] + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
&\qquad - \frac{1}{c^2}\cdot\frac{1}{1-\beta^2}\left(v^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x\,\partial t}\right) \\
&= \frac{\partial^2}{\partial x^2}\left\{\frac{1}{1-\beta^2} - \frac{v^2/c^2}{1-\beta^2}\right\} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{c^2\,\partial t^2}\left(\frac{1}{1-\beta^2} - \frac{v^2/c^2}{1-\beta^2}\right) \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad \text{since } \frac{v^2}{c^2} = \beta^2 \\
&= \Box^2.
\end{aligned}$$

Thus we may say that $\Box^2$ is invariant under Lorentz transformations.

15.4. Invariance of charge

According to the principle of invariance of charge *the total charge in an isolated system is unchanged by the motion of its carriers.*

The means that the quantity of electric charge remains exactly the same if it is measured from any other frame of reference.

The principle may also be stated as :

The total electric charge of an isolated sytem is relativistically invariant.

The experimental evidence of the principle is that the atoms and molecules are exactly electrically neutral even when they are in motion.

Recently this fact has been experimentally verified in the case of hydrogen and helium molecules as follows :

The large quantity of hydrogen gas was compressed into a tank. The tank was itself highly insulated from the surroundings. Then the gas was allowed to escape from the tank in such a way that ordinary ions were prevented to escape from the tank.

Suppose due to motion, the charge on the proton differed from that on electron by one part in a billion. Thus each hydrogen molecule (composed of the protons) would carry a charge of $2 \times 10^{-9}\,e$ where e is the elementary charge unit ($= 1{\cdot}6 \times 10^{-19}$ coulomb). The experimental arrangement could measure a residual charge as small as $10^{-20}\,e$; but no charge was observed on hydrogen molecules in motion. Thus the experiment indicated that the hydrogen was electrically neutral even in motion. A similar experiment was performed with Helium atoms in which the motion of protons and electrons is very different. The protons move with kinetic energy of the order of million electron volt ; but Helium atoms were also found electrically neutral. Thus we may conclude that if the motion had any effect on the quantity of electrical charge, then the nuclear and electronic charges in both Hydrogen and Helium atoms, could not have exact cancellation.

This experiment indicates *that the motion has not any effect on the amount of charge and hence the charge is relativistically invariant.*

The principle of invariance of charge can also be experimentally verified by the optical spectra of isotopes of the same element. The motion of protons inside the nucleus is different in different isotopes, but no discrepancy is observed on comparing the spectral lines of different isotopes of the same element in the optical spectra.

15.5. Transformation of Charge Density

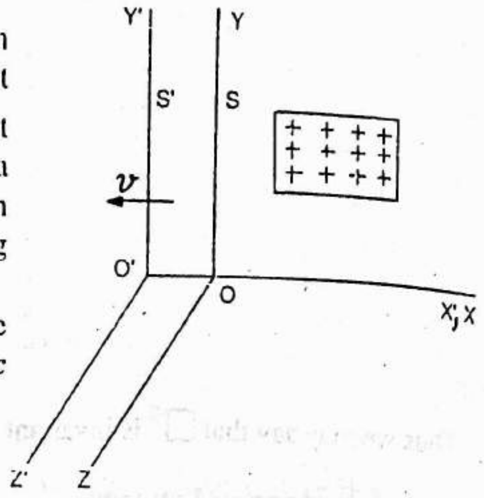


Fig. 15.2

Let there be two systems S and S', the latter moving with uniform velocity v relative to former along negative direction of X-axis. Let there be a stationary sheet of uniform charge density σ coulomb/m^2 at rest in system S having one edge parallel to X-axis. Let the sheet be a square of side l and placed parallel to X-Y plane. The observer in system S' will observe that the sheet is moving with velocity v along positive X-axis.

For getting the transformation relation of charge density we use the principle of invariance of charge which states. *"The total electric charge in an isolated system is relativistically invariant."*

The total charge as observed in system S.

$$= l^2\sigma \qquad \text{...(1)}$$

The observer in system S' will notice that the side of the square along X-axis has contracted from l to $l\sqrt{\left\{1-\frac{v^2}{c^2}\right\}}$; while the length along Y-axis has remained unchanged. Therefore the total charge as observed by observer in system S'

$$= l \cdot l\sqrt{\left[\left\{1-\frac{v^2}{c^2}\right\}\right]}\,\sigma' = l^2\sigma'\sqrt{\left\{1-\frac{v^2}{c^2}\right\}} \qquad \text{...(2)}$$

where σ' is the charge density in system S'.

According to principle of invariance of charge, from (1) and (2),

$$l^2\sigma = l^2\sigma'\sqrt{\left[\left\{1-\frac{v^2}{c^2}\right\}\right]}$$

or

$$\sigma' = \frac{\sigma}{\sqrt{\left[\left\{1-\frac{v^2}{c^2}\right\}\right]}} \qquad \text{...(3)}$$

This represents the transformation equation for charge density.

15.6. Electric Field Measured in Different Frames of Reference

Consider two system S and S', the latter moving with velocity v relative to former along negative direction of X-axis. Let there be two stationary sheets of charge of uniform charge densities $+\sigma$ and $-\sigma$ coulomb/m^2 respectively. Let these sheets be squares of side l and placed parallel to X-Y plane at such a small distance apart that the field between them may be taken to be uniform. The electric field exists only in the region between the sheets. The magnitude of the electric field everywhere between the sheets is σ/ε_0 and its direction is from positive to negative charged sheet *i.e.* along Z-axis.

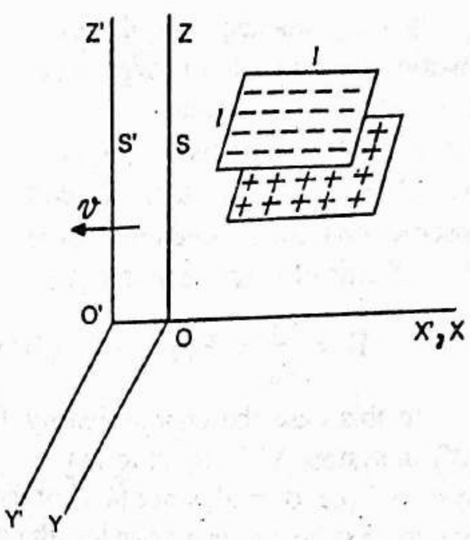


Fig. 15.3

Thus the electric field between the sheets has only Z-component given by

$$E_z = \frac{\sigma}{\varepsilon_0}$$

Hence the electric field in the region between the sheets is expressed as

$$\mathbf{E} = \frac{\sigma \mathbf{k}}{\varepsilon_0} = \mathbf{E}_x \mathbf{k} \qquad \text{...(1)}$$

Now let us think about the electric field $\mathbf{E}'$ in system S'. It is sure that the electric field $\mathbf{E}'$ is zero everywhere except the region between the sheets and there too it is uniform in the limit as the extent of the sheets become infinite. The electric field of an infinite uniform sheet neither depends on the distance from the sheet nor on the position along the sheet. As system S' is moving relative to sheets with velocity v along negative direction of X-axis, we may think it as sheets are moving relative to system S' with velocity v along positive direction of X-axis. Therefore if σ' and $-\sigma'$ are charge densities of positive and negative sheets in system S', then by the principle of invariance of charge.

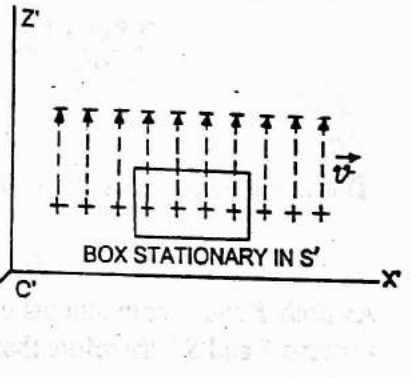


Fig. 15.4

$$\sigma' = \frac{\sigma}{\sqrt{\left\{1 - \frac{v^2}{c^2}\right\}}} \qquad \text{...(2)}$$

Also the field of infinite positive sheet of charge may be represented as in fig. (15·5a). Likewise the field of a sheet of negative charge would be represented as in fig. (15·5b). The superposition of these two fields would give the field of pattern shown in fig. 15·4.

Now consider a box type surface stationary in system S' (fig. 15·4). Applying Gauss's theorem to this surface and remembering that the electric field outside the sandwich is zero, we get E'_z, which is the only component inside the sandwich, to be given by

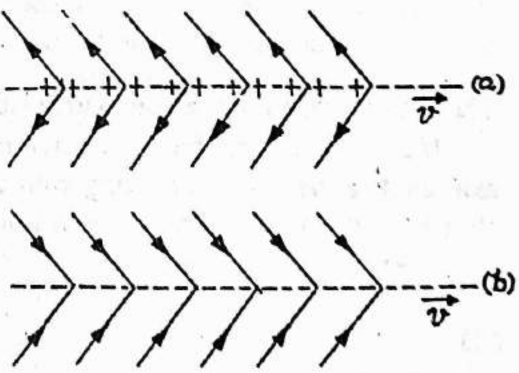


Fig. 15.5

$$E'_z = \frac{\sigma'}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0 \sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{E_z}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{...(3)}$$

Thus the electric field between the charge sheets as measured in system S' is expressed as

$$\mathbf{E}' = E'_z \mathbf{k} = \frac{\mathbf{E}_z}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \mathbf{k} = \frac{\sigma}{\varepsilon_0 \sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \mathbf{k} \qquad \text{...(4)}$$

Now consider a different situation in which the charge sheet stationary in system S is perpendicular to X axis. In this case the observer in system S will observe the field between the plates along X-axis of magnitude σ/ε_0 *i.e.*

$$\mathbf{E} = \frac{\sigma \mathbf{i}}{\varepsilon_0} = \mathbf{E}_x \mathbf{i} \qquad \text{...(5)}$$

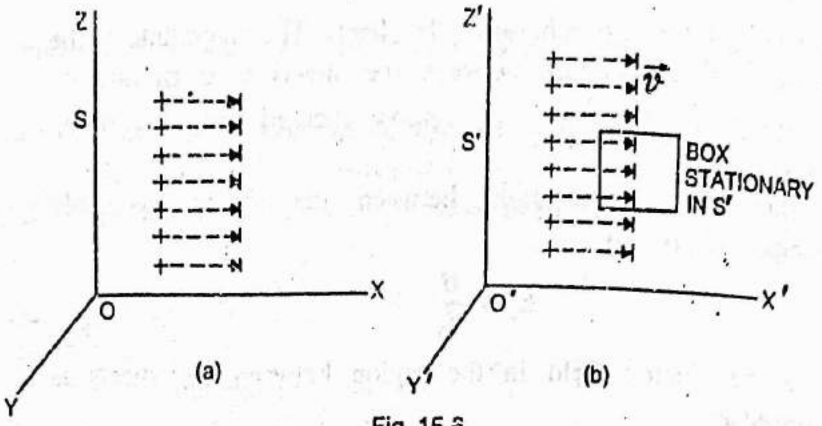


Fig. 15.6

In this case the charge density (σ') in system S' is equal to that in system S (*i.e.* $\sigma' = \sigma$), since none of the sides of the sheets is along the direction of motion (*i.e.* along X-axis) and there is no contraction in length perpendicular to the direction of motion.

Thus in this case due to motion only the distance between the sheets is contracted, which does not occur in the calculation of the field.

Now applying Gauss's theorem to a box stationary in system S' we get

$$\mathbf{E}' = \mathbf{E}_x' \mathbf{i} = \frac{\sigma'}{\varepsilon_0} \mathbf{i}$$

$$= \frac{\sigma \mathbf{i}}{\varepsilon_0} = E_x \mathbf{i} = \mathbf{E} \qquad \text{...(6)}$$

Thus,

$$E_x' = E_x = \frac{\sigma}{\varepsilon_0} \qquad \text{...(7)}$$

As both Y and Z components of electric field are perpendicular to the direction of relative motion between systems S and S', therefore they will transform in the same manner. Hence for Y-component of electric field we have

$$E_y' = \frac{E_y}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{...(8)}$$

The above results may be summarized as follows :

If **E** is the electric field measured in system S due to charges stationary in it and **E′** is the electric field measured in system S', moving with velocity **v** relative to system S in any direction, then resolving the electric field along and transverse to velocity **v**, we have

$$E_\parallel' = E_\parallel .$$

and

$$E_\perp' = \frac{E_\perp}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad \text{...(10)}$$

where $E_\parallel$ and $E_\parallel'$ are the components of electric field along the direction of velocity **v** in systems S and S' respectively ; and $E_\perp$ and $E_\perp'$ are the corresponding components perpendicular to velocity **v**. Equations (9) and (10) are used to transform $E_\parallel$ and $E_\perp$ arising only due to stationary charges into $E_\parallel'$ and $E_\perp'$ respectively.

Ex. 3. *An electric field arising due to stationary charge is given by*

$$\mathbf{E} = 400\,\mathbf{i} + 500\,\mathbf{j} \text{ volt/metre.}$$

How will you represent this field in a frame moving relative to charges with velocity $\mathbf{v} = (4\mathbf{i}+3\mathbf{j}) \times 10^7$ *metres/sec ?*

Sol. The unit vector along velocity **v** is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(4\,\mathbf{i}+3\,\mathbf{j}) \times 10^7}{5 \times 10^7} = \frac{4\,\mathbf{i}+3\,\mathbf{j}}{5}$$

The component of electric field **E** parallel to v is given by

$$E_{\parallel} = \mathbf{E} \cdot \hat{\mathbf{v}} = (400\,\mathbf{i}+500\,\mathbf{j}) \cdot \frac{(4\,\mathbf{i}+3\,\mathbf{j})}{5} = 620 \text{ volt/metre.}$$

Thus the component or **E** parallel to **v** is vectorially represented as

$$\mathbf{E}_{\parallel} = 620\left(\frac{4\,\mathbf{i}+3\,\mathbf{j}}{5}\right) = (496\,\mathbf{i}+372\,\mathbf{j}) \text{ volt/metre.}$$

Therefore

$$\mathbf{E}_{\perp} = \mathbf{E} - \mathbf{E}_{\parallel} = (420\,\mathbf{i}+500\,\mathbf{j}) - (496\,\mathbf{i}+372\,\mathbf{j})$$
$$= (-96\,\mathbf{i}+128\,\mathbf{j}) \text{ volts/metre.}$$

Now using transformation equations the component of **E′** parallel and perpendicular to velocity **v** are given by

$$E_{\parallel}' = E_{\parallel} = 496\,\mathbf{i} + 372\,\mathbf{j}$$

and

$$\mathbf{E}_{\perp}' = \frac{E_{\perp}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} = \frac{(-96\,\mathbf{i}+128\,\mathbf{j})}{\left[1-\left|\frac{(4\mathbf{i}+3\,\mathbf{j}) \times 10^7}{5 \times 3 \times 10^8}\right|^2\right]} = \frac{(-96\,\mathbf{i}+128\,\mathbf{j})}{\left\{1-\frac{0{\cdot}25}{9}\right\}}$$
$$= 1{\cdot}028\,(-96\,\mathbf{i}+128\,\mathbf{j}) = (-98{\cdot}74\,\mathbf{i}+131{\cdot}6\,\mathbf{j}) \text{ volt/metre.}$$

Thus electric field in a frame moving with velocity **v** is represented as

$$\mathbf{E}' = \mathbf{E}_{\parallel}' + \mathbf{E}_{\perp}' = (496\,\mathbf{i}+372\,\mathbf{j}) + (-98{\cdot}74\,\mathbf{i}+131{\cdot}6\,\mathbf{j})$$
$$= (397{\cdot}26\,\mathbf{j}+523{\cdot}6\,\mathbf{j}) \text{ volt/metre.}$$

15.7. Minkowski Space : Introduction

According to classical physics (Newtonian Mechanics), the time co-ordinate is unaffected by a tranformation from one inertial system to another. The time co-ordinate t' of one inertial system does not depend on the space co-ordinates x, y, z, of another inertial system, the transformation equation for time being $t' = t$. In relativistic physics, however, space and time are interdependent. The time co-ordinates of one inertial system depends on both the time and the space co-ordinates of another inertial system, the transformation equation being $t' = \dfrac{t-(vx/c^2)}{\sqrt{\{1-(v^2/c^2)\}}}$ Hence, instead of treating space and time co-ordinates separately, as is quite properly done is classical physics, it is natural in relativistic physics to treat them together. H. Minkowski was first to show clearly how this could be done.

According to Minkowski, the external world is not formed of ordinary three dimensional space known as Euclidean space ; but it is four dimensional space time continuum known as *Minkowski space*, where the time or more conventially *ict* may be regarded to be fourth dimension. Thus an event in Minkowski space can be represented by four co-ordinates (x_1, x_2, x_3, x_4) out of which the first three are space co-ordinates. This four dimensional Minkowski space can more conveniently be represented $(3+1)$ dimensional space-time continuum. The events in this four dimensional space or Minkowski space are represented by the points known as *world points* and each particle corresponds to a certain line known as *world line*.

For simplicity we shall consider only one space axis, the X-axis and shall ignore the Y and Z-axes. By this simplification no generality is lost and by this procedure we are able to focus more clearly on the interdependence of space and time and its geometrical representation. The co-ordinates of an event are then represented by (x, t). All possible space-time co-ordinates can be represented on a space time diagram in which the space axis is horizontal and the time-axis is vertical. For convenience the dimensions of the co-ordinates are kept the same ; this is easily done by using $m = ct$ instead of c. Then the Lorentz transformation equations for x and t can be written as follows :

$$\left.\begin{aligned} x' &= \frac{x-\beta m}{\sqrt{(1-\beta^2)}}, \ x = \frac{x'+\beta m'}{\sqrt{(1-\beta^2)}} \\ m' &= \frac{m-\beta x}{\sqrt{(1-\beta^2)}}, \ m = \frac{m'+\beta x'}{\sqrt{(1-\beta^2)}} \end{aligned}\right] \quad \text{with } \beta = v/c \qquad ...(1)$$

Obviously Lorentz transformation equations in this form possess symmetry. In Minkowski space if x and m axes of a frame are orthogonal to one another, X-axis being horizontal and m axis being vertical in a frame S, then the motion of a particle in this frame is represented by a curve known as world line, which gives the loci of space-time points, corresponding to the motion. The tangent to the world line at any point is always inclined at an angle less than 45° with the time axis ; since this angle is given by

$$\tan\alpha = \frac{1}{c}\frac{dx}{dt} = \frac{u}{c}$$

and we must have $u < c$ for a material particle. The world line for light signal (*i.e.* $u = c$) is a striaght line making an angle 45° with the axes.

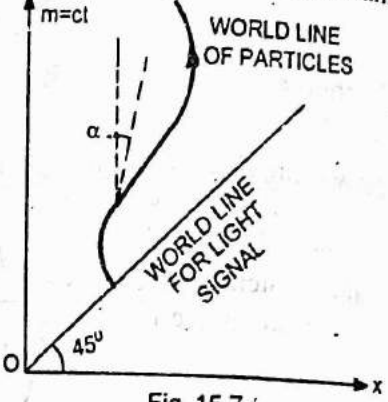


Fig. 15.7

15.8. Geometrical Interpretation of Lorentz Transformations of Space and time.

Consider two axes Ox and Op, O being the origin where $(p = ict)$.

Since we know that $x^2 - c^2t^2$ is invariant under Lorentz transformations, therefore $x^2 + p^2$ is invariant under Lorentz transformations. This means that the distance of variable point P in (x, p) plane from origin is unchanged. We have to show that a rotation of the (x, p) system of axes gives Lorentz transformations.

If Ox and Op are rotated through an angle θ, such that On and Op take the positions Ox' and Op', then radius vector in both systems is given by

$$\mathbf{r} = \mathbf{i}x + \mathbf{p}\hat{p} = \mathbf{i}'x' + \hat{\mathbf{p}}'p' \qquad ...(1)$$

where $\mathbf{i}, \hat{\mathbf{p}}, \mathbf{i}'$ and $\hat{\mathbf{p}}'$ are unit vectors along their respective directions.

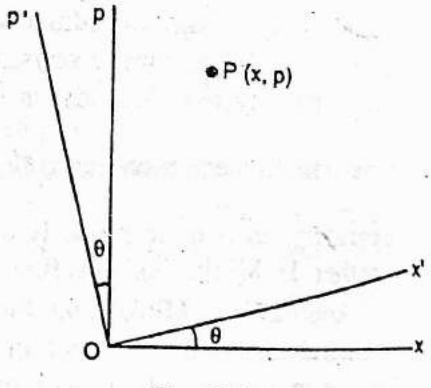


Fig. 15.8

Now from fig. 15·8 we have

$$\mathbf{i} = \mathbf{i}'\cos\theta - \hat{p}'\sin\theta.$$

and

$$\hat{p} = \hat{p}'\cos\theta + \mathbf{i}'\sin\theta.$$

Putting these values in equation (1) and equating the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{p}}'$, on either sides, we have

$$\left.\begin{aligned} p' &= -x\sin\theta + p\cos\theta \quad ...(a) \\ \text{and} \quad x' &= x\cos\theta + p\sin\theta \quad ...(b) \end{aligned}\right\} \qquad ...(2)$$

which satisfy the condition $x'^2 + p'^2 = x^2 + p^2$.

Now if we put $\tan\theta = +i\dfrac{v}{c} = +i\beta$

i.e. $$\sin\theta = \frac{+i\beta}{\sqrt{(1-\beta^2)}} \text{ and } \cos\theta = \frac{1}{\sqrt{(1-\beta^2)}}$$

in equation (2), we get from (2a)

$$p' = -\frac{ix\beta}{\sqrt{(1-\beta^2)}} + \frac{p}{\sqrt{(1-\beta^2)}} = \frac{p - ix\beta}{\sqrt{(1-\beta^2)}}$$

or $$ict' = \frac{ict - ix\,(v/c)}{\sqrt{(1-\beta^2)}} \quad (\text{since } p = ict \text{ and } p' = ict')$$

$$\therefore \quad t' = \frac{t - (xv/c^2)}{\sqrt{(1-\beta^2)}}$$

From (2b) $$x' = \frac{x}{\sqrt{(1-\beta^2)}} + \frac{ip\beta}{\sqrt{(1-\beta^2)}}$$

$$\therefore \quad x' = \frac{x + i\,ict\,.\,v/c}{\sqrt{(1-\beta^2)}} = \frac{x - vt}{\sqrt{(1-\beta^2)}}$$

The equations (3) and (4) gives Lorentz transformation equations.

Thus Minkowski discovered a very important relationship for transformation from one system S to another system S', moving with velocity v relative to system S. According to this relationship Lorentz transformations are equivalent to rotation of axes in four dimensional space (x, y, z, p) or (x, y, z, ict) through an hypothetical angle $\theta \left\{ = \tan^{-1}\frac{iv}{c} = \tan^{-1} i\beta \right\}$.

15.9. Four Vectors

Having introducing the idea of four dimensional space, it is possible to extend ordinary vector analysis to four dimensions to derive generally valid laws in the form of equations between four dimensional vectors. These four-dimensional vectors are called *four-vectors* or *world vectors.*

As the four-dimensional co-ordinates are orthogonal, we must have

$$\hat{\mathbf{i}}\,.\,\hat{\mathbf{j}} = 1,\ \hat{\mathbf{j}}\,.\,\hat{\mathbf{j}} = 1,\ \hat{\mathbf{k}}\,.\,\hat{\mathbf{k}} = 1,\ \hat{p}\,.\,\hat{p} = 0$$

and $$\hat{\mathbf{i}}\,.\,\hat{\mathbf{j}} = 0,\ \hat{\mathbf{i}}\,.\,\hat{\mathbf{k}} = 0,\ \hat{\mathbf{i}}\,.\,\hat{p} = 0,\ \hat{\mathbf{j}}\,.\,\hat{\mathbf{p}} = 0,\ \hat{\mathbf{k}}\,.\,\hat{\mathbf{p}} = 0$$

and so on. The world vectors are denoted by bars beneath their symbols.

If $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are two world vectors, then

$$\underline{\mathbf{A}} = \hat{\mathbf{i}}A_x + \hat{\mathbf{j}}A_y + \hat{\mathbf{k}}A_z + \hat{p}A_p$$
$$\underline{\mathbf{B}} = \hat{\mathbf{i}}B_x + \hat{\mathbf{j}}B_y + \hat{\mathbf{k}}B_z + \hat{p}B_p$$

The transformation law in Minkowski four dimensional space for the set of four co-ordinates (x_1, x_2, x_3, p) in one frame of reference to the set of four co-ordinates (x_1', x_2', x_3', p') in another frame moving with velocity v relative to the first frame is given by

$$x_\mu' = \sum_\nu a_{\mu\nu} x_\nu \qquad \text{...(1)}$$

with the constraint

$$x_1^2 + x_2^2 + x_3^2 + p^2 = x_1'^2 + x_2'^2 + x_3'^2 + p'^2$$

Any set of four components A_μ which transform under Lorentz transformations like the four-components (x_1, x_2, x_3, p) is called a *four vector.* The components A_μ of four vector $\underline{\mathbf{A}}$ transform under Lorentz transformation according to (with $A_0 = iA_4$)

$$A_\mu' = \sum_{\nu=0}^{3} a_{\mu\nu} A_\nu \qquad ...(2)$$

the quantities $a_{\mu\nu}$ being given by eqn. (1).

The scalar product of two world vectors $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ is defined as

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z + A_p B_p$$

which may also be written as

$$\sum_{\mu} A_\mu B_\mu = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_0 B_0$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3 - A_4 B_4 = \mathbf{A} \cdot \mathbf{B} - A_4 B_4$$

where

$$A_\mu = (A_1, A_2, A_3, A_0) \text{ or } (A_1, A_2, A_3, iA_4) \qquad ...(3)$$

It can easily be verified that the scalar product given by eqn. (3) is invariant quantity, *i.e.*,

$$\sum_{\mu} A_\mu' B_\mu' = A_\lambda B_\lambda = \text{constant} \qquad ...(4)$$

Since

$$\sum_{\mu} A_\mu' B_\mu' = \sum_{\mu} \sum_{\lambda} \sum_{\nu} a_{\mu\lambda} A_\lambda a_{\mu\nu} B_\nu$$

$$= \sum_{\lambda} \sum_{\nu} \left(\sum_{\mu} a_{\mu\nu} a_{\mu\lambda}\right) A_\lambda B_\nu$$

$$= \sum_{\lambda} \sum_{\nu} \delta_{\lambda\nu} A_\lambda B_\nu \qquad \left(\text{as } \sum_{\mu} a_{\mu\nu} a_{\mu\lambda} = \delta_{\lambda\nu}\right)$$

$$= \sum_{\lambda} A_\lambda B_\lambda$$

(Here $\delta_{\lambda\nu}$ is Kronecker delta symbol)

***i.e.* the scalar product of two four vectors in invariant quantity.**

The definition of vector product of two world vectors is somewhat difficult, because the combination forms an antisymmetric tensor which in three dimensions is represented by a vector. In four dimensions an antisymmetric tensor has six components while a vector has only four.

The vector product of two world vectors $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$, *i.e.*, $\underline{\mathbf{A}} \times \underline{\mathbf{B}}$, is defined as an antisymmetric world tensor ϕ whose components are the determinants of the matrix

$$\begin{bmatrix} A_1 & A_2 & A_3 & A_0 \\ B_1 & B_2 & B_3 & B_0 \end{bmatrix}$$

Corresponding to three dimensional ∇-operator (del operator) we have a four dimensional operator $\Box$ (D' Alembertian) defined as

$$\Box = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} + \hat{p}\frac{\partial}{\partial p} \qquad ...(5)$$

Thus in four-dimensions, we have

$$\underline{\textbf{grad A}} = \Box \mathbf{A} = \mathbf{i}\frac{\partial A_x}{\partial x} + \mathbf{j}\frac{\partial A_y}{\partial y} + \mathbf{k}\frac{\partial A_z}{\partial z} + \hat{p}\frac{\partial A_p}{\partial p} \qquad ...(6)$$

$$\underline{\textbf{div A}} = \Box \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial A_p}{\partial p} \qquad ...(7)$$

$$\underline{\textbf{Curl A}} = \Box \times \mathbf{A} = \begin{Bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial p} \\ A_x & A_y & A_z & A_p \end{Bmatrix} \qquad ...(8)$$

As one of the components of any four vector is imaginary, square of four-vectors may be (+) ve or (−) ve, unlike ordinary vectors. If the square of magnitude of a four vector is greater than or equal to zero, the four-vector is said to be **space like** ; but if the magnitude is negative the four-vector is said to be **time like.** The position vector of a point in four-dimensional space is given by

$$\underline{\mathbf{r}} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z + \hat{p}\,p \qquad \text{...(9)}$$

Let us now consider a difference vector between two world points which may be either (+) ve or (−) ve. Let **R** be the difference vector defined as

$$\underline{\mathbf{R}} = \underline{\mathbf{r}_2} - \underline{\mathbf{r}_1}$$

where $\underline{\mathbf{r}_1}$ and $\underline{\mathbf{r}_2}$ are the position vectors of two world points respectively.

Then the magnitude of $\underline{\mathbf{R}}$ is given by

$$\underline{\mathbf{R}}^2 = |\underline{\mathbf{r}_2} - \underline{\mathbf{r}_1}|^2 = |\mathbf{r}_2 - \mathbf{r}_1|^2 + (p_2 - p_1)^2$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are ordinary position vectors.

But $\qquad p = ict$, *i.e.*, $p_1 = ict_1$ and $p_2 = ict_2$.

$$\therefore \qquad \mathbf{R}^2 = |\mathbf{r}_2 - \mathbf{r}_1|^2 - c^2 (t_2 - t_1)^2. \qquad \text{...(10)}$$

Thus $\underline{\mathbf{R}}$ will be **space like** if

$$|\mathbf{r}_2 - \mathbf{r}_1|^2 > c^2 (t_2 - t_1)^2 \qquad \text{...(11)}$$

and $\underline{\mathbf{R}}$ will be **time like** if.

$$|\mathbf{r}_2 - \mathbf{r}_1|^2 < c^2 (t_2 - t_1)^2 \qquad \text{...(12)}$$

Let the spatial axes be so oriented that the difference vector $(\mathbf{r}_2 - \mathbf{r}_1)$ is along x-axis and therefore fourth component of **R** may be written as

$$c\,(t_2' - t_1') = \frac{c\,(t_2 - t_1) - \dfrac{v}{c}(x_1 - x_1)}{\sqrt{\left\{1 - \dfrac{v^2}{c^2}\right\}}}$$

like

$$t' = \frac{t - \dfrac{vx}{c^2}}{\sqrt{\left\{1 - \dfrac{v^2}{c^2}\right\}}}$$

If $\underline{\mathbf{R}}$ is space like, then $c\,(t_2 - t_1) < x_2 - x_1$.

Therefore it is possible to find a velocity $v << c$ such that $ic\,(t_2' - t_1')$ vanishes. Then vanishing of $ic\,(t_2' - t_1')$, *i.e.* fourth component of $\underline{\mathbf{R}}'$, means that if the distance between two events is space-like it is possible to find reference system in which the two events are simultaneous.

15.10. Lorentz transformations of Space and Time in Four-vector Form

Consider two systems S and S', the latter moving with velocity v relative to former along positive direction of X axis. In Minkowski space let the co-ordinates of an event be represented by the quantities (x, y, z, ict) or x_μ ($\mu = 1, 2, 3, 4$) where

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \\ x_4 &= ict \end{aligned}\right\} \qquad \text{...(1)}$$

Lorentz transformations of space and time with $\beta = v/c$ are written as

$$x' = \frac{x - vt}{\sqrt{(1-\beta^2)}}$$

$$y' = y$$

$$z' = z$$

and
$$t' = \frac{t - vx/c^2}{\sqrt{(1-\beta^2)}}$$

or equivalently

$$\left.\begin{aligned} x_1' &= \frac{x_1 - vt}{\sqrt{(1-\beta^2)}} = \gamma(x_1 - i\beta x_4) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ \text{and} \quad x_4' &= \frac{\left(t - \frac{vx_1}{c^2}\right)}{\sqrt{(1-\beta^2)}} = \gamma(x_4 - i\beta x_1) \end{aligned}\right\} \text{with } \gamma = \frac{1}{\sqrt{(1-\beta^2)}} \qquad \text{...(2)}$$

Above transformations can be expressed as

$$\left.\begin{aligned} x_1' &= \gamma x_1 + 0.x_2 + 0.x_3 + i\beta\gamma\,.\,x_4 \\ x_2' &= 0.x_1 + 1.x_2 + 0.x_3 + 0.x_4 \\ x_3' &= 0.x_1 + 0.x_2 + 1.x_3 + 0.x_4 \\ x_4' &= +i\beta\,\gamma x_1 + 0.x_2 + 0.x_3 + \gamma.x_4 \end{aligned}\right] \qquad \text{...(3)}$$

These results can be written compactly by a single equation

$$x_\mu' = \alpha_{\mu\nu}\, x_\nu\,. \qquad \text{...(4)}$$

Where

$$\alpha_{\mu\nu} = \begin{array}{c|cccc} \mu \diagdown \nu \rightarrow & 1 & 2 & 3 & 4 \\ \hline \downarrow 1 & \gamma & 0 & 0 & i\beta\gamma \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & -i\beta\gamma & 0 & 0 & \gamma \end{array} \qquad \text{...(5)}$$

Equation (4) represents the Lorentz transforamtions of space and time in four-vector form.

Similarly Lorentz transforamtions of any four-vector A_μ ($\mu = 1, 3, 4$) may be expressed as

$$A_\mu' = a_{\mu\nu} A_\nu \qquad \text{...(6)}$$

15.11. Transformations for Charge and Current Densities

In relativistic physics the current density and the charge density can not be distinct and completely separable entities; because a charge distribution static in one inertial reference frame will appear as current distribution in the other moving inertial frame. In Minkowski space the charge density ρ and current density **j** are grouped together by current four-vector **j** or j_μ according to

$$j_\mu = (j\,,\, ic\rho)$$

which may be seen as follows :

Consider a volume element $d\tau = dx_1\, dx_2\, dx_3$, then the charge contained in this small volume is

$$dq = \rho\, dx_1\, dx_2\, dx_3 \qquad \text{...(1)}$$

Multiplying above equation by a four vector dx_μ, we get

$$dq\, dx_\mu = \rho\, dx_\mu\, dx_1\, dx_2\, dx_3 = \rho \frac{dx_\mu}{dt} dx_1\, dx_2\, dx_3\, dt \qquad ...(2)$$

As charge is invariant (*i.e.* a scalar); the L.H.S. of eqn. (2) is a four vector, so R.H.S. of this equation must be a four vector.

Further we know that four dimensional volume element is invariant *i.e.*

$$dx_1\, dx_2\, dx_3\, dx_4 = \text{invariant}$$

or

$$dx_1\, dx_2\, dx_3\, (icdt) = \text{invariant}$$

i.e.

$$dx_1\, dx_2\, dx_3\, dt = \text{invariant (or scalar)}$$

Therefore for R.H.S. of eqn. (2) to be a four-vector, $\rho \frac{dx_\mu}{dt}$ must be a four-vector. Let this four-vector be represented by j_μ, *i.e.*

$$j_\mu = \rho \frac{dx_\mu}{dt} \qquad ...(3)$$

So that

$$\left.\begin{aligned} j_1 &= \rho \frac{dx_1}{dt} = \rho u_1 \\ j_2 &= \rho \frac{dx_2}{dt} = \rho u_2 \\ j_3 &= \rho \frac{dx_3}{dt} = \rho u_3 \\ j_4 &= \rho \frac{dx_4}{dt} = \rho \frac{d\,(ict)}{dt} = ic\rho \end{aligned}\right\} \qquad ...(4)$$

i.e. the four-vector j_μ, known as **current four vector**, is represented as

$$j_\mu = (\mathbf{j}\,,\, ic\rho) \qquad ...(5)$$

As j_μ has been specified as a four-vector, it must transform from one inertial frame S to the other inertial frame S', moving with velocity v relative to S along x_1-axis under Lorentz transformations as

$$j_\mu' = \alpha_{\mu\nu} j_\nu \qquad ...(6a)$$

where

$$\alpha_{\mu\nu} = \begin{array}{c|cccc} \mu \backslash \nu \to & 1 & 2 & 3 & 4 \\ \downarrow\ 1 & \gamma & 0 & 0 & i\beta\gamma \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & -i\beta\gamma & 0 & 0 & \gamma \end{array} \qquad ...(6b)$$

So that

$$\begin{aligned} j_1' = \alpha_{1\nu} j_\nu &= \alpha_{11} j_1 + \alpha_{12} j_2 + \alpha_{13} j_3 + \alpha_{14} j_4 \\ &= \gamma j_1 + 0\,.\,j_2 + 0\,.\,j_3 + i\,\beta\gamma\, j_4 \\ &= \gamma \left[j_1 + i\frac{v}{c}(ic\rho) \right] \\ &= \frac{j_1 - v\rho}{\sqrt{(1-\beta^2)}} \end{aligned} \qquad ...(7a)$$

$$j_2' = \alpha_{2\nu} j_\nu = \alpha_{21} j_1 + \alpha_{22} j_2 + \alpha_{23} j_3 + \alpha_{24} j_4$$

$$= 0.j_1 + 1.j_2 + 0.j_3 + 0.j_4$$

$$= j_2 \qquad \ldots(7b)$$

$$j_3' = \alpha_{3\nu} j_\nu = \alpha_{31} j_1 + \alpha_{32} j_2 + \alpha_{33} j_3 + \alpha_{34} j_4$$

$$= j_2 \qquad \ldots(7c)$$

and

$$j_4' = \alpha_{4\nu} j_\nu = \alpha_{41} j_1 + \alpha_{42} j_2 + \alpha_{43} j_3 + \alpha_{44} j_4$$

$$= -i\beta\gamma j_1 + 0\,.\,j_2 + 0.j_3 + \gamma \cdot j_4$$

$$= \gamma (j_4 - i\beta j_1) \qquad \ldots(7d\ i)$$

or

$$ic\rho' = \gamma\left[ic\rho - \frac{iv}{c} j_1\right]$$

i.e.

$$\rho' = \frac{\rho - \frac{v}{c^2} j_1}{\sqrt{(1-\beta^2)}} \qquad \ldots(7d\ ii)$$

Thus the transformation equations for current and charge densities are

$$\left.\begin{aligned} j_1' &= \frac{j_1 - v\rho}{\sqrt{(1-\beta^2)}}\,; j_2' = j_2\,, j_3' = j_3 \\ \text{and}\quad \rho' &= \frac{\rho - \frac{v}{c^2} j_1}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad \ldots(8)$$

The *inverse transformations for current and charge densities* are obtained by replacing v by $-v$ and interchanging primed and unprimed quantities, *i.e.*,

$$\left.\begin{aligned} j_1 &= \frac{j_1' + v\rho'}{\sqrt{(1-\beta^2)}}\,, j_2 = j_2'\,, j_3 = j_3' \\ \text{and}\quad \rho &= \frac{\rho' + \frac{v}{c^2} j_1'}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad \ldots(9)$$

Discussion of Results.

(a) Equation of continuity in covariant form.

The continuity equation is

$$\operatorname{div} \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

This can be written as

$$\operatorname{div} \mathbf{j} + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

i.e.

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{\partial j_4}{\partial x_4} = (0 \text{ since } j_4 = ic\rho \text{ and } x_4 = ict)$$

i.e.

$$\frac{\partial j_\mu}{\partial x_\mu} = \square \cdot j_\mu = 0$$

where $\square \cdot = \dfrac{\partial}{\partial x_\mu}$ is the four dimensional divergence operator.

Equation (10) is covariant equation, *i.e.*, its form is unaltered under Lorentz transformations. This equation, expresses that the four divergence of the current four vector j_μ vanishes.

(b) Special case. When charge distribution is at rest in system S, then $\mathbf{j} = 0$. Therefore the transformation equations (8) take the form

$$j_1' = -v\rho\,, j_2' = 0\,, j_3' = 0 \text{ and } \rho' = \frac{\rho}{\sqrt{(1-\beta^2)}} \qquad \text{...(11)}$$

(c) Invariance of charge. If $d\tau' = dx_1'\, dx_2'\, dx_3'$ is the volume element in system S', then the charge contained in this volume in system S' is

$$\begin{aligned} dq' &= \rho'\, dx'_1\, dx_2'\, dx_3' = \frac{\rho'\, dx_1\, dx_2\, dx_3}{\sqrt{(1-\beta^2)}} \\ &= \rho\, dx_1\, dx_2\, dx_3 = dq \end{aligned}$$

i.e. charge measured in system S' is the same as that in system S *i.e. electric charge is invariant under Lorentz transformations.*

as $\rho' \left[= \frac{\rho}{\sqrt{(1-\beta^2)}} \right]$ the *electric charge density is not relativistically invariant.*

15.12. Transformations of Electromagnetic Potentials A and φ

The Lorentz condition is

$$\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \qquad \text{...(1)}$$

This equation suggests that we can define a four vector A_μ as

$$A_\mu = \left(\mathbf{A}\,, \frac{i\phi}{c}\right) \qquad \text{...(2)}$$

To prove that A_μ is a four vector, let us consider Maxwell's field equations*

$$\Box^2 \mathbf{A} = \mu_0 \mathbf{j} \qquad \text{...(3)}$$

and

$$\Box^2 \phi = -\frac{\rho}{\varepsilon_0} \qquad \text{...(4)}$$

These equations can be written as

$$\Box^2 A_1 = -\mu_0 j_1 \qquad \text{...(5a)}$$

$$\Box^2 A_2 = -\mu_0 j_2 \qquad \text{...(5b)}$$

$$\Box^2 A_3 = -\mu_0 j_3 \qquad \text{...(5c)}$$

$$\Box^2 \phi = -\frac{\rho}{\varepsilon_0} \qquad \text{...(5d)}$$

The last equation can be written as

$$\Box^2 A_3 = -\frac{i\rho}{c\varepsilon_0} = -\frac{ic\rho}{c^2\varepsilon_0} = -\mu_0\,(ic\rho)\left(\text{since } \mu_0\,\varepsilon_0 = \frac{1}{c^2}\right)$$

i.e.

$$\Box^2 A_4 = -\mu_0 j_4 \,(\text{since } j_4 = ic\rho)$$

---

* We have already seen that Maxwell's four equations are equivalent to equations in terms of electromagnetic potentials **A** and φ given by equations (3) and (4).

Thus complete set equation (5) is written as

$$\Box^2 A_\mu = -\mu_0 j_\mu \qquad (\mu = 1, 2, 3, 4) \quad \text{...(6)}$$

Now as $\Box^2$ is Lorentz invariant and j_μ is a four vector, A_μ must be a four vector. This four vector is known as *four vector potential* or *electromagnetic four potential.* This must transform according to

$$A_\mu' = \alpha_{\mu\nu} A_\nu \quad \text{...(7a)}$$

Where

$$\alpha_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \quad \text{...(7b)}$$

so that

$$\begin{aligned} A_1' = x_{1\nu} A_\nu &= \alpha_{11} A_1 + \alpha_{12} A_2 + \alpha_{13} A_3 + \alpha_{14} A_4 \\ &= \gamma A_1 + 0.A_2 + 0.A_3 + i\beta\gamma . A_4 \\ &= \gamma (A_1 + i\beta A_4) \\ &= \frac{A_1 + i\frac{v}{c}\left\{\frac{i\phi}{c}\right\}}{\sqrt{(1-\beta^2)}} \end{aligned}$$

or

$$A_1' = \frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}} \quad \text{...(8a)}$$

$$\begin{aligned} A_2' = \alpha_{2\nu} A_\nu &= \alpha_{21} A_1 + \alpha_{22} A_2 + \alpha_{23} A_3 + \alpha_{24} A_4 \\ &= 0.A_1 + 1.A_2 + 0.A_3 + 0.A_4 \end{aligned}$$

or

$$A_2' = A_2 \quad \text{...(8b)}$$

$$\begin{aligned} A_3' = \alpha_{3\nu} A_\nu &= \alpha_{31} A_1 + \alpha_{32} A_2 + \alpha_{33} A_3 + \alpha_{34} A_4 \\ &= 0.A_1 + 0.A_2 + 1.A_3 + 0.A_4 \end{aligned}$$

or

$$A_3' = A_3 \quad \text{...(8c)}$$

and

$$\begin{aligned} A_4' = \alpha_{4\nu} A_\nu &= (\alpha_{41} A_1 + \alpha_{42} A_2 + \alpha_{43} A_3 + \alpha_{44} A_4) \\ &= -i\beta\gamma A_1 + 0.A_2 + 0.A_3 + \gamma A_4 \\ &= \gamma (A_4 - i\beta A_1) \quad \text{...(8d i)} \end{aligned}$$

or

$$\frac{i\phi'}{c} = \frac{\frac{i\phi}{c} - i\frac{v}{c} A_1}{\sqrt{(1-\beta^2)}}$$

or

$$\phi' = \frac{\phi - v A_1}{\sqrt{(1-\beta^2)}} \quad \text{...(8d ii)}$$

Thus the transformation laws for electromagnetic potentials **A** and ϕ are written as

$$\left.\begin{aligned} A_1' &= \frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}},\ A_2' = A_2,\ A_3' = A_3 \\ \text{and} \qquad \phi' &= \frac{\phi - v A_1}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad ...(8)$$

The inverse transformation for electromagnetic potentials A and ϕ are obtained replacing v by $-v$ and interchanging primed and unprimed quantities, *i.e.*,

$$\left.\begin{aligned} A_1 &= \frac{A_1' + \frac{v}{c^2}\phi'}{\sqrt{(1-\beta^2)}},\ A_2 = A_2',\ A_3 = A_3' \\ \text{and} \qquad \phi &= \frac{\phi' + v A_1'}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad ...(9)$$

Equations (8) and (9) represent required transformations for four vector potential.

15.13. Lorentz Condition in Covariant Form

The Lorentz condition is

$$\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

This can be written as

$$\text{div}\,\mathbf{A} + \frac{\partial\,(i\phi/c)}{\partial\,(ict)} = 0$$

i.e.
$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0$$

i.e.
$$\frac{\partial A_\mu}{\partial x_\mu} = \square \cdot A_\mu = 0 \qquad ...(1)$$

where $\square \cdot = \dfrac{\partial}{\partial x^\mu}$ is four dimensional divergence operator.

Equation (1) is covariant equation and is known as *Lorentz condition in covariant form*. This equation expresses that the four-divergence of the electromagnetic four-potential vanishes.

15.14. Invariance (or covariance) or Maxwell Field Equations in terms of Four Vectors.

By invariance (or covariance) of Maxwell field equations we mean that these equations do not change their form by the change of frame of reference. Here we shall prove the invariance of Maxwell equations in terms of four vectors.

Maxwell field equations are

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \rho & &...(a) \\ \text{div}\,\mathbf{B} &= 0 & &...(b) \\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & &...(c) \\ \text{curl}\,\mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} & &...(d) \end{aligned}\right\} \begin{aligned} &\text{with} && \mathbf{D} = \varepsilon_0 \mathbf{E} \\ &\text{and} && \mathbf{B} = \mu_0 \mathbf{H} \\ & && \\ & && \end{aligned} \qquad ...(2)$$

In terms of electromagnetic potentials A and ϕ, these equations are written as

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{j}$$

$$\Box^2 \phi = -\frac{\rho}{\varepsilon_0} \qquad \text{...(3)}$$

with the Lorentz condition

$$\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \qquad \text{...(4)}$$

The electromagnetic four potential A_μ and the current four potential j_μ are defined as

$$A_\mu = \left(\mathbf{A}, \frac{i\phi}{c}\right) \qquad \text{...(5)}$$

$$j_\mu = (\mathbf{j}, ic\rho). \qquad \text{...(6)}$$

These four vectors obey the Lorentz transformations

$$A_\mu' = \alpha_{\mu\nu} A_\nu \qquad \text{...(7)}$$

$$j_\mu' = \alpha_{\mu\nu} j_\nu . \qquad \text{...(8)}$$

where

$$\alpha_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \qquad \text{...(9)}$$

i.e. the components of electromagentic four potential and current four potential transform according to the relations

$$\left.\begin{aligned} A_1' &= \frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}}, A_2' = A_2, A_3' = A_3 \\ \text{and} \quad A_4' &= \frac{A_4 - i\beta A_1}{\sqrt{(1-\beta^2)}} \text{ or } \phi' = \frac{\phi - vA_1}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad \text{...(10)}$$

$$\left.\begin{aligned} j_1' &= \frac{j_1 - v\rho}{\sqrt{(1-\beta^2)}}, j_2' = j_2, j_3' = j_3 \\ \text{and} \quad j_4' &= \frac{j_4 - i\beta j_1}{\sqrt{(1-\beta^2)}} \text{ or } \rho' = \frac{\rho - \frac{v}{c^2} j_1}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad \text{...(11)}$$

In terms of electromagnetic four potential A_μ and current four potential j_μ Maxwell's field equations take the form

$$\Box^2 A_\mu = -\mu_0 j_\mu \qquad \text{...(12)}$$

with the Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0. \qquad \text{...(13)}$$

The invariance, of Maxwell's equations requires that in any inertial frame S', equations (12) and (13) must retain the same form *i.e.* Maxwell's field equations will be invariant if

$$\Box'^2 A_\mu' = -\mu_0 j_\mu' .$$

with the condition ...

$$\frac{\partial A_\mu'}{\partial x_\mu'} = 0.$$

Let us now consider

(a) $\Box'^2 A_1' = \Box^2 A_1'$. [since $\Box'^2 = \Box^2$ from section 15·3]

$$= \Box^2 \left[\frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}}\right]$$ (from 10)

$$= \frac{\Box^2 A_1 - \frac{v}{c^2}\,\Box^2 \phi}{\sqrt{(1-\beta^2)}} = \frac{\left[-\mu_0 j_1 - \frac{v}{c^2}\left\{-\frac{\rho}{\varepsilon_0}\right\}\right]}{\sqrt{(1-\beta^2)}} = \frac{\mu_0\,(j_1 - v.\rho)}{\sqrt{(1-\beta^2)}}$$

(since $\mu_0\,\varepsilon_0 = 1/c^2$)

$= -\mu_0 j_1'$ (from 11)

i.e., $\Box'^2 A_1' = -\mu_0 j_1'$ ...(15 a)

(b) $\Box'^2 A_2' = \Box^2 A_2'$ (since $\Box'^2 = \Box^2$)

$= \Box^2 A_2$ (since $A_2' = A_2$ from 10)

$= -\mu_0 j_2$ (from 12)

i.e. $\Box'^2 A_2' = -\mu_0 j_2'$ (since $j_2' = j_2$ from 11) ...(15b)

(c) $\Box'^2 A_3' = -\Box^2 A_3$ (since $\Box'^2 = \Box^2$ and $A_3' = A_3$)

$= -\mu_0 j_3$ (from 12)

$= -\mu_0 j_3'$ [(since $j_3 = j_3'$ from (11)] ...(15c)

(d) $\Box'^2 A_4' = \Box^2 A_4'$ (since $\Box'^2 = \Box^2$)

$$= \Box^2 \left[\frac{A_4 - i\beta A_1}{\sqrt{(1-\beta^2)}}\right]$$ (from 10)

$$= \frac{\Box^2 A_4 - i\beta\,\Box^2 A_1}{\sqrt{(1-\beta^2)}} = \frac{-\mu_0 j_4 - i\beta\,(-\mu_0 j_1)}{\sqrt{(1-\beta^2)}}$$ (from 11)

$$= -\frac{\mu_0\,(j_4 - i\beta j_1)}{\sqrt{(1-\beta^2)}} = -\mu_0 j_4'$$ from (11)

i.e. $(\Box')^2 A_4' = -\mu_0 . j_4'$. ...(15d)

Equations (15 a), (15 b), (15 c) and (15 d) can be written in the form of a single equation as

$$\Box'^2 A_\mu' = -\mu_0 j_\mu'. \qquad ...(16)$$

Further we have

$$A_\mu' = \alpha_{\mu\nu} A_\nu$$

$$A_\mu' = \frac{\partial x_\mu'}{\partial x_\nu} A_\nu$$

or

$$\frac{\partial A_\mu'}{\partial x_\mu'} = \frac{\partial}{\partial x_\mu'}\frac{\partial x_\mu'}{\partial x_\nu} A_\nu$$

i.e.

$$\frac{\partial A_\mu'}{\partial x_\mu'} = \frac{\partial A_\nu}{\partial x_\nu} \qquad ...(17)$$

But from Lorentz condition (13) $\frac{\partial A_\nu}{\partial x_\nu} = 0.$

$$\frac{\partial A_\mu'}{\partial x_\mu'} = 0. \qquad ...(18)$$

Equations (16) and (18) are the same as equations (14). Thus equations (12) and (13) retain the same form when expressed in terms of primed quantities. Consequently ***Maxwell's field equations are invariant under Lorentz transformations.***

15.15. The Electromagnetic Field Tensor

The electromagnetic field vectors **E** and **B** are written in terms of electromagnetic potentials **A** and ϕ as

$$\mathbf{B} = \text{curl } \mathbf{A} \qquad ...(1)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi \qquad ...(2)$$

Here the electromagnetic vectors **E** and **B** are not four vectors; but their six components E_x, E_y, E_z (or E_1, E_2, E_3) and B_x, B_y, B_z (or B_1, B_2, B_3) may be used to develop an antisymmetric second rank tensor in the following manner.

From equation (2) *X*-component of **E** is written as

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x}. \qquad ...(3)$$

The R.H.S. of (3) can now be expressed in terms of the electromagnetic four potential A_μ as

$$E_x = -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial x_1}$$

or

$$\frac{iE_x}{c} = \frac{\partial A_1}{\partial (ict)} - \frac{\partial \left\{\frac{i\phi}{c}\right\}}{\partial x_1}$$

i.e.

$$\frac{iE_x}{c} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} \qquad ...(4)$$

Similarly

$$\frac{iE_y}{c} = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2} \qquad ...(5)$$

and

$$\frac{iE_z}{c} = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3}. \qquad ...(6)$$

Similarly the components of **B** can be expressed in terms of the electromagnetic four potential as

$$B_x = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \qquad ...(7)$$

$$B_y = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \qquad ...(8)$$

$$B_z = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \qquad ...(9)$$

From equations (4) — (9) it is clear that these equations can expressed by a single equation as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \quad \text{...(10a)}$$

with

$$\left.\begin{matrix} \frac{iE_x}{c} = F_{41} & F_{12} = B_z \\ \frac{iE_y}{c} = F_{42} & F_{23} = B_x \\ \frac{iE_z}{c} = F_{43} & F_{31} = B_y \end{matrix}\right\} \quad \text{...(10b)}$$

Also $F_{\mu\nu} = -F_{\nu\mu}$ and $F_{\mu\mu} = 0$.

i.e.,

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

$$= \begin{pmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -E_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_z & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{pmatrix} \quad \text{...(11)}$$

This tensor is called the ***electromagnetic antisymmetric tensor of rank two***. Equation (10 along with equation (11) represents equations from (4) to (9) as may be seen as follows :

Suppose we want to write equation for E_y. From equation (11) it is clear that E_y corresponds to

$$\mu = 4 \text{ and } \nu = 2 \text{ or } \mu = 2 \text{ and } \nu = 4.$$

So that equation (10a) becomes

$$F_{42} = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2} \text{ or } F_{24} = \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}$$

In the light of equation (11) both these equations yield

$$\frac{iE_y}{c} = \frac{\partial A_2}{\partial x_2} - \frac{\partial A_4}{\partial x_2}$$

which is same as field equation (5). In the similar manner we may write field equation for any component of **E** and **B**.

15.16. Maxwell's Equations In Covariance Four Tensor Form

Maxwell's field equations are

$$\left.\begin{matrix} \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{...(a)} \\ \text{curl } \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} & \text{...(b)} \\ \text{div } \mathbf{D} = \rho & \text{...(c)} \\ \text{div } \mathbf{B} = 0 & \text{...(d)} \end{matrix}\right\} \quad \text{...(1)}$$

and

In free space $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$; hence *Maxwell's field equations in free space* may be expressed as

$$\left.\begin{aligned} \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{...(a)} \\ \text{curl } \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{...(b)} \\ \text{div } \mathbf{E} &= \frac{\rho}{\varepsilon_0} && \text{...(c)} \\ \text{and} \qquad \text{div } \mathbf{B} &= 0 && \text{...(d)} \end{aligned}\right\} \qquad \text{...(2)}$$

Introducing the coordinates

$x = x_1$, $y = x_2$, $z = x_3$ and $ict = x_4$. Maxwell's field equations may be written as

$$\text{curl } \mathbf{E} + ic \frac{\partial \mathbf{B}}{\partial x_4} = 0 \qquad \text{...(3a)}$$

$$\text{curl } \mathbf{B} = \mu_0 \left\{ ic\varepsilon_0 \frac{\partial \mathbf{E}}{\partial x_4} + \mathbf{j} \right\}$$

or

$$\text{curl } \mathbf{B} - \frac{i}{c} \frac{\partial \mathbf{E}}{\partial x_4} = \mu_0 \mathbf{j} \qquad \text{...(3b)}$$

$$\text{div } \mathbf{E} = \rho/\varepsilon_0 \qquad \text{...(3c)}$$

and

$$\text{div } \mathbf{B} = 0 \qquad \text{...(3d)}$$

Now writing non-homogeneous pair of equations *i.e.*, equations (3b) and (3c) in terms of components, we get

$$0 + \frac{\partial B_z}{\partial x_2} - \frac{\partial B_y}{\partial x_3} - \frac{i}{c} \frac{\partial E_x}{\partial x_4} = \mu_0 j_1 \qquad \text{...(4a)}$$

$$-\frac{\partial B_z}{\partial x_1} + 0 + \frac{\partial B_x}{\partial x_3} - \frac{i}{c} \frac{\partial E_y}{\partial x_4} = \mu_0 j_2 \qquad \text{...(4b)}$$

$$\frac{\partial B_y}{\partial x_1} - \frac{\partial B_x}{\partial x_2} + 0 - \frac{i}{c} \frac{\partial E_z}{\partial x_3} = \mu_0 j_3 \qquad \text{...(4c)}$$

and

$$\frac{\partial E_x}{\partial x_1} + \frac{\partial E_y}{\partial x_2} + \frac{\partial E_z}{\partial x_3} + 0 = \frac{\rho}{\varepsilon_0}.$$

or

$$\frac{i}{c} \frac{\partial E_x}{\partial x_1} + \frac{i}{c} \frac{\partial E_y}{\partial x_2} + \frac{i}{c} \frac{\partial E_z}{\partial x_3} + 0 = \frac{i\rho}{c\varepsilon_0} = \mu_0 ic\rho \qquad \left(\text{since } \frac{1}{\mu_0 \varepsilon_0} = c^2 \right)$$

or

$$\frac{i}{c} \frac{\partial E_x}{\partial x_1} + \frac{i}{c} \frac{\partial E_y}{\partial x_2} + \frac{i}{c} \frac{\partial E_z}{\partial x_3} + 0 = \mu_0 j_4 \qquad \text{...(4d)}$$

Consider j_1, j_2, j_3, j_4 in the R.H.S. of equations (4) as the components of current four vector j_μ and introducing electromagnetic antisymmetric tensor defined by

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{pmatrix} 0 & B_z & -B_y & -\dfrac{iE_x}{c} \\ -B_z & 0 & B_x & -\dfrac{iE_y}{c} \\ B_y & -B_z & 0 & -\dfrac{iE_z}{c} \\ \dfrac{iE_x}{c} & \dfrac{iE_y}{c} & \dfrac{iE_z}{c} & 0 \end{pmatrix} \qquad ...(5)$$

Now equations (4a), (4b), (4c) and (4d) may be written in a more compact form by a single equation

$$\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu \qquad ...(6)$$

For example if $\mu = 1$, eqn. (6) takes the form

$$\sum_{\nu=1}^{4} \frac{\partial F_{1\nu}}{\partial x_\nu} = \mu_0 j_1$$

i.e.

$$\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4} = \mu_0 j_1$$

i.e.

$$0 + \frac{\partial B_z}{\partial x_2} - \frac{\partial B_y}{\partial x_3} - \frac{i}{c}\frac{\partial E_x}{\partial x_4} = \mu_0 j_1$$

which is same as equation (4a). Similarly for $\mu = 2, 4$ and 3 ; we get equations (4b), (4c) and (4d) respectively.

Now writing homogeneous pair of Maxwell's field equations *i.e.* equations (3a) and (3d) in terms of components, we get

$$0 + \frac{\partial E_z}{\partial x_2} - \frac{\partial E_y}{\partial x_3} + ic\frac{\partial B_x}{\partial x_4} = 0 \qquad ...(7a)$$

$$-\frac{\partial E_z}{\partial x_1} + 0 + \frac{\partial E_x}{\partial x_3} + ic\frac{\partial B_y}{\partial x_4} = 0 \qquad ...(7b)$$

$$\frac{\partial E_y}{\partial x_1} - \frac{\partial E_x}{\partial x_2} + 0 + ic\frac{\partial B_z}{\partial x_4} = 0 \qquad ...(7c)$$

and

$$\frac{\partial B_x}{\partial x_1} + \frac{\partial B_y}{\partial x_2} + \frac{\partial B_z}{\partial x_3} + 0 = 0 \qquad ...(7d)$$

Dividing eqn. (7a), (7b) and (7c) by ic , these equations may be rewritten as

$$0 - \frac{i}{c}\frac{\partial E_z}{\partial x_2} + \frac{i}{c}\frac{\partial E_y}{\partial x_3} + \frac{\partial B_x}{\partial x_4} = 0 \qquad ...(a)$$

$$\frac{i}{c}\frac{\partial E_z}{\partial x_1} + 0 - \frac{i}{c}\frac{\partial E_x}{\partial x_3} + \frac{\partial B_y}{\partial x_4} = 0 \qquad ....(b) \qquad ...(8)$$

$$-\frac{i}{c}\frac{\partial E_y}{\partial x_1} + \frac{i}{c}\frac{\partial E_x}{\partial x_2} + 0 + \frac{\partial B_z}{\partial x_4} = 0 \qquad ...(c)$$

and

$$\frac{\partial B_x}{\partial x_1} + \frac{\partial B_y}{\partial x_2} + \frac{\partial B_z}{\partial x_3} + 0 = 0 \qquad ...(d)$$

Using electromagnetic field tensor $F_{\mu\nu}$, these equations can be written as

$$0+\frac{\partial F_{34}}{\partial x_2}+\frac{\partial F_{42}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_4}=0 \qquad \text{...(a)}$$

$$\frac{\partial F_{43}}{\partial x_1}+0+\frac{\partial F_{14}}{\partial x_2}+\frac{\partial F_{31}}{\partial x_4}=0 \qquad \text{...(b)} \qquad \text{...(9)}$$

$$\frac{\partial F_{24}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_2}+0+\frac{\partial F_{13}}{\partial x_4}=0 \qquad \text{...(c)}$$

and

$$\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{13}}{\partial x_2}+\frac{\partial F_{12}}{\partial x_3}+0=0 \qquad \text{...(d)}$$

All the equations (8) can be written by a single equation in tensor form as

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}=0 \qquad \text{...(10)}$$

for example if we take $\lambda=1, \mu=2, \nu=3$ we get

$$\frac{\partial F_{12}}{\partial x_3}+\frac{\partial F_{43}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}=0$$

or

$$\frac{\partial B_z}{\partial x_3}+\frac{\partial B_x}{\partial x_1}+\frac{\partial B_y}{\partial x_2}+0$$

i.e.

$$\frac{\partial B_x}{\partial x_1}+\frac{\partial B_y}{\partial x_2}+\frac{\partial B_z}{\partial x_3}=0$$

which is same as equation (8d).

As another example if we take $\lambda=1, \mu=2$ and $\nu=4$, we get

$$\frac{\partial F_{12}}{\partial x_4}+\frac{\partial F_{24}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_2}=0$$

i.e.

$$\frac{\partial B_z}{\partial x_4}-\frac{i}{c}\frac{\partial E_y}{\partial x_1}+\frac{i}{c}\frac{\partial E_x}{\partial x_2}=0$$

i.e.

$$-\frac{i}{c}\frac{\partial E_y}{\partial x_1}+\frac{i}{c}\frac{\partial E_x}{\partial x_2}+\frac{\partial B_z}{\partial x_4}=0$$

which is same as equation (8c). Similarly we get equations (8a) and (8b) from equation (10).

Thus we see that equations (6) and (10) represent Maxwell's field equations in terms of electromagnetic field tensor $F_{\mu\nu}$ defined by equation (5). As tensor equations are invariant under coordinate transformations, Maxwell's equations (6) and (10) and hence equations (2) are invariant under Lorentz transformations. Obviously equations (6) and (10) and consequently equations (4) and (8) represent *Maxwell's field equations in covariant form.*

Ex. 4. *Show that continuity equation is self contained in the homogeneous pair of Maxwell's field equations.*

Solution. The inhomogeneous pair of Maxwell's field equations in terms of electromagnetic tensor $F_{\mu\nu}$ is expressed as

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu}=\mu_0 j_\mu \qquad \text{(from eqn. 6 of section 15·16)} \qquad \text{...(1)}$$

where j_μ is the current four vector.

Differentiating eqn. (1) with respect to x_μ we get

$$\frac{\partial^2 F_{\mu\nu}}{\partial x_\mu \partial x_\nu} = \mu_0 \frac{\partial j_\mu}{\partial x_\mu} \qquad \text{...(2)}$$

Since $F_{\mu\nu}$ is antisymmetric, *i.e.* $F_{\mu\nu} = -F_{\nu\mu}$, we write equation (2) as

$$-\frac{\partial^2 F_{\nu\mu}}{\partial x_\mu \partial x_\nu} = \mu_0 \frac{\partial j_\mu}{\partial x_\mu} \qquad \text{...(3)}$$

Interchanging dummy indices* μ and ν in above equation, we get

$$-\frac{\partial^2 F_{\mu\nu}}{\partial x_\nu \partial x_\mu} = \mu_0 \frac{\partial j_\mu}{\partial x_\mu} \qquad \text{...(4)}$$

Using the property of perfect differentials

$$\left(i.e.\ \frac{\partial^2}{\partial x_\mu \partial x_\nu} = \frac{\partial^2}{\partial x_\nu \partial x_\mu} \right),$$

equation (4) can be written as

$$-\frac{\partial^2 F_{\mu\nu}}{\partial x_\mu \partial x_\nu} = \mu_0 \frac{\partial j_\mu}{\partial x_\mu} \qquad \text{...(5)}$$

Adding equations (2) and (5), we get

$$2\mu_0 \frac{\partial j_\mu}{\partial x_\mu} = 0$$

i.e.

$$\frac{\partial j_\mu}{\partial x_\mu} = 0$$

i.e.

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{\partial j_4}{\partial x_4} = 0$$

i.e.

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

i.e.

$$\text{div}\,\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

which is *continuity equation.*

15.17. Lorentz Transformation of Electric and Magnetic Fields

If the Maxwell's field equations are to be invariant under Lorentz transformations, then it is necessary that the electromagnetic field tensor $F_{\mu\nu}$ have the same form in all inertial frames,

i.e. if

$$F_{\mu\nu} = \frac{\partial A_\nu'}{\partial x_\mu'} - \frac{\partial A_\mu'}{\partial x_\nu'} \quad \text{in system } S'.$$

then we must have

$$F'_{\mu\nu} = \frac{\partial A_\nu'}{\partial x_\mu'} - \frac{\partial A_\mu'}{\partial x_\nu'} \quad \text{in system } S'. \qquad \text{...(2)}$$

This condition specifies the manner in which the electromagnetic field tensor components $F_{\mu\nu}$ behave under Lorentz transformation.

The transformations of x_μ and A_ν are written as

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* Please refer chapter 1, section 1·14 d.

$$x_\mu' = \alpha_{\mu\lambda} x_\lambda \qquad ...(3)$$

and

$$A_\nu' = \alpha_{\nu\sigma} A_\sigma . \qquad ...(4)$$

These inverse transformation of x_μ is written as

$$x_\lambda = \alpha_{\mu\lambda} x_\mu' \qquad ...(5)$$

i.e.

$$\frac{\partial x_\lambda}{\partial x_\mu'} = \alpha_{\mu\lambda}. \qquad ...(6)$$

Therefore

$$\begin{aligned} F_{\mu\nu} &= \frac{\partial A_\nu'}{\partial x_\mu'} - \frac{\partial A_\mu'}{\partial x_\nu'} \\ &= \alpha_{\nu\sigma}\frac{\partial A_\sigma}{\partial x_\mu'} - \alpha_{\mu\lambda}\frac{\partial A_\lambda}{\partial x_\nu'} && \text{[using 4]} \\ &= \alpha_{\nu\sigma}\frac{\partial A_\sigma}{\partial x_\lambda}\frac{\partial x_\lambda}{\partial x_\mu'} - \alpha_{\mu\lambda}\frac{\partial A_\lambda}{\partial x_\sigma}\cdot\frac{\partial x_\sigma.}{\partial x_\nu'} \\ &= \alpha_{\nu\sigma}\alpha_{\mu\lambda}\frac{\partial A_\sigma}{\partial x_\lambda} - \alpha_{\mu\lambda}\alpha_{\nu\sigma}\frac{\partial A_\lambda}{\partial x_\sigma} && \text{*(using 6)} \\ &= \alpha_{\mu\lambda}\alpha_{\nu\sigma}\left[\frac{\partial A_\sigma}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\sigma}\right] \end{aligned}$$

i.e.

$$F'_{\mu\nu} = \alpha_{\mu\alpha}\alpha_{\nu\sigma}F_{\lambda\sigma}. \qquad ...(7)$$

The coefficients $\alpha_{\mu\nu}$, when system S' moves relative to system S along positive x_1 axis, are given by

$$\alpha_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \qquad ...(8a)$$

i.e.

$$\begin{aligned} \alpha_{11} &= \alpha_{44} = \gamma = \frac{1}{\sqrt{(1-\beta^2)}} \\ \alpha_{22} &= \alpha_{33} = 1 \qquad ...(8b) \\ \alpha_{14} &= -\alpha_{41} = i\beta\gamma = \frac{B}{\sqrt{(1-\beta^2)}} = \frac{v/c}{\sqrt{\left\{1-\left(\frac{v^2}{c^2}\right)\right\}}} \end{aligned}$$

and all other coefficients are zero.

The components of electric and magnetic field vectors **E** and **B** in terms of electromagnetic field tensor $F_{\mu\nu}$ are exoressed as

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{pmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{pmatrix} \qquad ...(9)$$

Transformation equations for magnetic field induction components

(i) In equation (7) taking $\mu = 2$ and $\nu = 3$, we get

$$
\begin{aligned}
F_{23}' &= \alpha_{2\lambda}\,\alpha_{3\sigma}\,F_{\lambda\sigma} \\
&= \alpha_{2\lambda}\,[\alpha_{31}\,F_{\lambda 1} + \alpha_{32}\,F_{\lambda 2} + \alpha_{33}\,F_{\lambda 3} + \alpha_{34}\,F_{\lambda 4}] \\
&= \alpha_{21}\,[\alpha_{31}\,F_{11} + \alpha_{32}\,F_{12} + \alpha_{33}\,F_{13} + \alpha_{34}\,F_{14}] + \alpha_{22}\,[\alpha_{31}\,F_{21} + \alpha_{32}\,F_{22} + \alpha_{33}\,F_{23} + \alpha_{34}\,F_{24}] \\
&\quad + \alpha_{23}\,[\alpha_{31}\,F_{31} + \alpha_{32}\,F_{32} + \alpha_{33}\,F_{33} + \alpha_{34}\,F_{34}] + \alpha_{24}\,[\alpha_{31}\,F_{41} + \alpha_{32}\,F_{42} + \alpha_{23}\,F_{43} + \alpha_{34}\,F_{44}] \\
&= \alpha_{22}\,\alpha_{33}\,F_{23}
\end{aligned}
$$

since all other coefficients are zero, *i.e.*,

$$F_{23}' = F_{22} \text{ [since } \alpha_{22} = \alpha_{33} = 1]$$

or

$$B_x' = B_x \qquad \text{...(10a)}$$

(ii) In equation (7) taking $\mu = 2$, $\nu = 1$, we get

$$F_{31}' = \alpha_{2\lambda}\,\alpha_{1\sigma}\,F_{\lambda\sigma},$$

λ and σ being summation indices*. The only surviving coefficients are those for which

$$\lambda = 3, \sigma = 1 \quad \text{and} \quad \lambda = 3, \sigma = 4$$

$\therefore$

$$
\begin{aligned}
F_{31}' &= \alpha_{33}\alpha_{11}\,F_{31} + \alpha_{33}\alpha_{14}\,F_{34} \\
&= \gamma F_{31} + i\beta\gamma F_{34}
\end{aligned}
$$

i.e.

$$B_y' = \frac{B_y}{\sqrt{(1-\beta^2)}} + \frac{\left\{\frac{iv}{c}\right\}\cdot\left\{-\frac{iE_z}{c}\right\}}{\sqrt{(1-\beta^2)}}$$

i.e.

$$B_y' = \frac{B_y + \frac{v}{c}E_z}{\sqrt{(1-\beta^2)}} \qquad \text{...(10b)}$$

(iii) Now taking $\mu = 1$ and $\nu = 2$, we get

$$F_{12}' = \alpha_{1\lambda}\,\alpha_{2\sigma}\,F_{\lambda\sigma}.$$

The only surviving coefficients are those for which

$$\lambda = 1, \sigma = 2 \quad \text{and} \quad \lambda = 4, \sigma = 2.$$

$\therefore$

$$
\begin{aligned}
F_{12}' &= \alpha_{11}\,\alpha_{22}\,F_{12} + \alpha_{14}\,\alpha_{22}\,F_{42} \\
&= \gamma F_{12} + i\beta\gamma F_{42}
\end{aligned}
$$

i.e.

$$B_z' = \frac{B_z}{\sqrt{(1-\beta^2)}} + \frac{\left\{\frac{iv}{c}\right\}\cdot\left\{\frac{iE_y}{c}\right\}}{\sqrt{(1-\beta^2)}}$$

i.e.

$$B_z' = \frac{B_z - \frac{v}{c^2}E_y}{\sqrt{(1-\beta^2)}} \qquad \text{...(10c)}$$

Transformation equation for electric field strength components

(i) Taking $\mu = 1$ and $\nu = 4$, we get

$$F_{14}' = \alpha_{1\lambda}\,\alpha_{4\sigma}\,F_{\lambda\sigma}.$$

The only surviving coefficients are those for which

$$\lambda = 1, \sigma = 4, \quad \text{and} \quad \lambda = 4, \sigma = 1$$

$$
\begin{aligned}
F_{11}' &= \alpha_{11}\alpha_{44}F_{14} + \alpha_{22}\alpha_{41}F_{41} \\
&= \gamma\,.\,\gamma\,.\,F_{11} + (i\beta\gamma)\,(-i\beta\gamma)\,F_{41}
\end{aligned}
$$

---

[*] Please refer chapter 1, Section 1·14d.

i.e.
$$\left(-\frac{iE_x'}{c}\right) = \frac{\left(-\frac{iE_x}{c}\right)}{1-\beta^2} - \frac{\beta^2 \cdot \left(\frac{iE_x}{c}\right)}{1-\beta^2}$$

i.e.
$$E_x' = E_x\left(\frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2}\right) = E_x \cdot \frac{(1-\beta^2)}{(1-\beta^2)}$$

i.e.
$$E_x' = E_x . \qquad \text{...(11a)}$$

(ii) Taking $\mu = 2$ and $\nu = 4$, we get

$$F_{24}' = \alpha_{2\lambda}\,\alpha_{4\sigma}\,F_{\lambda\sigma} .$$

The only surviving coefficients are those for which

$$\lambda = 2\,, \sigma = 4 \quad \text{and} \quad \lambda = 2\,, \sigma = 1.$$

$$\therefore \quad F_{24}' = \alpha_{22}\,\alpha_{34}\,F_{24} + \alpha_{32}\,\alpha_{41}\,F_{21}$$
$$= \gamma F_{24} + (-i\beta\gamma)\,F_{21}$$

i.e.
$$\left\{-\frac{iE_y'}{c}\right\} = \frac{\left\{-\frac{iE_y}{c}\right\}}{\sqrt{(1-\beta^2)}} - \frac{\left\{i \cdot \frac{v}{c} . B_z\right\}}{\sqrt{(1-\beta^2)}}$$

i.e.
$$E_y' = \frac{E_y - vB_z}{\sqrt{(1-\beta^2)}}, \qquad \text{...(11b)}$$

(iii) Taking $\mu = 3$, $\nu = 4$, we get

$$F_{34}' = \alpha_{3\lambda}\alpha_{4\sigma}F_{\lambda\sigma} .$$

The only surviving coefficients are those for which

$$\lambda = 3\,, \sigma = 4 \quad \text{and} \quad \lambda = 3\,, \sigma = 1.$$

$$\therefore \quad F_{34}' = \alpha_{33}\alpha_{44}F_{34} + \alpha_{33}\alpha_{41}F_{31}$$
$$= \gamma F_{34} - i\beta\gamma\,F_{31}$$

i.e.
$$\left\{-\frac{iE_z'}{c}\right\} = \frac{-\frac{iE_z}{c}}{\sqrt{(1-\beta^2)}} - \frac{\left\{\frac{i\,v}{c}\right\} \cdot B_y}{\sqrt{(1-\beta^2)}}$$

i.e.
$$E_z' = \frac{E_z}{\sqrt{(1-\beta^2)}} + \frac{v\,B_x}{\sqrt{(1-\beta^2)}}$$

i.e.
$$E_z' = \frac{E_z + vB_y}{\sqrt{(1-\beta^2)}} . \qquad \text{...(11c)}$$

Equations (10a), (10b), (10c), and (11a), (11b), (11c) represent required transformation equations for electric and magnetic fields **E** and **B**.

Equations (10) and (11) can be inverted to give **inverse transformations** of electric and magnetic field vectors, *i.e.*,

$$B_x = B_x'\,, E_x = E_x'$$

$$B_y = \frac{B_y' - \frac{v}{c^2}\,E_z'}{\sqrt{(1-\beta^2)}}\,, E_y = \frac{E_y' + v\,B_y'}{\sqrt{(1-\beta^2)}} \qquad \text{...(12)}$$

$$B_z = \frac{B_z' + \frac{v}{c^2} E_y'}{\sqrt{(1-\beta^2)}}, \quad E_z = \frac{E_z' - v\,B_y'}{\sqrt{(1-\beta^2)}}$$

Aliter. Maxwell's field equations are

$$\left.\begin{aligned} \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{...(a)} \\ \text{curl } \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{...(b)} \\ \text{div } \mathbf{E} &= \frac{\rho}{\varepsilon_0} && \text{...(c)} \\ \text{div } \mathbf{B} &= 0 && \text{...(d)} \end{aligned}\right\} \quad \text{...(13)}$$

The equations when solved for **E** and **B** in terms of electromagnetic potential **A** and ϕ yield

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \text{...(14)}$$

and

$$\mathbf{B} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi \quad \text{...(15)}$$

As Maxwell's field equations are invariant under Lorentz transformations, then in system S', we must have

$$\mathbf{B}' = \text{curl } \mathbf{A}' \quad \text{...(16)}$$

and

$$\mathbf{E}' = -\frac{\partial \mathbf{A}'}{\partial t} - \text{grad } \phi' \quad \text{...(17)}$$

where the primes indicates that the corresponding quantities are functions of the primed co-ordinates.

The transformation of differential operators, from section 15·2, in terms of co-ordinates (x_1, x_2, x_3, ict) are written as

$$\frac{\partial}{\partial x_1'} = \frac{\frac{\partial}{\partial x_1} + \frac{v}{c^2}\frac{\partial}{\partial t}}{\sqrt{(1-\beta^2)}}$$

$$\frac{\partial}{\partial x_2'} = \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3'} = \frac{\partial}{\partial x_3} \quad \text{...(18)}$$

and

$$\frac{\partial}{\partial t'} = \frac{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x_1}}{\sqrt{(1-\beta^2)}}$$

Also the transformations of current four vector $(\mathbf{A}, i\phi/c)$ are given by

$$A_1' = \frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}}, A_2' = A_2, A_3' = A_3$$

and

$$\phi' = \frac{\phi - vA_1}{\sqrt{(1-\beta^2)}} \quad \text{...(19)}$$

To find the transformations of electric and magnetic field vector **B** and **E** given by eqn. (16) and (17) we write them in terms of components and then express their R.H.S. in terms of unprimed quantities

Equations (16) and (17) in terms of components are written as :

$$\begin{aligned} B_x' &= \frac{\partial A_3'}{\partial x_2'} - \frac{\partial A_2'}{\partial x_3'} \\ B_y' &= \frac{\partial A_2'}{\partial x_3'} - \frac{\partial A_3'}{\partial x_1'} \\ B_z' &= \frac{\partial A_2'}{\partial x_1'} - \frac{\partial A_1'}{\partial x_2'} \end{aligned} \qquad \text{...(20)}$$

and

$$\begin{aligned} E_x' &= -\frac{\partial A_1'}{\partial t'} - \frac{\partial \phi'}{\partial x_1'} \\ E_y' &= -\frac{\partial A_2'}{\partial t'} - \frac{\partial \phi'}{\partial x_2'} \\ E_z' &= -\frac{\partial A_3'}{\partial t'} - \frac{\partial \phi}{\partial x_3'} \end{aligned} \qquad \text{...(21)}$$

Transformation of magnetic field induction components :

(i) We have

$$B_x' = \frac{\partial A_3'}{\partial x_2'} - \frac{\partial A_2'}{\partial x_3'} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \qquad \text{using (18) and (19)}$$

i.e.

$$B_x' = B_x \qquad \text{...(22a)}$$

(ii)

$$\begin{aligned} B_y' &= \frac{\partial A_1'}{\partial x_3'} - \frac{\partial A_3'}{\partial x_1} \\ &= \frac{\partial}{\partial x_3}\left[\frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}}\right] - \frac{\left(\frac{\partial}{\partial x_1} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)}{\sqrt{(1-\beta^2)}} A_3 \text{ (using 18 and 19)} \\ &= \frac{1}{\sqrt{(1-\beta^2)}}\left[\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) - \frac{v}{c^2}\left(\frac{\partial \phi}{\partial x_3} + \frac{\partial A_3}{\partial t}\right)\right] \\ &= \frac{1}{\sqrt{(1-\beta^2)}}\left[B_y + \frac{v}{c^2}E_z\right] \end{aligned}$$

i.e.,

$$B_y' = \frac{B_y + \frac{v}{c^2}E_z}{\sqrt{(1-\beta^2)}} \qquad \text{...(22b)}$$

(iii)

$$\begin{aligned} B_z' &= \frac{\partial A_2'}{\partial x_1'} - \frac{\partial A_1'}{\partial x_2'} \\ &= \frac{\left(\frac{\partial}{\partial x_1} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)A_2}{\sqrt{(1-\beta^2)}} - \frac{\partial}{\partial x_2}\left(\frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}}\right) \\ &= \frac{1}{\sqrt{(1-\beta^2)}}\left[\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) + \frac{v}{c^2}\left(\frac{\partial A_2}{\partial t} + \frac{\partial \phi}{\partial x_2}\right)\right] \\ &= \frac{1}{\sqrt{(1-\beta^2)}}\left[B_z - \frac{v}{c^2}E_y\right] \end{aligned}$$

i.e.
$$B_z' = \frac{B_z - \frac{v}{c^2}E_y}{\sqrt{(1-\beta^2)}} \qquad ...(22c)$$

Transformation of electric field strength components.

(i)
$$E_x' = -\frac{\partial A_1'}{\partial t'} - \frac{\partial \phi'}{\partial x_1'}$$

$$= -\left[\frac{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x_1}}{\sqrt{(1-\beta^2)}}\right]\left[\frac{A_1 - \frac{v}{c^2}\phi}{\sqrt{(1-\beta^2)}}\right] - \left[\frac{\frac{\partial}{\partial x_1} + \frac{v}{c^2}\frac{\partial}{\partial t}}{\sqrt{(1-\beta^2)}}\right] \times \left[\frac{\phi - vA_1}{\sqrt{(1-\beta^2)}}\right]$$

which on simplification yields

$$E_x' = -\frac{\partial A_1}{\partial x_1} - \frac{\partial \phi}{\partial x_1}$$

i.e.
$$E_x' = E_x. \qquad ...(23a)$$

(ii)
$$E_y' = -\frac{\partial A_2'}{\partial t'} - \frac{\partial \phi'}{\partial x_2'}$$

$$= -\left[\frac{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x'}}{\sqrt{(1-\beta^2)}}\right]A_2 - \frac{\partial}{\partial x_2}\left[\frac{\phi - vA_1}{\sqrt{(1-\beta^2)}}\right]$$

$$= \frac{1}{\sqrt{(1-\beta^2)}}\left[\left(-\frac{\partial A_3}{\partial t} - \frac{\partial \phi}{\partial x_2}\right) - v\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\right]$$

$$= \frac{1}{\sqrt{(1-\beta^2)}}[E_y - vB_z]$$

i.e.
$$E_y' = \frac{E_y - vB_z}{\sqrt{(1-\beta^2)}} \qquad ...(23)$$

(iii)
$$E_z = -\frac{\partial A_3}{\partial t'} - \frac{\partial \phi'}{\partial x_3}$$

$$= -\left[\frac{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x_1}}{\sqrt{(1-\beta^2)}}\right]A_3 - \left\{\frac{\partial}{\partial x_3}\frac{\phi - vA_1}{\sqrt{(1-\beta^2)}}\right\}$$

$$= \frac{1}{\sqrt{(1-\beta^2)}}\left[\left(-\frac{\partial A_3}{\partial t} - \frac{\partial \phi}{\partial x_3}\right) + v\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right)\right]$$

$$= \frac{1}{\sqrt{(1-\beta^2)}}[E_z + vB_y]$$

i.e.
$$E_z' = \frac{E_z + vB_y}{\sqrt{(1-\beta^2)}}$$

The sets of equations (22) and (23) represent the required transformations of

15.18. Lorentz Force on a Charged Particle

Let S and S' be two systems, the latter moving with velocity v relative to former along positive direction of X-axis. Let a charged particle of charge q be placed in system S'. Thus the charged particle is moving with velocity $\mathbf{v} = v\mathbf{i}$ relative to system S.

As the particle is rest in system S', therefore the field in S' is purely electrostatic, *i.e.* $B_x' = B_y' = B_z' = 0$.

But the particle is moving relative to S, therefore the same field appears to be electromagnetic in system S.

According to Lorentz transformations, the force components transform as

$$F_x' = F_x, F_y' = \frac{F_y}{\sqrt{\left\{1-\frac{v^2}{c^2}\right\}}}, F_z' = \frac{F_z}{\sqrt{\left\{1-\frac{v^2}{c^2}\right\}}} \quad ...(1)$$

If $\mathbf{F}'$ is the electric field strength in system S', then force in system S' is $\mathbf{F}' = q\mathbf{E}$ (since charge is invariant under Lorentz transformations)

or in terms of components

$$F_x' = qE_x', F_y' = qE_y', F_z' = qE_z'. \quad ...(2)$$

Using (1), we get

$$F_x = F_x' = qE_x'$$

$$F_y = F_y'\sqrt{\left\{1-\frac{v^2}{c^2}\right\}} = qE_y'\sqrt{\left\{1-\frac{v^2}{c^2}\right\}}$$

$$F_z = F_z'\sqrt{\left\{1-\frac{v^2}{c^2}\right\}} = qE_z'\sqrt{\left\{1-\frac{v^2}{c^2}\right\}} \quad ...(3)$$

But according to Lorentz transformations of electric field components, we have

$$E_x' = E_x, E_y' = \frac{E_y - vB_z}{\sqrt{\left\{1-\frac{v^2}{c^2}\right\}}}, E_z' = \frac{E_z + vB_y}{\sqrt{\left\{1-\frac{v^2}{c^2}\right\}}} \quad ...(4)$$

Using (4), components of force from (3) are written as

$$\left.\begin{aligned} F_x &= qE_x & ...(i) \\ F_y &= q(E_x - vB_z) & ...(ii) \\ F_z &= q(E_z + vB_y) & ...(iii) \end{aligned}\right\} \quad ...(i)$$

Multiplying 5 (i), 5 (ii) and 5 (iii) by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively and adding, we get

$$(\mathbf{i}F_x + \mathbf{j}F_x + \mathbf{k}F_z) = qE_x\mathbf{i} + q(E_y - vB_z)\mathbf{j} + q(E_z + vB_y)\mathbf{k}$$

$$= (\mathbf{i}E_x + E_y\mathbf{j} + \mathbf{k}E_z) + q(vB_y\mathbf{k} - vB_z\mathbf{j})$$

or

$$\mathbf{E} = q\mathbf{E} + q\{v\mathbf{i} \times (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z)\} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

i.e.

$$\mathbf{E} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

This is required Lorentz force on a particle of charge q

15.19. The Invariants of the Electromagnetic Field

There are two invariants of the electromagnetic field which are

(i) $c^2B^2 - E^2$ (ii) $\mathbf{E} \cdot \mathbf{B}$.

(i) Invariance of $\mathbf{c}^2\mathbf{B}^2 - \mathbf{E}^2$.

According to transformations of electric and magnetic field components

$$B_x' = B_x \qquad E_x' = E_x$$

$$B_y' = \frac{B_y + \frac{v}{c^2}E_z}{\sqrt{(1-\beta^2)}}\ ;\ E_y' = \frac{E_y - vB_z}{\sqrt{(1-\beta^2)}}$$

$$B_z' = \frac{B_z - \frac{v}{c^2}E_y}{\sqrt{(1-\beta^2)}}\ ,\ E_z' = \frac{E_z + vB_y}{\sqrt{(1-\beta^2)}}$$

Therefore

$$\begin{aligned}
\mathbf{c}^2\mathbf{B}'^2 - \mathbf{E}'^2 &= c^2B_x'^2 + c^2B_y'^2 + c^2B_z'^2 - (E_x'^2 + E_y'^2 + E_z'^2) \\
&= c^2B_x^2 + c^2\left[\frac{B_y + \frac{v}{c^2}E_z}{\sqrt{(1-\beta^2)}}\right]^2 + c^2\left[\frac{B_z - \frac{v}{c^2}E_y}{\sqrt{(1-\beta^2)}}\right]^2 \\
&\qquad - \left[E_x^2 + \left\{\frac{E_y - vB_z}{\sqrt{(1-\beta^2)}}\right\}^2 + \left\{\frac{E_z + vB_y}{\sqrt{(1-\beta^2)}}\right\}^2\right] \\
&= c^2B_x^2 + c^2\left(\frac{B_y^2 + \frac{v^2}{c^4}E_z^2 + 2\frac{v}{c^2}B_yE_z}{1-\beta^2}\right) + c^2\left(\frac{B_z^2 + \frac{v^2}{c^4}E_y^2 - 2\frac{v}{c^2}E_yB_z}{1-\beta^2}\right) \\
&\qquad - \left[E_x^2 + \left(\frac{E_y^2 + v^2B_z^2 - 2vE_yB_z}{1-\beta^2}\right) + \left(\frac{E_z^2 + v^2B_y^2 + 2vE_zB_y}{1-\beta^2}\right)\right] \\
&= e^2B_x^2 + \frac{c^2B_y^2 + c^2B_z^2 - E_y^2 - E_z^2}{1-\beta^2} - \frac{v^2}{c^2}\left(\frac{-E_z^2 - E_y^2 + c^2B_z^2 + c^2B_y^2}{1-\beta^2}\right) - E_x^2 \\
&= c^2B_x^2 + \frac{c^2B_y^2 + c^2B_z^2 - E_y^2 - E_z^2}{1-\beta^2}\left(1 - \frac{v^2}{c^2}\right) - E_x^2 \\
&= c^2B_x^2 + c^2B_y^2 + c^2B_z^2 - E_x^2 - E_y^2 - E_z^2 \qquad \text{...(1)} \\
&= c^2B^2 - \mathbf{E}^2.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } \mathbf{E}' \cdot \mathbf{B}' &= (\mathbf{i}\,E_x' + \mathbf{j}\,E_y' + \mathbf{k}\,E_z') \cdot (\mathbf{i}\,B_x' + \mathbf{j}\,B_y' + \mathbf{k}\,B_z') \\
&= E_x'B_x' + E_y'B_y' + E_z'B_z'
\end{aligned}$$

$$= E_xB_x + \left\{\frac{E_y - vB_z}{\sqrt{(1-\beta^2)}}\right\}\cdot\left(\frac{B_y + \frac{v}{c^2}E_z}{\sqrt{(1-\beta^2)}}\right) + \left\{\frac{E_z + vB_y}{\sqrt{(1-\beta^2)}}\right\}\cdot\left(\frac{B_z - \frac{v}{c^2}E_y}{\sqrt{(1-\beta^2)}}\right)$$

$$= E_xB_x + \frac{E_yB_z + \frac{v}{c}E_yE_z - vB_zB_y - \frac{v^2}{c^2}B_zE_z}{1-\beta^2} + \frac{E_zB_z - \frac{v}{c^2}E_yE_z + vB_yB_z - \frac{v^2}{c^2}B_yE_y}{1-\beta^2}$$

$$= E_xB_x + \frac{E_yB_y + E_zB_z}{(1-\beta^2)}\left\{1 - \frac{v^2}{c^2}\right\}$$

$$= E_xB_x + E_yB_y + E_zB_z = \mathbf{E}\cdot\mathbf{B} \qquad ...(2)$$

From equation (1) and (2) it is obvious that $c^2\mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{E}\cdot\mathbf{B}$ are invariant under Lorentz transformations. *The invariance of these quantities indicates that the electromagnetic field of a plane wave appear similar to all inertial frames.*

15.20. Electromagnetic Field due to a Moving Charge

Consider two systems S and S', the latter moving with velocity v relative to former along positive direction of common X-axis. Consider that the origins of the two frames coincide at $t = t' = 0$. Let a charge q be situated at the origin in system S'. Thus the charge q is at rest in system S' and therefore it produces merely electric field $\mathbf{E}'$ in system S′. If there is any point P whose position vector is $\mathbf{r}'$ relative to system S' then electric and magnetic fields at P is system S' are given by

$$\mathbf{E}' = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}'}{r'^3} \qquad ...(1)$$

and

$$\mathbf{B}' = 0$$

or in terms of components

$$E_x' = \frac{1}{4\pi\varepsilon_0}\frac{qx'}{r'^3};\ B_x' = 0$$

$$E_y' = \frac{1}{4\pi\varepsilon_0}\frac{qy'}{r'^3};\ B_y' = 0 \qquad ...(2)$$

$$E_z' = \frac{1}{4\pi\varepsilon_0}\frac{qz'}{r'^3};\ B_z' = 0$$

As the charge q is moving relative to S with velocity v and the moving charge produces electromagnetic field, the charge q will produce an electromagnetic field if observed from system S.

The components of electric and magnetic fields (**E** and **B**) at point P as observed from system S are given by

$$\left.\begin{aligned} E_x &= E_x' \\ E_y &= \frac{E_y' + vB_z'}{\sqrt{(1-\beta^2)}} = \frac{E_y'}{\sqrt{(1-\beta^2)}} \\ E_z &= \frac{E_z' - vB_y'}{\sqrt{(1-\beta^2)}} = \frac{E_z'}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad ...(3)$$

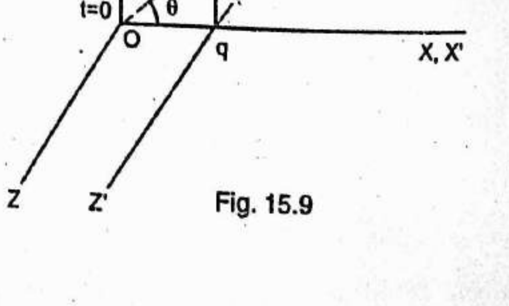

Fig. 15.9

$$B_x = B_x' = 0$$

$$B_y = \frac{B_y' - \frac{v}{c^2}E_z'}{\sqrt{(1-\beta^2)}} = -\frac{\frac{v}{c^2}E'_z}{\sqrt{(1-\beta^2)}}$$

$$B_z = \frac{B_z' + \frac{v}{c^2}E_y'}{\sqrt{(1-\beta^2)}} = \frac{\frac{v}{c^2}E_y'}{\sqrt{(1-\beta^2)}} \qquad \text{(since } B_x' = B_y' = B_z' = 0.)$$

Electric Field. According to Lorentz transformations of space and time, we have

$$x' = \frac{x - vt}{\sqrt{(1-\beta^2)}},\ y' = y,\ z' = z,\ t' = \frac{t - (vx/c^2)}{\sqrt{(1-\beta^2)}}, \qquad ...(5)$$

At time $t = 0$, when the charge q is instantaneously at the origin of system S, equation (5) yields

$$x' = \frac{x}{\sqrt{(1-\beta^2)}};\ y' = y,\ z' = z \text{ and } t' = -\frac{vx/c^2}{\sqrt{(1-\beta^2)}} \qquad ...(6)$$

If θ is the angle between X-axis and the line joining the point P to the charge q at $t = 0$, then

$$x = r\cos\theta$$

also

$$r^2 = x^2 + y^2 + z^2.$$

So that

$$r^2 = r^2\cos^2\theta + y^2 + z^2$$

i.e.

$$y^2 + z^2 = r^2 - r^2\cos^2\theta = r^2\sin^2\theta.$$

We have

$$r'^2 = x'^2 + y'^2 + z'^2.$$

So that at $t = 0$;

$$r'^2 = \frac{x^2}{1-\beta^2} + y^2 + z^2 = \frac{r^2\cos^2\theta}{1-\beta^2} + r^2\sin^2\theta$$

$$= \frac{r^2}{1-\beta^2}[1 - \beta^2\sin^2\theta]$$

i.e.

$$r' = \frac{r}{\sqrt{(1-\beta^2)}}[1 - \beta^2\sin^2\theta]^{1/2} \qquad ...(7)$$

Using (2), (3), (6) and (7), the components of electric field of charge q at $t = 0$ as seen in system S' are given by

$$E_x = E_x' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qx'}{r'^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{\{qx/\sqrt{(1-\beta^2)}\}}{\{r^3(1-\beta^2\sin^2\theta)^{3/2}/(1-\beta^2)^{3/2}\}}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{qx(1-\beta^2)}{r^3(1-\beta^2\sin^2\theta)^{3/2}} \qquad ...(8a)$$

$$E_y = \frac{E_y'}{\sqrt{(1-\beta^2)}} = \frac{1}{\sqrt{(1-\beta^2)}}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{qy'}{r'^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{qy(1-\beta^2)}{r^3(1-\beta^2\sin^2\theta)^{3/2}} \qquad ...(8b)$$

and
$$E_z = \frac{E'_z}{\sqrt{(1-\beta^2)}} = \frac{1}{\sqrt{(1-\beta^2)}} \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{qz'}{r'^3}$$
$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{qz\,(1-\beta^2)}{r^3\,(1-\beta^2 \sin^2\theta)^{3/2}} \qquad \text{...(8c)}$$

Combining equations (8a), (8b) and (8c), we get the electric field strength at point P in system S when the charge q is instantaneously at the origin of system S, as

$$\mathbf{E} = \mathbf{i}\,E_x + \mathbf{j}\,E_y + \mathbf{k}\,E_z$$
$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q\,(1-\beta^2)\,\mathbf{r}}{r^3\,(1-\beta^2 \sin^2\theta)^{3/2}} \qquad \text{...(9)}$$

where **r** is the position vector of point P relative to charge q which at this instant is at the origin of system S.

Magnetic Field.

The components of magnetic field induction at point P in system S, when q is instantaneously at the origin of system S

$$B_x = B'_x = 0, \qquad \text{...(10a)}$$
$$B_y = \frac{(v/c^2)\,E'_z}{\sqrt{(1-\beta^2)}} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{v/c^2}{\sqrt{(1-\beta^2)}} \cdot \frac{qz'}{r'^3}$$
$$= \frac{1}{4\pi\varepsilon_0 c^2} \cdot \frac{v}{\sqrt{(1-\beta^2)}} \cdot \frac{qz\,(1-\beta^2)^{3/2}}{(1-\beta^2 \sin^2\theta)^{3/2}}$$
$$= \frac{\mu_0}{4\pi} \frac{qvz\,(1-\beta^2)}{r^2\,(1-\beta^2 \sin^2\theta)^{3/2}} \qquad \text{...(10b)}$$
$$\left(\text{since } \mu_0\varepsilon_0 = \frac{1}{c^2} \text{ i.e. } \mu_0 = \frac{1}{c^2\varepsilon_0}\right)$$
$$B_z = \frac{(v/c^2)\,E'_y}{\sqrt{(1-\beta^2)}} = \frac{v/c^2}{\sqrt{(1-\beta^2)}} \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{qy}{r'^3}$$
$$= \frac{1}{4\pi\varepsilon_0 c^2} \cdot \frac{v}{\sqrt{(1-\beta^2)}} \frac{qy\,(1-\beta^2)^{3/2}}{r^3\,(1-\beta^2 \sin^2\theta)^{3/2}}$$
$$= \frac{\mu_0}{4\pi} \cdot \frac{qyv\,(1-\beta^2)}{r^3\,(1-\beta^2 \sin^2\theta)^{3/2}} \qquad \text{...(10c)}$$

Equations (10) may be written, in similar, form as

$$B_x = \frac{\mu_0}{4\pi} \cdot \frac{q\,(1-\beta^2)}{1-\beta^2 \sin^2\theta} \cdot \frac{(\mathbf{v}\times\mathbf{r})_x}{r^3}$$
$$B_y = \frac{\mu_0}{4\pi} \cdot \frac{q\,(1-\beta^2)}{(1-\beta^2 \sin^2\theta)} \cdot \frac{(\mathbf{v}\times\mathbf{r})_y}{r^3} \qquad \text{...(11)}$$

and
$$B_z = \frac{\mu_0}{4\pi} \cdot \frac{q\,(1-\beta^2)}{(1-\beta^2 \sin^2\theta)} \cdot \frac{(\mathbf{v}\times\mathbf{r})_z}{r^3}$$

Combining equations (11), the magnetic field induction **B** at point P in system S, when the charge q is instantaneously at the origin of system S, is given by

$$\mathbf{B} = \mathbf{i}\,B_x + \mathbf{j}\,B_y + \mathbf{k}\,B_z$$

$$= \frac{\mu_0}{4\pi}\frac{q(1-\beta^2)}{(1-\beta^2\sin^2\theta)}\cdot\frac{(\mathbf{v}\times\mathbf{r})}{r^3} \qquad \text{...(12)}$$

This is *relativistic Biot-Savart law for the magnetic field of a moving charge.*

When $\beta \ll 1$, eqn. (12) takes the form

$$\mathbf{B} = \frac{\mu_0}{4\pi}\cdot\frac{q(\mathbf{v}\times\mathbf{r})}{r^3} \qquad \text{...(13)}$$

This is *usual non-relativistic Biot-Savart law for the magnetic field of moving charge.*

15.21. Lorentz Force in Covariant Form

When a charged particle is placed in an electromagnetic field, it experiences a mechanical force given by Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \qquad \text{...(1)}$$

where q is electric charge and **u** is the velocity of the particle.

This equation can not be derived by Maxwell's equations and rather constitutes an extra postulate supported by experiments. But this equation is a natural consequence of the relativistic formulation of Maxwell's equations. To write Lorentz force in covariant from instead of eqn. (1) we consider the force acting on a unit volume of charge density ρ ; if **f** is the force per unit volume then eqn. (1) yields.

$$\mathbf{f} = \rho(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

i.e.

$$\mathbf{f} = \rho\,\mathbf{E} + \rho\,\mathbf{u}\times\mathbf{B}$$

i.e.

$$\mathbf{f} = \rho\,\mathbf{E} + \mathbf{j}\times\mathbf{B} \qquad \text{...(2)}$$

(since $\mathbf{j} = \rho\,\mathbf{u}$)

In terms of components above equation may be written as

$$\begin{aligned} f_1 &= \rho\,E_x + j_2B_z - j_3B_y \\ f_2 &= \rho\,E_y + j_3B_x - j_1B_z \\ f_3 &= \rho\,E_z + j_1B_y - j_2B_x \end{aligned} \qquad \text{...(3)}$$

Using electromagnetic field tensor $F_{\mu\nu}$ given by

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{pmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ B_z & 0 & B_x & -\frac{iE_y}{c} \\ -B_y & B_x & 0 & \frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{pmatrix} \qquad \text{...(4)}$$

equation (3) may be written as

$$\left.\begin{aligned} f_1 &= F_{11}j_1 + F_{12}j_2 + F_{13}j_3 + F_{14}j_4 \\ f_2 &= F_{21}j_1 + F_{22}j_2 + F_{23}j_3 + F_{24}j_4 \\ f_3 &= F_{31}j_1 + F_{32}j_2 + F_{33}j_3 + F_{34}j_4 \end{aligned}\right\} \qquad \text{..(5)}$$

where current four vector,

$$j_\nu = (\mathbf{j}, ic\rho).$$

Equation (5) can be written in the form of a single equation as

$$f_k = F_{k\nu} j_\nu \text{ (with } k = 1, 2, 3) \qquad \text{...(6)}$$

The right hand side of this equation is evidently the space component of a four vector : so f_μ must be a space component of a four vector f_μ ; such that

$$f_\mu = F_{\mu\nu} j_\nu \qquad \text{...(7)}$$

Here f_μ is called the *force-density vector.*

$$\left(\text{using eqn. 6 of section 15·6 } i.e. \frac{\partial F_{\mu\lambda}}{\partial x_\lambda} = \mu_0 j_\nu\right)$$

$$f_\mu = F_{\mu\nu} \cdot \frac{1}{\mu_0}\left(\frac{\partial F_{\nu\lambda}}{\partial x_\lambda}\right) = \frac{1}{\mu_0} F_{\mu\nu}\left(\frac{\partial F_{\nu\lambda}}{\partial x_\lambda}\right) \qquad \text{...(8)}$$

Equation (7) or (8) is a tensor equation ; hence it is invariant under Lorentz transformation, *i.e.* eqn. (7) or (8) represents the covariance form of the Lorentz force equation.

Physical meaning of the fourth component of the force density four vector.

The fourth component of the force density four vector is written as

$$\begin{aligned} f_4 &= f_{4\nu} j_\nu = f_{41} j_1 + f_{42} j_2 + f_{43} j_3 + f_{44} j_4 \\ &= \frac{i}{c} E_x j_1 + \frac{i}{c} E_y j_2 + \frac{i}{c} E_z j_3 + 0 \\ &= \frac{i}{c}(E_x j_1 + E_y j_2 + E_z j_3) \\ &= \frac{i}{c}(\mathbf{E} \cdot \mathbf{j}) = \frac{i}{c}\rho\,(\mathbf{E} \cdot \mathbf{u}) \end{aligned} \qquad \text{...(9)}$$

Obviously *fourth component of four-density four vector is* imaginary and along with the factor i/c, it *represents the amount of work done by the electric field on the charge per unit volume per unit time.*

Thus the Lorentz force equation in covariant form gives the rate of change of mechanical momentum per unit volume as its space part and mechanical energy per unit time as its time part.

Ex. 5. *Using the Lorentz force equation in covariant form, derive the transformation law for the force.*

Solution. The Lorentz force equation in covariant form is

$$f_\mu = F_\mu j_\nu \qquad \text{...(1)}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, j_μ is force density four vector and j_ν is current density four vector.

As in the rest frame of charges no work is done on moving charges, *i.e.*

$$f_4 = 0$$

So according to law of transformation of force density four vector,

i.e.,

$$f_\mu' = \alpha_{\mu\nu} f_\nu, \qquad \text{...(2)}$$

where

$$\alpha_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

we have

$$\left.\begin{aligned} f_1' &= \gamma f_1 \\ f_2' &= f_2 \\ f_3' &= f_3 \end{aligned}\right] \qquad \text{...(3)}$$

Now the components of the total force exerted on a given volume of the charge distribution are given by

$$F_k' = \int_{\tau'} f_k'\, d\tau'$$

So that

$$F_x' = F_1' = \int_{\tau'} f_1'\, d\tau' = \int_{\tau'} \gamma f_1\, d\tau \sqrt{(1-\beta^2)} \qquad [\text{since } d\tau' = d\tau\sqrt{(1-\beta^2)}]$$

$$\bullet = \int_{\tau} f_1\, d\tau = F_1 = F_x$$

$$F_y' = F_2' = \int_{\tau'} f_2'\, d\tau' = \int_{\tau} f_2\, d\tau \sqrt{(1-\beta^2)}$$

$$= \sqrt{(1-\beta^2)}\, F_2 = \sqrt{(1-\beta^2)}\, F_y$$

and

$$F_z' = F_3' = \int_{\tau'} f_2'\, d\tau' = \int_{\tau} f_3\, d\tau \sqrt{(1-\beta^2)}$$

$$= \sqrt{(1-\beta^2)}\, F_3 = \sqrt{(1-\beta^2)}\, F_z$$

i.e. the transformation law for force is

$$\left.\begin{aligned} F_x' &= F_x \\ F_y' &= F_y\sqrt{(1-\beta^2)} \\ F_z' &= F_z\sqrt{(1-\beta^2)} \end{aligned}\right] \qquad \text{...(4)}$$

Equation (4) can be written more convenietnly as

$$F_{\parallel}' = F_{\parallel}$$

and

$$F_{\perp}' = F_{\perp}\sqrt{(1-\beta^2)} \qquad \text{...(5)}$$

where $F_{\parallel}$ and $F_{\perp}$ are the components of the Lorentz force parallel and perpendicular to the direction of motion of the frame S'.

Equation (4) represent the required transformation law for the Lorentz force.

QUESTIONS AND PROBLEMS

1. (a) Prove that the Laplacian operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is not invariant under Lorentz transformations while the D' Alembertian operator

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

is invariant.

(b) Prove that three-dimensional volume element $dx\,dy\,dz$ is not invariant under Lorentz transformations while the four-dimensional volume element $dx\,dy\,dz$ is invariant. **(Meerut 1991; Rohilkhand 74)**

2. Show that the Maxwell's electromagnetic field equations are invariant under the Lorentz transforamtions. **(Meerut 1998, 82; Vikram 69; Agra 75, 67)**

3. If two reference frames are related to each other by the Lorentz transformations

$$x' = \frac{x - v\,t}{\sqrt{\left[1 - \dfrac{v^2}{c^2}\right]}},\ y' = y,\ z' = z$$

$$t' = \frac{t-(v\,x/c^2)}{\sqrt{\left[1-\frac{v^2}{c}\right]}}$$

how are the electromagnetic potential (ϕ , **A**) and electromagnetic field strength (**E** , **H**) in two systems related. **(Agra 1985)**

4. "All intertial frames are equivalent for the descritpion of nature". Explain and comment on the above statement with special reference to Maxwell's electromagnetic field equations. How did Einstein resolve to impasse between theory and experiment ? **(Kanpur 1984)**
5. State Maxwell's equations and show that they are invariant under Lorentz transformation. **(Kanpur 1984)**
6. Express Maxwell's equations in covariant form and derive the transformation equations for the electric and magnetic fields under Lorentz transformation. **(Meerut 1997; Kanpur 74; Agra 71)**
7. Show that Maxwell's equations are covariant in four dimensions. **(Meerut 1992)**
8. Writing Maxwell's equations in four dimensional formalism, show that they are invariant under Lorentz transformations.

   Show that the potential-four vector satisfies the wave equation under certain conditions. **(Banaras 1970)**
8. What is an inertial frame ? Show that under Lorentz transforamtions all inertial frames are equivalent for the description of electromagnetic phenomena. **(Agra 1976)**
10. (a) Show that Maxwell's equations in empty space are invariant with respect to Lorentz transformations. **(Kanpur 1975)**

    (b) Prove that the charge is not altered by the motion relative to an observer. **(Banaras 1995)**
11. (a) Prove that Lorentz covariance of the first pair of Maxwell's equations of the electromagnetic field

    $$\text{div } \mathbf{B} = 0 \text{ and curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

    (b) Using the tensor expression for the electromagnetic field components or otherwise find the transformation laws of the electric and magnetic fields. **(Agra 1993)**
12. Use the transformational properties of the field strength tensor to find the Lorentz transformation equations for the electric and magnetic fields. **(Meerut 1971)**
13. Give the Lorentz transformation of electrical and magnetic fields. **(Meerut 1992)**
14. (a) Write down the equation for the field tensor and express Maxwell equations in 4-tensor form.

    (b) Show that the equations in the above form in (a) are invariant under Lorentz transformations. **(Meerut 1972, 70)**
15. (a) Using the Lorentz transformation equations for the electric and magnetic fields, obtain the fields of a point charge q moving in a straight line path with a velocity v.

    (b) Show that $c^2\mathbf{B}^2 - \mathbf{E}^2$ in an invariant under Lorentz transformations. What is its form in four dimensional notation ? **(Meerut 1973)**
16. (a) Define the electromagnetic field four strength tensor and write its components explicitly. Show how Maxwell's equations can be written in covariant form. **(Meerut 1991, 74)**

    (b) Determine the Lorentz transformation equations for the field vectors **E** and **B** and show that the scalar product **E.B** is unchanged under a Lorentz transformation. **(Meerut 1974)**
17. What is a four vector ? Obtain Lorentz transformations of the components of current density and the charge density and thus show that they form a four vector. **(Rohilkhand 1980)**
18. What is meant by proper time of a particle ? Why is it introduced in the definition of four velocity ? How are the components of a four velocity vector related to the ordinary velocity of particle.

    Derive covariant form of Lorentz force equation. **(Meerut 1982)**
19. What are the invariants of the electromagnetic field ? Prove their invariance.
20. Show that $c^2B^2 - \mathbf{E}^2$ and **E.B** are invariant under Lorentz transformations. **(Meerut 1989)**
21. Express Lorentz condition and equation of continuity in covariant form.
22. Derive the Lorentz transformations for electromagentic potential four vector and current density four vector.
23. Write the Lorentz force equation in covariant form and explain the meaning of the fourth component of the force density four vector. **(Meerut 1972)**
24. Define electromagnetic field tensor and derive Maxwell's equations in tensor form. **(Banaras 1998)**

25. Show that the electromagnetic forces can be exhibited as components of a skew-symmetric tensor of second order in space-time co-ordinates. Hence find transformation for electromagnetic forces under Lorentz transformations. Show that Maxwell's equations are invariant under these transformations. **(Kanpur 1971)**
26. Write the Maxwell's equations in a covariant form and show that the electric and the magnetic induction are elements of a second rank antisymmetric field strength tensor. **(Meerut 1988)**
27. (a) Show that the Lorentz force on a single charged particle moving in an electromagnetic field with velocity **v** is given by

$$\mathbf{F} = e\,[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

where the symbols have their usual meaning.

(b) Derive the transformation equations for **E** and **B.** **(Agra 1977)**

28. Obtain the Lorentz force experienced by a moving charged particle in the presence of electric and magnetic fields by performing Lorentz transformation on the electrostatic force experienced by it in its rest frame. **(Meerut 1975)**
29. Derive an expression for the Lorentz force on a charged particle in an electromagnetic field with the help of Lorentz transformation method. **(Meerut 1981; 80)**
30. Derive the covariant form of Lorentz force equation and then show the three dimensional integrals of this equation. Describe the conservation laws of momentum and energy. **(Meerut 1988; 83)**
31. Write Maxwell's equations in the presence of a point charge in motion. Show that the total charge, energy and momentum of the system are conserved. **(Meerut 1987)**
32. Give the expression for electromagnetic field tensor in detail. **(Meerut 1992)**

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12

Electrodynamics of a Moving Charge and Radiating Systems

12.1. Introduction

A stationary charge produces electrostatic field only. A uniformly moving charge produces electromagnetic field of short range, so they can not radiate electromagnetic energy. We shall see that *the electromagnetic energy can be radiated only if a charged particle is accelerated.* The fields produced by moving charges can be computed either by the concept of retarded potentials or the theory of relativity. In the former case the field point parameters are related to the source point. Since the cause occurs at the source point and the effect follows at the field point; the procedure to relate the cause and effect is known as *retardation.* In the relativistic method two inertial frames of reference are chosen, with field point in one, the source point in the other and the charge is assumed to be relativistically invariant.

In this chapter we shall find the electromagnetic fields by the concept of retarded potentials and shall calculate quantities of interest such as total radiation emitted by an accelerated charge and the angular distribution of radiation to discuss the behaviour of radiated energy. Finally we shall discuss the fields and radiations produced by an electric dipole with simple applications to short linear antenna and antenna arrays.

12.2. Solution of Inhomogeneous Wave Equation by Fourier Analysis.

The scalar and vector wave equations in *freee space* are

$$\left.\begin{array}{ll} \left(\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\phi = -\dfrac{\rho}{\varepsilon_0} \quad \text{or} \quad \Box^2 \phi = -\dfrac{\rho}{\varepsilon_0} & \text{...(a)} \\ \left(\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0 \mathbf{J} \quad \text{or} \quad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J} & \text{...(b)} \end{array}\right\} \qquad \text{...(1)}$$

These two equations may be replaced by a single equation of the form

$$\Box^2 \psi(\mathbf{r}, t) = -g(\mathbf{r}, t) \qquad \text{...(2)}$$

where $\psi(\mathbf{r}, t)$ is the source function at the field point $(\mathbf{r}, t)$ and covers both ϕ and $\mathbf{A}$; while $g(\mathbf{r}, t)$ is the source function and covers both ρ and $\mathbf{J}$.

According to Fourier's theorem a complex single harmonic periodic function may be expressed as a sum of simple harmonic periodic functions of all possible frequencies. Accordingly the source function $g(\mathbf{r}, t)$ may be analysed by the Fourier integral

$$g(\mathbf{r}, t) = \int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega \qquad ...(3)$$

which has the *Fourier inversion*

$$g_\omega(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\mathbf{r}, t)\, e^{i\omega t}\, dt \qquad ...(4)$$

Similarly the general complex potential function $\psi(\mathbf{r}, t)$ may be analysed as

$$\psi(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega \qquad ...(5)$$

and the corresponding Fourier inversion is

$$\psi_\omega(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(\mathbf{r}, t)\, e^{i\omega t}\, dt \qquad ...(6)$$

Substituting values of $g(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$ for (3) and (5) in equation (2), we get

$$\Box^2 \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega = \int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega = -\int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

or

$$\nabla^2 \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega + \frac{\omega^2}{c^2}\int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega = -\int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

or

$$\int_{-in}^{+\infty} \left\{\nabla^2 \psi_\omega(\mathbf{r}) + \frac{\omega^2}{c^2}\psi_\omega(\mathbf{r}) + g_\omega(\mathbf{r})\right\} e^{-i\omega t}\, d\omega = 0,$$

This equation is true for any arbitrary instant and any arbitrary ω; hence this is satisfied only if

$$\nabla^2\psi_\omega(\mathbf{r}) + \frac{\omega^2}{c^2}\psi_\omega(\mathbf{r}) + g_\omega(\mathbf{r}) = 0$$

i.e.

$$\nabla^2 \psi_\omega + \frac{\omega^2}{c^2}\psi_\omega = -g_\omega \qquad ...(7)$$

This equation is similar to Poisson's equation $\nabla^2\psi = -\rho/\varepsilon_0$, except for the second term on L.H.S. The solution of equation (7) may be assumed as the superposition of unit point solutions corresponding to a source at the point $\mathbf{r}'$ given by $g(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$: where $\delta(\mathbf{r} - \mathbf{r}')$ is the *Dirac-function.* Then if G is a unit source potential called the *Green's function,* then equation (7) in terms of G takes the form

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \qquad ...(8)$$

where G is a function of both $\mathbf{r}$ and $\mathbf{r}'$. According to property of Dirac δ-function $\delta(\mathbf{r} - \mathbf{r}') = 0$ for $\mathbf{r} \neq \mathbf{r}'$. This means when the field and source points are different, the R.H.S. of equation (8) is zero and the relation is then homogeneous.

The partial solution cooresponding to frequency ω of the total source is given by the superposition :

$$\Psi_\omega(\mathbf{r}) = \int g_\omega(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, dv'. \qquad ...(9)$$

To find the value of G, we note that the solution of equation (8) is spherically symmetric in R, the distance between points $\mathbf{r}$ and $\mathbf{r}'$, except at $R = 0$ where $\mathbf{r} = \mathbf{r}'$. For spherically symmetric situation

$$\nabla^2 G = \frac{1}{R}\frac{d^2}{dR^2}(RG).$$

Hence at all points other than $R = 0$, equation (8) is identical with the equation

$$\frac{1}{R}\frac{d^2(RG)}{dR^2} + \frac{\omega^2}{c^2} G = 0. \qquad ...(10)$$

Writing $k = \frac{\omega}{c}$ as usual, equation (10) takes the form

$$\frac{1}{R}\frac{d^2(RG)}{dR^2} + k^2 G = 0$$

i.e.

$$\frac{d^2(RG)}{dR^2} + k^2 RG = 0. \qquad ...(11)$$

On integration this equation immediately gives

$$RG = A\, e^{\pm i\omega R}, \qquad \text{where } A \text{ is constant}$$

or

$$G = \frac{A}{R} e^{\pm i\omega R} \qquad ...(12)$$

It is obvious that in the neighbourhood of singular point $R = 0$, G behaves as A/R.

The Gauss divergence theorem applicable to any vector $\mathbf{V}$ is

$$\int \nabla \cdot \mathbf{V}\, dv = \int \mathbf{V} \cdot d\mathbf{S}.$$

Letting $\mathbf{V} = \nabla\left[\frac{1}{R}\right]$, we get

$$\int \nabla \cdot \nabla\left[\frac{1}{R}\right] dv' = \int \nabla\left[\frac{1}{R}\right] \cdot d\mathbf{S}'$$

or

$$\int \nabla^2\left[\frac{1}{R}\right] dv' = -\int \frac{\mathbf{R} \cdot d\mathbf{S}'}{R^3} = -\int d\Omega. \qquad ...(13)$$

where Ω is the solid angle subtended at point $\mathbf{r}$ by the surface of integration $\mathbf{S}'$ over the variables $\mathbf{r}'$. Since $\mathbf{S}'$ includes $\mathbf{r}$, we have simply $\int d\Omega = 4\pi$. Hence equation (13) gives

$$\int \nabla^2\left[\frac{1}{R}\right] dv' = -4\pi.$$

The implies

$$\int \nabla^2\left(\frac{A}{R}\right) dv' = -4\pi A \qquad \text{(since } A \text{ is constant)}$$

i.e.

$$\int \nabla^2 G\, dv' = 4\pi A \qquad ...(14)$$

in the neighbourhood of singular point $R = 0$.

Now considering volume integral of equation (8) over the neighbourhood of the singular point $R = 0$ where G behaves as A/R, we get

$$\int \nabla^2 G(\mathbf{r}, \mathbf{r}')\, dv' + \frac{\omega^2}{c^2}\int G(\mathbf{r}, \mathbf{r}')\, d' = -\int \delta(\mathbf{r} - \mathbf{r}')\, dv'. \qquad ...(15)$$

Noting that second term vanishes in the limit $R \to 0$ while considering the volume integral in neighbourhood of singular point $R = 0$ and using (14) along with the property of Dirac δ-function, *viz.* $\int \delta(\mathbf{r} - \mathbf{r}')\, dv' = 1$ for $R = 0$, equation (15) gives

$$-4\pi A = -1$$

or

$$A = \frac{1}{4\pi}. \qquad \text{...(16)}$$

Substituting this value of A in equation (12), we get

$$G = \frac{1}{4\pi R} e^{\pm ikR} \qquad \text{...(17)}$$

as the solution of equation (8).

Substituting value of G from (17) in equation (9), we get

$$\psi_\omega(\mathbf{r}) = \frac{1}{4\pi} \int \frac{g_\omega(\mathbf{r}')}{R(\mathbf{r}, \mathbf{r}')} e^{\pm ikR(\mathbf{r}, \mathbf{r}')}\, dv'. \qquad \text{...(18)}$$

We note that the effect of second term in equation (7) is simply to introduce a complex exponential factor in the familiar Coulomb potential expression *viz.*

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{R(\mathbf{r}, \mathbf{r}')}\, dv' = \frac{1}{4\pi} \int \frac{g(\mathbf{r}')}{R(\mathbf{r}, \mathbf{r}')}\, dv'. \qquad \text{...(19)}$$

Now substituting value of $\psi_\omega(\mathbf{r})$ from (18) in equation (5), we get

$$\psi(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{4\pi} \iint \frac{g_\omega(\mathbf{r}')\, e^{-i(\omega t \pm kR)}}{R(\mathbf{r}, \mathbf{r}')}\, d\omega\, dv'. \qquad \text{...(20a)}$$

Since $k = \omega/c$, equation (20a) may be expressed as

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \iint \frac{g_\omega(\mathbf{r}')\, e^{i\omega(t \pm R/c)}}{R(\mathbf{r}, \mathbf{r}')}\, d\omega\, dv' \qquad \text{...(20b)}$$

Not let us introduce a new time t' given by

$$t'(\mathbf{r}, \mathbf{r}') = t \pm \frac{R}{c}. \qquad \text{...(21)}$$

which corresponds to shifting of origin of time by an amount equal to the time taken by the light signal to travel a distance R from the source point r' to the field point $\mathbf{r}$. Then we may represent

$$\int_{-\infty}^{+\infty} g(\mathbf{r})\, e^{-i\omega(t \pm R/c)}\, d\omega = g\left(\mathbf{r}', t \pm \frac{R}{c}\right) \qquad \text{...(22)}$$

So that equation (20b) takes the form

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{g(\mathbf{r}', t \pm R/c)}{R(\mathbf{r}, \mathbf{r}')}\, dv' \qquad \text{...(23)}$$

Mathematically both plus and minus signs in above equation are valid; but only the minus sign appears to have physical significance. The minus sign indicates that the signal arriving at $(\mathbf{r}, t)$ should have started from $\mathbf{r}'$ at instant $(t - R/c)$. In other words the minus sign corrsponds to the cause preceding the effect. Equation (23) with minus sign in known as the *retarded potential solution of inhomogeneous wave equation.* The plus sign in (23) indicates that the signal arriving at $(\mathbf{r}, t)$ should have started from $\mathbf{r}'$ at instant $(t + R/c)$; it implies that the effect precedes the cause in time. That is why we shall ignore the solution with plus sign, since it clearly violates experience; the field does not appear to precede the charges and currents which cause it. The solution with plus sign in known as *abvanced solution.* Here we shall restrict ourselves to the minus sign *i.e.* to the use of retarded solutions.

The retarded solution for scalar and vector potentials may now be written as

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0}\int \frac{[\rho(\mathbf{r}')]\,dv'}{R(\mathbf{r}, \mathbf{r}')} && \text{...(a)} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}(\mathbf{r}')]\,dv'}{R(\mathbf{r}, \mathbf{r}')} && \text{...(b)} \end{aligned}\right\} \qquad \text{...(24)}$$

where $[\rho(\mathbf{r}')]$ and $[\mathbf{J}(\mathbf{r}')]$ within square brackets represent, however, not the values of charge and current densities at the time of observation t, but at the earlier time $t' = t - R/c$. In other words the retarded potentials ϕ and $\mathbf{A}$ are calculated from charge and current densities at a time which is *set back* by R/c relative to the time of observation.

Having known the electromagnetic potentials $\mathbf{A}$ and ϕ, the electromagnetic field vectors, in general may be calculated by the relations;

$$\mathbf{B} = \text{curl } \mathbf{A} \text{ and } \mathbf{E} = -\text{ grad } \phi - \frac{\partial \mathbf{A}}{\partial t}. \qquad \text{...(25)}$$

12.3. Lienard-Wiechert Potentials

The Lienard-Wiechert potentials are scalar and vector potentials produced by a moving point charge. To obtain these potentials one of the most instructive procedures is to consider a moving volume carrying with it a fixed charge distribution, for example a uniformly charged spherical volume, moving through space along a prescribed trajectory. The field due to a point charge may then be obtained by taking properly the limit of the field due to such a charge distribution.

The scalar and vector potentials due to moving charge and current distributions at point $\mathbf{r}$ and time t are given by the *retarded potentials*

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\mathbf{r}', t - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right) dV'}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\mathbf{r}', t - \dfrac{R}{c}\right)}{\mathbf{R}(\mathbf{r}, \mathbf{r}')}\, dV' \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\left(\mathbf{r}', t - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|}\, dV' \\ &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\left(\mathbf{r}', t - \dfrac{R}{c}\right)}{R(\mathbf{r}, \mathbf{r}')}\, dV' \end{aligned}$$

where

$$\mathbf{R}(\mathbf{r}, \mathbf{r}') = (\mathbf{r} - \mathbf{r}'). \qquad \text{...(1)}$$

The difficulty in integration arises because time $t' = \left(t - \dfrac{R}{c}\right)$ is not fixed and hence the volume of integration. In other words the volume in which ρ and $\mathbf{J}$ are different from zero, is not specified. To remove the difficulty let us introduce one dimensional Dirac delta function having properties

$$\left.\begin{aligned}&\text{(i)}\quad \delta(x) = 0 \quad \text{for} \quad x \neq 0 \text{ and } \delta(0) \neq 0\\&\text{(ii)}\quad \int_{-\infty}^{+\infty} \delta(x)\,dx = 1\\&\text{(iii)}\quad f(x)\,\delta(x-a) = f(a)\,\delta(x-a)\\&\text{(iv)}\quad \int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)\end{aligned}\right\} \qquad ...(2)$$

where $f(x)$, is regular, continuous and differentiable function of x.

With this δ function the scalar and vector potentials given by (1) may be written in the form

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\int_{-\infty}^{+\infty} \frac{\rho(\mathbf{r}', t')\,\delta\left[t' - \left(t - \frac{R}{c}\right)\right]}{R(\mathbf{r}, \mathbf{r}')}\,dV'\,dt'$$

and
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{V'}\int_{-\infty}^{+\infty} \frac{\mathbf{J}(\mathbf{r}', t')}{R(\mathbf{r}, \mathbf{r}')}\,\delta\left[\left(t' - \left(t - \frac{R}{c}\right)\right)\right] dV'\,dt' \qquad ...(3)$$

Now $\rho(\mathbf{r}', t')\,dV' = dq$ is the charge in the volume element dV' and similarly $\mathbf{J}(\mathbf{r}, t') = \rho(\mathbf{r}', t')\,\mathbf{v}(t')$ is the *current density produced by an element of charge density* $\rho(\mathbf{r}', t')$ *moving with velocity* $\mathbf{v}(t')$. This gives

$$\mathbf{J}(r', t')\,dV' = \rho(r', t')\,\mathbf{v}(t')\,dV' = \mathbf{v}(t')\,dq. \qquad ...(4)$$

Thus, we get

$$\left.\begin{aligned}\phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0}\int_{V}\int_{-\infty}^{+\infty} \frac{dq\,\delta(t' - t + R/c)\,dt'}{R(\mathbf{r}, \mathbf{r}')}\\\mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int_{V'}\int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\,dq\,\delta(t' - t + R/c)\,dt'}{R(\mathbf{r}, \mathbf{r}')}\end{aligned}\right] \qquad ...(5)$$

If the *charge distribution limits to point charge* q, we can immediately write equation (5) as

$$\left.\begin{aligned}\phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\,dt'}{R(\mathbf{r}, \mathbf{r}')}\\\mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\,\delta(t' - t + R/c)\,dt'}{R(\mathbf{r}, \mathbf{r}')}\end{aligned}\right] \qquad ...(6)$$

Thse integrals can be evaluated if we choose instead of t' a new fixed time t_1 give by

$$t_1 = t' - t + \frac{R}{c} \qquad ...(7a)$$

We get
$$\frac{dt_1}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'} \qquad ...(7b)$$

We have $R^2 = \mathbf{R}\cdot\mathbf{R}$

or
$$2R\frac{dR}{dt'} = \mathbf{R}\cdot\frac{d\mathbf{R}}{dt'} + \frac{d\mathbf{R}}{dt'}\cdot\mathbf{R}$$

i.e.
$$\frac{dR}{dt'} = \frac{\mathbf{R}}{R}\cdot\frac{d\mathbf{R}}{dt'} \qquad ...(8)$$

From fig. 12.1 we have

$$\mathbf{R} = (\mathbf{R} + d\mathbf{R}) + \mathbf{v}\,dt'$$

i.e.
$$\frac{d\mathbf{R}}{dt'} = -\mathbf{v}.$$

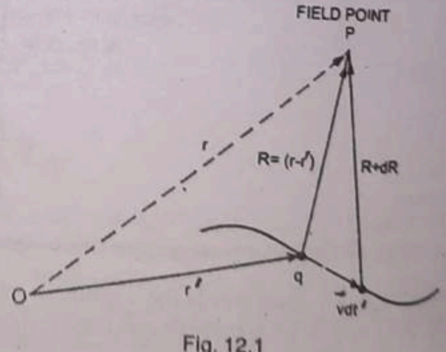


Fig. 12.1

...(2)

n in the form

...(3)

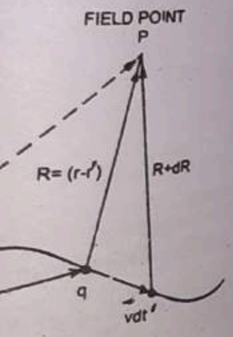
(r, t') = ρ (r', t') v (t') is
elocity v (t'). This gives

...(4)

...(5)

n (5) as

...(6)



12.1

Using this; equation (8) gives

$$\frac{dR}{dt'} = -\,\mathbf{v}\cdot\frac{\mathbf{R}}{R} \qquad \text{...(9)}$$

Substituting this value of dR/dt in equation (7b), we get

$$\frac{dt_1}{dt'} = 1 - \frac{\mathbf{v}}{c}\cdot\frac{\mathbf{R}}{R} = \frac{R - (\mathbf{v}\cdot\mathbf{R}/c)}{R}$$

$$dt' = \frac{R\,dt_1}{R - \left\{\frac{\mathbf{v}}{c}\right\}\cdot\mathbf{R}} \qquad \text{...(10)}$$

Making substitutions from (7a) and (10) in (6), we get

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{\delta(t_1)\,dt_1}{R - \left\{\frac{\mathbf{v}\cdot\mathbf{R}}{\mathbf{c}}\right\}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\frac{\mathbf{v}(t')\,\delta(t_1)\,dt_1}{R - \left\{\frac{\mathbf{v}\cdot\mathbf{R}}{\mathbf{c}}\right\}} \end{aligned}\right] \qquad \text{...(11)}$$

Now using property of δ function, we have

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}\right]_{\text{at } t_1 = C} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}\right]_{\text{at } t_1 = 0} \end{aligned}\right] \qquad \text{...(12)}$$

But $t_1 = 0$ implies that $t' - t + R/c = 0$ or $t' = t - R/c$, the *retarded time.*

Thus equations (12) may be expressed as

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}\right]_{\text{retarded}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}\right]_{\text{retarded}} \end{aligned}\right] \qquad \text{...(13)}$$

These potentials are *velocity dependent* and were first of all given by *A. Lienard and E Wiechert* for a moving point charge and after their names they are known as Lienard-Weichert potentials. These potentials may be expressed in a more compact form if we substitute

$$k = 1 - \frac{\mathbf{v}\cdot\mathbf{n}}{c} = 1 - \vec{\beta}\cdot\mathbf{n} \qquad \text{...(14)}$$

where $\vec{\beta} = \frac{\mathbf{v}}{c}$ and $\mathbf{n} = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{\mathbf{R}}{R}$, is a unit vector directed from source point q to the field point P. Then we may write equations (13) as,

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{kR}\right]_{\text{retarded}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\left[\frac{\vec{\beta}c}{kR}\right]_{\text{retarded}} \end{aligned}\right\}_{\text{where } \vec{\beta} = \mathbf{v}/c} \qquad \text{...(15)}$$

The square brackets with subscript retarded mean that the quantity in brackets is to be evaluated at the retarded time $t' = t - R(t')/c$. For non-relativitic motion $\beta \to 0$; so that $k \to 1$, then the potentials (15) reduce to the well known *non-relativistic results* given by

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{q}{[R]_{\text{retarded}}} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r', t')\, dV'}{[R]_{\text{retarded}}}$$

$$= \frac{1}{4\pi\varepsilon_0} \int \left[\frac{\rho}{R}\right]_{\text{retarded}} dV' \qquad \text{...(16a)}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \left[\frac{\mathbf{v}}{R}\right]_{\text{retarded}} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')\, dV'}{[R]_{\text{retarded}}}$$

$$= \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}}{R}\right]_{\text{retarded}} dV' \qquad \text{...(16b)}$$

12.4. The Electromagnetic Fields from Lienard-Wiechert Potentials of a Moving Point Charge.

So far we have found the electromagnetic potentials **A** and ϕ for a charged particle undergoing arbitrary motion. Now we want to find the electromagnetic fields **E** and **B** given by

$$\mathbf{B} = \text{curl}\, \mathbf{A} \qquad \text{...(1)}$$

and

$$\mathbf{E} = -\,\text{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{...(2)}$$

Let us start with *Lienard-Wiechert potential* expressions (6) of section 12.3, given by

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')} \qquad \text{...(3)}$$

$$\mathbf{A}(\mathbf{r}, t) - \frac{\mu_0 q}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')} \qquad \text{...(4)}$$

Substituting these values in equation (2), we get

$$\mathbf{E}(\mathbf{r}, t) = -\left[\frac{q}{4\pi\varepsilon_0} \nabla \int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')} + \frac{\mu_0 q}{4\pi} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\right] \qquad \text{...(5)}$$

Now we note that the only dependence on the spatial coordinates r of the observation (field) point is through R. Hence the gradient operation is equivalent to

$$\text{grad} = \nabla \to \mathbf{n} \frac{\partial}{\partial R} \qquad \text{...(6)}$$

where **n** is a unit vector along $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$.

Therefore equation (5) takes the form

$$\mathbf{E}(\mathbf{r}, t) = -\left[\frac{q}{4\pi\varepsilon_0} \mathbf{n} \int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt}{R(\mathbf{r}, \mathbf{r}')} + \frac{\mu_0 q}{4\pi} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\right]$$

$$= \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \left[\frac{\mathbf{n}}{R^2} \delta(t' - t + R/c) - \frac{\mathbf{n}}{Rc} \delta'(t' - t + R/c)\right] dt'$$

$$+ \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta'(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\Bigg]$$

The prime on the delta function means differentiation with respect to its argument.

Now using $\mu_0\varepsilon_0 = 1/c^2$ and hence substituting $\mu_0 = 1/c^2\varepsilon_0$ and $\vec{\beta} = \mathbf{v}/c$, we get

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{R^2}\,\delta(t'-t+R/c) + \left(\frac{\vec{\beta}-\mathbf{n}}{cR}\right)\delta'(t'-t+R/c)\right]dt' \qquad ...(7)$$

Similarly substituting value of **A** from (4) in equation (1) we get .

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\,\nabla\times\int_{-\infty}^{+\infty}\frac{\mathbf{v}(t')\,\delta(t'-t+R/c)\,dt'}{R(\mathbf{r},\mathbf{r}')}\\ &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial R}\times\frac{\{\mathbf{v}(t')\,\delta(t'-t+R/c)\}}{R(\mathbf{r},\mathbf{r}')}\right]dt'\\ &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\mathbf{n}\times\mathbf{v}\left\{\frac{\delta(t'-t+R/c)}{R^2}+\frac{\delta'(t'-t+R/c)}{Rc}\right\}\Bigg]dt'\\ &= \frac{\mu_0 qc}{4\pi}\int_{-\infty}^{+\infty}\left[\mathbf{n}\times\vec{\beta}\left\{\frac{\delta(t'-t+R/c)}{R^2}+\frac{\delta'(t'-t+R/c)}{R/c}\right\}\right]dt' \qquad ...(8)\end{aligned}$$

Since $t_1 = t' + R/c$

i.e.

$$\frac{dt_1}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'} = 1 = \frac{\mathbf{v}\cdot\mathbf{R}}{c\mathbf{R}} \qquad ...(9)$$

$$\left[\text{since } \frac{dR}{dt} = -\frac{\mathbf{v}\cdot\mathbf{R}}{cR} \text{ using eqn. (9) of section 12.3}\right]$$

$$= 1 - \vec{\beta}\cdot\mathbf{n} = \kappa \qquad (\text{where } \vec{\beta} = \mathbf{v}/c \text{ and } \kappa = 1 - \vec{\beta}\cdot\mathbf{n})$$

so that

$$dt' = \frac{dt_1}{\kappa} \qquad ...(10)$$

Substituting (9) and (10) in equations (7) and (8), we get

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{\kappa R^2}\,\delta(t_1-t) + \frac{(\vec{\beta}-\mathbf{n})}{\kappa cR}\,\delta'(t_1-t)\right]dt_1 \qquad ...(11)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\int_{-\infty}^{+\infty}\left[(\mathbf{n}\times\vec{\beta})\left\{\frac{-\delta(t_1-t)}{\kappa R^2} + \frac{1}{\kappa cR}\,\delta'(t_1-t)\right\}\right]dt_1 \qquad ...(12)$$

Integrating by parts on the derivative of delta functinon, we readily get

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa R^2} - \frac{d}{dt_1}\left(\frac{\vec{\beta}-\mathbf{n}}{\kappa cR}\right)\right] \qquad ...(13)$$

and

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\left[\frac{\mathbf{n}\times\vec{\beta}}{\kappa R^2} - \frac{d}{dt_1}\left(\frac{\mathbf{n}\times\vec{\beta}}{\kappa cR}\right)\right]. \qquad ...(14)$$

From (10), $dt_1 = \kappa\, dt'$, therefore equations (13) and (14) take the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa}\,\frac{1}{dt'}\left(\frac{\mathbf{n}-\vec{\beta}}{\kappa R}\right)\right] \qquad ...(15)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\left[\frac{\vec{\beta}\times\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa}\,\frac{d}{dt'}\left(\frac{\vec{\beta}\times\mathbf{n}}{\kappa R}\right)\right]. \qquad ...(16)$$

Referring fig. 12.1, we have

$$\frac{dR}{dt'} = -\mathbf{v}\cdot\mathbf{n} \text{ and } \frac{d\mathbf{R}}{dt'} = -\mathbf{v}. \qquad ...(17)$$

We have

$$\mathbf{R} = \mathbf{n}R$$

$$\because \qquad \frac{d\mathbf{R}}{dt'} = \mathbf{n}\frac{dR}{dt'} + R\frac{d\mathbf{n}}{dt'}$$

$\therefore$ Rate of change of unit vector **n** with time is given by

$$\frac{d\mathbf{n}}{dt'} = -\frac{\mathbf{v}}{R} + \frac{\mathbf{n}(\mathbf{v}\cdot\mathbf{n})}{R}$$

$$= -\frac{\vec{\beta}c}{R} + \frac{\mathbf{n}(\vec{\beta}c\cdot\mathbf{n})}{R} = \frac{c}{R}\{\mathbf{n}(\vec{\beta}\cdot\mathbf{n}) - (\mathbf{n}\cdot\mathbf{n})\vec{\beta}\}$$

Therefore
$$\frac{1}{c}\frac{d\mathbf{n}}{dt'} = \frac{\mathbf{n}\times(\mathbf{n}\times\vec{\beta})}{R}. \qquad \text{...(18)}$$

Making this substitution, wherever it appears explicity in (15) and (16), we get

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa^2R^2} + \frac{\mathbf{n}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right) - \frac{\vec{\beta}}{\kappa^2R^2} - \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right] \qquad \text{...(19)}$$

and
$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\left[\left\{\frac{\vec{\beta}}{\kappa^2R^2} + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right\}\times\mathbf{n}\right] \qquad \text{...(20)}$$

From equations (19) and (20), we observe that the magnetic induction is related simply to the electric field by the relation

$$\mathbf{B} = \frac{\mathbf{n}\times\mathbf{E}}{c} \qquad \text{...(21)}$$

where the equation is understood to be in terms of the retarded quantities in square brakets.

The remaining derivatives needed in equations (19) and (20) are

$$\frac{d\vec{\beta}}{dt'} = \dot{\vec{\beta}} \qquad \text{...(22)}$$

and
$$\frac{1}{c}\frac{d}{dt'}(\kappa R) = \frac{R}{c}\frac{dk}{dt'} + \frac{\kappa}{c}\frac{dR}{dt'}$$

$$= \frac{R}{c}\frac{d}{dt'}(1 - \mathbf{n}\cdot\vec{\beta}) + (1 - \mathbf{n}\cdot\vec{\beta})(-\mathbf{n}\cdot\vec{\beta})$$

$$\left(\text{since } \frac{1}{c}\frac{dR}{dt'} = -\mathbf{n}\cdot\vec{\beta}\right)$$

$$= \frac{R}{c}\left\{\frac{d\mathbf{n}}{dt'}\cdot\vec{\beta} - \mathbf{n}\cdot\frac{d\vec{\beta}}{dt'}\right\} - \mathbf{n}\cdot\vec{\beta} + (\mathbf{n}\cdot\vec{\beta})^2$$

$$= -\{\mathbf{n}\times(\mathbf{n}\times\vec{\beta})\}\cdot\vec{\beta} - \frac{(\mathbf{n}\cdot\dot{\vec{\beta}})R}{c} - \mathbf{n}\cdot\vec{\beta} + (\mathbf{n}\cdot\vec{\beta})^2$$

$$= -\{(\mathbf{n}\cdot\vec{\beta})\mathbf{n} - (\mathbf{n}\cdot\mathbf{n})\vec{\beta}\}\cdot\vec{\beta} - \frac{(\mathbf{n}\cdot\dot{\vec{\beta}})R}{c} - \mathbf{n}\cdot\vec{\beta} + (\mathbf{n}\cdot\vec{\beta})^2$$

$$= \beta^2 - \mathbf{n}\cdot\vec{\beta} - \frac{R}{c}(\mathbf{n}\cdot\dot{\vec{\beta}}) \qquad \text{...(23)}$$

Then equation (14) takes the form

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa^2R^2} + \frac{\mathbf{n}}{c\kappa}\left(-\frac{1}{\kappa^2R^2}\right)\frac{d}{dt'}(\kappa R) - \frac{\vec{\beta}}{c\kappa^2R} - \frac{\dot{\vec{\beta}}}{c\kappa^2R} - \frac{\vec{\beta}}{c\kappa}\left(-\frac{1}{\kappa^2R^2}\right)\frac{d}{dt'}(\kappa R)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^2R^2} - \frac{(\mathbf{n}-\vec{\beta})}{c\,\kappa^3R^2}\frac{d}{dt'}(\kappa R) - \frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^2R^2} - \frac{(\mathbf{n}-\vec{\beta})}{\kappa^3R^2}\left\{\beta^2 - \mathbf{n}\cdot\vec{\beta} - \frac{R}{c}(\mathbf{n}\cdot\dot{\vec{\beta}})\right\} - \frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^3R^2}\left\{\kappa - \beta^2 + \mathbf{n}\cdot\vec{\beta} + \frac{R}{c}(\mathbf{n}\cdot\dot{\vec{\beta}})\right\} - \frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^3R^2}\left\{(1-\mathbf{n}\cdot\vec{\beta}) - \beta^2 + \mathbf{n}\cdot\vec{\beta} + \frac{R}{c}\,\mathbf{n}\cdot\dot{\vec{\beta}}\right\} - \frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2} + \frac{\mathbf{n}-\vec{\beta}}{\kappa^3Rc}\{(\mathbf{n}\cdot\dot{\vec{\beta}}) - \kappa\dot{\vec{\beta}}\}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2} + \frac{(\mathbf{n}-\vec{\beta})}{\kappa^3Rc}\{(\mathbf{n}\cdot\dot{\vec{\beta}}) - (1-\mathbf{n}\cdot\vec{\beta})\dot{\vec{\beta}}\}\right] \qquad ...(24)$$

using

$$\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}}) = (\mathbf{n}\cdot\dot{\vec{\beta}})(\mathbf{n}-\vec{\beta}) - \{\mathbf{n}\cdot(\mathbf{n}-\vec{\beta})\}\dot{\vec{\beta}}$$

$$= (\mathbf{n}\cdot\dot{\vec{\beta}})(\mathbf{n}-\vec{\beta}) - (1-\mathbf{n}\cdot\vec{\beta})\dot{\vec{\beta}}$$

We may write equation (24) as

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2} + \frac{1}{\kappa^3Rc}\{\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\right] \qquad ...(25)$$

Now using $n = \frac{\mathbf{R}}{R}$ and $\kappa = 1 - \mathbf{n}\cdot\vec{\beta} = 1 - \frac{\mathbf{R}\cdot\vec{\beta}}{R}$ *i.e.* $\kappa R = R - \mathbf{R}\cdot\vec{\beta}$

we may write equation (25) as

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}-(R\vec{\beta})(1-\beta^2)}{R-\mathbf{R}\cdot\vec{\beta}} + \frac{\mathbf{R}\times(\mathbf{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \qquad ...(26)$$

and the *magnetic field induction is* given by

$$= \frac{q}{4\pi\varepsilon_0Rc}\left[\frac{\mathbf{n}\times(\mathbf{R}-R\vec{\beta})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{n}\times\{\mathbf{R}\times(\mathbf{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c(R-\mathbf{R}\cdot\vec{\beta})^2}\right]$$

Using $\mu_0\varepsilon_0 = 1/c^2$ and $\mathbf{R} \times \mathbf{R} = 0$, we may write above equation

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0\varepsilon_0 q}{4\pi}\left[\frac{(\vec{\beta} \times \mathbf{n})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{n} \times \{\mathbf{R} \times \{\mathbf{R} - R\vec{\beta}\} \times \vec{\dot{\beta}}\}\}}{cR\,(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \quad \text{...(27)}$$

Equation (26) and (27) represent required *electromagnetic* fields due to a moving point charge.

From equations (26) and (27) it is clear that electromagnetic fields **E** and **B** contain two terms in square brackets : the first term depends on velocity $\mathbf{v} = \vec{\beta} c$ and is independent of acceleration while the second term depends on acceleration $\mathbf{a} = c\vec{\dot{\beta}}$. Thus electromagnetic fields **E** and **B** divide themselves naturally into two types of fields namely *velocity fields* which are independent of acceleration and *acceleration fields* which depend linearly on $\vec{\dot{\beta}}$. The velocity fields are essentially *static field* falling off as R^{-2} ; whereas the acceleration fields are typical radiation fields, both **E** and **B** being transverse to radius vector and varying as R^{-1}.

12.5. The Electromagnetic Fields of a Uniformly Moving Point Charge.

The electric and magnetic fields of a point charge in motion are given by equation (26) and (27) of section (12.4) *i.e.*

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R} - R\vec{\beta})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\{R \times (\mathbf{R} - \mathbf{R}\vec{\beta}) \times \vec{\dot{\beta}}\}}{c\,(\mathbf{R}-\mathbf{R}\cdot\vec{\dot{\beta}})^3}\right]_{retarded} \quad \text{...(1)}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{n} \times \mathbf{E}}{c} = \frac{\mu_0 c q}{4\pi}\left[\frac{(\vec{\beta} \times \mathbf{n})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{n} \times \{\mathbf{R} \times (\mathbf{R} - R\vec{\beta}) \times \vec{\dot{\beta}}\}}{cR\,(R-\mathbf{R}\cdot\vec{\dot{\beta}})^3}\right]_{retarded} \quad \text{...(2)}$$

For a *uniformly moving point charge*, $\mathbf{v} = c\beta =$ constant.

This implies $\dot{\beta} = 0$.

Therefore for a *uniformly moving point charge*, equation (1) takes the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R} - R\vec{\beta})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \quad \text{...(3)}$$

Now as the particle is moving with constant velocity, we have

$$\left.\begin{aligned} [\vec{\beta}]_{retarded} &= \vec{\beta} \\ \text{and hence } [1-\beta^2]_{retarded} &= 1-\beta^2 \end{aligned}\right] \quad \text{...(4)}$$

Hence equation (3) may be expressed as

$$\mathbf{E}(\mathbf{r}, t) = \frac{q(1-\beta^2)}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R} - R\vec{\beta})}{(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \quad \text{...(5)}$$

The magnetic field may be computed by the relation

$$\mathbf{B}(\mathbf{r}, t) = \left[\frac{\mathbf{n} \times \mathbf{E}(\mathbf{r}, t)}{c}\right]_{retarded} \qquad ...(6)$$

Equations (5) and (6) represent electromagnetic fields in terms of retarded position of the point charge. Now for a uniformly moving point charge, it is desirable to represent these fields in terms of the present position of the charge. For the purpose we consider the geometrical configuration of fig. 12.2. relating the present and retarded positions of the charge. The positions A and A' occupied by the charge represent respectively the retarded and present positions of the charge with reference to field point (or point of observation) P.

The retarded distance of field point $P(\mathbf{r})$ from the source point $A(\mathbf{r}')$ is shown by $|R(t')| = R'$ and the present distance of field point P from the source point A is shown by $R(t) = R$. The electromagnetic signal starting from the source point A' reaches the field point P in time R'/c. In the meanwhile the charge itself moving with velocity $\mathbf{v}$ reaches the point A, the present position therefore the distance $A'A = vR'/c = \beta R'$; θ' is the angle between $\mathbf{v}$ and $\mathbf{R}'$ and θ is the angle between $\mathbf{v}$ and $\mathbf{R}$, AN is the perpendicular drawn from the present position on $A'P$ and PM is the perpendicular drawn from the field point P on the direction of velocity $\mathbf{v}$ of the moving point charge.

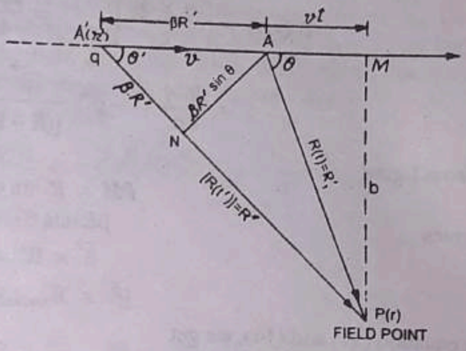


Fig. 12.2. Present and retarded positions of a point charge in uniform motion.

Now referring to fig. 12.2, we have

$$[\mathbf{R} - R\vec{\beta}]_{retarded} = \mathbf{R}' - R'\,\vec{\beta} = \mathbf{R} \qquad \text{(from triangle } AA'P) \;...(7)$$

and

$$b = R' \sin\theta' = R \sin\theta.$$

This implies $\beta R' \sin\theta' = \beta R \sin\theta$

i.e.
$$\beta^2 R'^2 \sin^2\theta' = \beta^2 R^2 \sin^2\theta$$

i.e.
$$\beta^2 R'^2 (1 - \cos^2\theta') = \beta^2 R^2 (1 - \cos^2\theta)$$

i.e.
$$\beta^2 R'^2 - (\beta R' \cos\theta')^2 = \beta^2 R^2 - (\beta R \cos\theta)^2$$

i.e.
$$\beta^2 R'^2 - (\vec{\beta}\cdot\mathbf{R}')^2 = \beta^2 R^2 - (\vec{\beta}\cdot\mathbf{R})^2 \qquad ...(8)$$

Squaring equation (7), we get

$$R'^2 - 2R'(\vec{\beta}\cdot\mathbf{R}') + R'^2\beta^2 = R^2 \qquad ...(9)$$

Substracting equation (8) from (9), we get

$$R'^2 - 2R'(\vec{\beta}\cdot\mathbf{R}') + (\vec{\beta}\cdot\mathbf{R}')^2 = R^2 - \beta^2 R^2 + (\vec{\beta}\cdot\mathbf{R})^2 \qquad ...(10)$$

Now

$$\begin{aligned}[\{R - \mathbf{R}\cdot\vec{\beta}\}^2]_{retarded} &= (R' - \mathbf{R}'\cdot\vec{\beta})^2 = R'^2 - 2R'(\mathbf{R}'\cdot\vec{\beta}) + (\mathbf{R}'\cdot\vec{\beta})^2 \\ &= R'^2 - 2R'(\vec{\beta}\cdot\mathbf{R}') + (\vec{\beta}\cdot\mathbf{R}')^2 \\ &= R^2 - \beta^2 R^2 + (\vec{\beta}\cdot\mathbf{R})^2 \qquad \text{using (10)} \\ &= R^2 - \beta^2 R^2 + \beta^2 R^2 \cos^2\theta \\ &= R^2 - \beta^2 R^2 (1 - \cos^2\theta)\end{aligned}$$

$$[(R-\mathbf{R}\cdot\vec{\beta})^2]_{retarded} = R^2(1-\beta^2\sin^2\theta) \qquad ...(11)$$

Substituting values from (7) and (11) in equation (5), we get

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{\mathbf{R}(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}} \qquad ...(12)$$

Now from (2) using $\dot{\vec{\beta}} = 0$, we have

$$\mathbf{B}(\mathbf{r},t) = \left[\frac{\mathbf{n}\times\mathbf{E}}{c}\right]_{retarded} = \frac{\mu_0 cq}{4\pi}\left[\frac{(\vec{\beta}\times\mathbf{R})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded}$$

$$= \frac{\mu_0 cq}{4\pi}(1-\beta^2)\,\frac{[\vec{\beta}\times\mathbf{R}]_{retarded}}{[(R-\mathbf{R}\cdot\vec{\beta})_{retarded}]^3} \qquad ...(13)$$

Now from figure

$$PM = R'\sin\theta' = R\sin\theta.$$

This gives

$$\beta R'\sin\theta' = \beta R\sin\theta$$

$\therefore$

$$\vec{\beta}\times\mathbf{R}' = \vec{\beta}\times\mathbf{R}$$

i.e.

$$[\vec{\beta}\times\mathbf{R}]_{retarded} = \vec{\beta}\times\mathbf{R} \qquad ...(14)$$

Using equation (11) and (14), we get

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 cq}{4\pi}\,\frac{(\vec{\beta}\times\mathbf{R})(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}} \qquad ...(15)$$

Equations (12) and (15) represent desired values of electromagnetic field vectors **E** and **B** for a uniformly moving charge.

Discussion. From equations (12) and (15), we come to the following conclusions.

(i) **Static Case.** For smaller velocities such that $\beta \to 0$; the electromagnetic field vectors **E** and **B** take the form

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\,\frac{q\mathbf{R}}{R^3} \qquad ...(16)$$

which is Coulomb's law. This shows that for non-relativistic velocities the field is *spherically symmetric* ; and

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 cq}{4\pi}\,\frac{\vec{\beta}\times\mathbf{R}}{R^3} = \frac{\mu_0 q}{4\pi}\,\frac{\mathbf{v}\times\mathbf{R}}{R^3} \qquad \left[\text{since } \vec{\beta} = \frac{\mathbf{v}}{c}\right]$$

$$= \frac{\mu_0 I\,dt}{4\pi}\,\frac{\mathbf{v}\times\mathbf{R}}{R^3} = \frac{\mu_0}{4\pi}\,\frac{I(\mathbf{v}\,dt)\times\mathbf{R}}{R^3}$$

i.e.

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\,\frac{I\,\vec{dl}\times\mathbf{R}}{R^3} \qquad ...(17)$$

which is *Biot-Savart field.*

(ii) The electromagnetic fields vary inversely as the square of distance R.

(iii) For high velocities $\beta \to 1$ of $v \to c$, both field magnitudes depend on the angles θ between the direction of motion of charge and radius vector **R**. From equation (12) it is obvious that electric field is greater for greater values of θ, *i.e.* the field is not spherically symmetric as in the static case ; but it is stronger in the direction perpendicular to velocity. If $\mathbf{E}_\perp$ and $E_\parallel$ are electric fields perpendicular and parallel to velocity, then

$$\left.\begin{aligned} \mathbf{E}_\perp\ (\theta = 90^\circ) &= \frac{1}{4\pi\varepsilon_0}\,\frac{q\mathbf{R}}{\mathbf{R}^3\sqrt{(1-\beta^2)}} \\ \text{and}\qquad \mathbf{E}_\parallel\ (\theta = 0^\circ) &= \frac{1}{4\pi\varepsilon_0}\,\frac{q\mathbf{R}\,(1-\beta^2)}{R^3} \end{aligned}\right] \qquad \text{...(18)}$$

This means that the electric field is increased in a direction perpendicular to direction of motion in the ratio of $1/\sqrt{(1-\beta^2)}$; while in the direction of motion the field is decreased in the ratio $(1-\beta^2)$. Thereby indicating that *the electric field of a moving charge is* $[1/(1-\beta^2)^{3/2}]$ *times stronger in the direction perpendicular to its motion than the field at the same distance along the direction of motion.* At very high velocities the field thus resembles more and more the field in a plane wave. For a short time, as a high velocity charge passes an observer, he sees a purely transverse electric and magnetic fields.

(iv) The magnetic field **B** is related to **E** through the relation*

$$\mathbf{B}\,(\mathbf{r}, t) = \frac{\vec{\beta} \times \mathbf{E}\,(\mathbf{r}, t)}{c} = \frac{\mathbf{v} \times \mathbf{E}\,(\mathbf{r}, t)}{c^2}.$$

This equation indicates that **B** is at right angles to both **v** and **E**.

(v) The Poynting vector is given by

$$\begin{aligned} \mathbf{S} = \mathbf{E} \times \mathbf{H} &= \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \\ &= \frac{\mathbf{E} \times (\mathbf{v} \times \mathbf{E})}{\mu_0 c^2} = \frac{(\mathbf{E}\cdot\mathbf{E})\,\mathbf{v} - (\mathbf{E}\cdot\mathbf{v})\,\mathbf{E}}{\mu_0 c^2} \end{aligned}$$

When v approaches c ; we have seen that *electric field is perpendicular to* **v** and so $\mathbf{E}\cdot\mathbf{v} = 0$

Consequently

$$\mathbf{S} = \frac{E^2\mathbf{v}}{\mu_0 c^2} = \varepsilon_0 E^2 \mathbf{v}$$

Since $\mathbf{E} \propto \dfrac{1}{R^2}$; we note that

$$\mathbf{S} \propto \frac{1}{R^4},$$

In order to calculate the energy radiated by the charged particle, we must integrate the normal component of **S** over the surface of sphere. If we take the radius of sphere to be R, then the energy radiated will be proportional to $1/R^2$ since the element of surface area involves R^2. Therefore as $R \to \infty$, the energy radiated tends to zero. Thus we conclude that *the charged particles moving with constant velocity can not radiate energy.*

12.6. Radiation from an Accelerated Charge at Low Velocity – Larmor's Formula.

The electromagnetic fields of a point charge in motion are

$$\mathbf{E}\,(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R} - R\vec{\beta})(1-\beta^2)}{(R - \mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{R} \times (\mathbf{R} - R\vec{\beta}) \times \dot{\vec{\beta}}}{c\,(R - \mathbf{R}\cdot\vec{\beta})^3}\right] \qquad \text{...(1)}$$

---

*Comparing equation (12) and (15) and keeping in mind $\mu_0\varepsilon_0 = 1/c^2$, we immediately get $\mathbf{B} = (\vec{\beta} \times \mathbf{E})/c$.

and
$$\mathbf{B}(\mathbf{r}, t) = \left[\frac{\mathbf{n} \times \mathbf{E}}{c}\right] = \left[\frac{\mathbf{R} \times \mathbf{E}}{Rc}\right]. \qquad ...(2)$$

In equation (1) the first term in square bracket represents velocity field ; while the second term represents acceleration field. In preceding section, we have seen that velocity fields are essentially static fields and they are incapable of radiating energy. Hence for radiation we shall consider only *acceleration fields* given by

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R} \times (\mathbf{R} - R\vec{\beta}) \times \dot{\vec{\beta}}}{c\,(R - R\vec{\beta})^3}\right] \qquad ...(3)$$

with
$$\mathbf{B}_a(\mathbf{r}, t) = \left[\frac{\mathbf{n} \times \mathbf{E}_a}{c}\right] = \left[\frac{\mathbf{R} \times \mathbf{E}_a}{Rc}\right] \qquad ...(4)$$

where the square bracket notation reminds us that all quantities in square brackets must be evaluated at retarded time. If the velocity of charge is small compared with the velocity of light, then

$$\beta = \frac{v}{c} \ll 1\text{ ; so that}$$

$$\mathbf{R} - R\vec{\beta} \to \mathbf{R}$$

Then the *acceleration field equation* (3) takes the form

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\vec{\mathbf{R}} \times (\mathbf{R} \times \dot{\vec{\beta}})}{R^3}\right] \qquad ...(3a)$$

$$= \frac{q}{4\pi\varepsilon_0 c}\,\frac{(\mathbf{R} \cdot \dot{\vec{\beta}})\,\mathbf{R} - (\mathbf{R} \cdot \mathbf{R})\,\dot{\vec{\beta}}}{R^3}$$

$$= \frac{q}{4\pi\varepsilon_0 c}\left\{\frac{(\mathbf{R} \cdot \dot{\vec{\beta}})\,\mathbf{R} - [R^2\,\dot{\vec{\beta}}]}{R^3}\right] \qquad ...(3b)$$

Using $\mathbf{n} = \mathbf{R}/R$ where $\mathbf{n}$ is a unit vector along $\mathbf{R}$, we get

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{(\mathbf{n} \cdot \dot{\vec{\beta}}\,\mathbf{n} - \dot{\vec{\beta}}}{R}\right] \qquad ...(4)$$

If $\mathbf{a}$ is the acceleration of the particle, then $\mathbf{a} = \dot{\mathbf{v}} = c\dot{\vec{\beta}}$ or $\dot{\vec{\beta}} = \mathbf{a}/c$ and we get

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{(\mathbf{n} \cdot \mathbf{a})\,\mathbf{n} - \mathbf{a}}{R}\right] \qquad ...(5)$$

Now
$$\mathbf{B}_a = \left[\frac{\mathbf{n} \times \mathbf{E}_a}{\mu_0 c}\right] \qquad ...(6)$$

Therefore the *instantaneous energy flux given by Poynting vector* is

$$\mathbf{S}_a^* = \mathbf{E}_a \times \mathbf{H}_a = \frac{\mathbf{E}_a \times \mathbf{B}_a}{\mu_0}$$

---

*All values on R.H.S. correspond to retarded position of charge.

$$= \left[\frac{\mathbf{E}_a \times (\mathbf{n} \times \mathbf{E}_a)}{\mu_0 c}\right] = \frac{1}{\mu_0 c}\,[(\mathbf{E}_a \cdot \mathbf{E}_a) - (\mathbf{E}_a \cdot \mathbf{n})\,\mathbf{E}_\alpha]$$

But $\mathbf{E}_a$ is perpendicular to $\mathbf{R}$ or $\mathbf{n}$ as is obvious from (3), thus $\mathbf{E}_a \cdot \mathbf{n} = 0$; hence we get

$$\mathbf{S}_a = \frac{1}{\mu_0 c}\,|E_a|^2\,\mathbf{n}$$

Substituting value of E_a from (5) we get

$$\mathbf{S}_a = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0 c^2}\right)^2 \left|\left\{\frac{(\mathbf{n}\cdot\mathbf{a})\,\mathbf{n} - \mathbf{a}}{R}\right\}^2\right|\,\mathbf{n}$$

$$= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0 c^2}\right)^2 \left[\frac{(\mathbf{n}\cdot\mathbf{a})^2 - 2\,(\mathbf{n}\cdot\mathbf{a})\,(\mathbf{a}\cdot\mathbf{n}) + a^2}{R^2}\right]$$

Using $u_0\varepsilon_0 = 1/c^2$ and $\mathbf{n}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{n}$, we obtain

$$\mathbf{S}_a = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{4\pi c^3}\left[\frac{a^2 - (\mathbf{n}\cdot\mathbf{a})}{R}\right]\mathbf{n}$$

If θ is the angle between $\mathbf{n}$ and $\mathbf{a}$, then $\mathbf{n}\cdot\mathbf{a} = a\cos\theta$; So that we get

$$\mathbf{S}_a = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c^3}\left[\frac{a^2 - a^2\cos^2\theta}{R^2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3 . R^2}\,\sin^2\theta\,\mathbf{n} \qquad ...(6)$$

This expression represents the power flow per unit area. The total power radiated may be obtained by integrating this Poynting vector over a closed surface surrounding the charge. A convenient choice for such a surface is a sphere centred at the retarted position of the charge. It is convenient to choose z. axis along acceleration a (fig. 12.3). Then

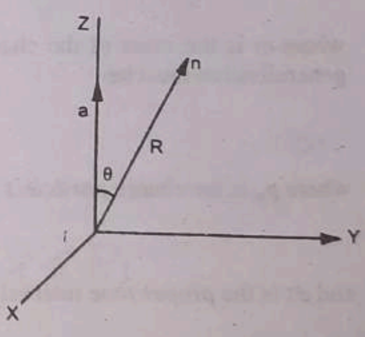


Fig. 12.3

$$P = \int_S \mathbf{S}_a \cdot \mathbf{n}\, da$$

(where da is an element of area)

$$= \int_S \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3 R}\,\sin^2\theta\ \mathbf{n}\cdot\mathbf{n}\,da$$

But $\mathbf{n}\cdot\mathbf{n} = 1$ and

$da = (R\,d\theta)\,(R\sin\theta\,d\phi)$; therefore

$$P = \int_0^\pi \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2 \sin^2\theta}{4\pi c^3 R^2}\,(R\,d\theta)\quad(R\sin\theta\,d\phi)$$

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3}\int_0^\pi \sin^3\theta\,d\theta \int_0^{2\pi} d\phi$$

But $\int_0^{2\pi} d\phi = 2\pi$ and $\int_0^\pi \sin^3\theta\,d\theta = \frac{4}{3}$; therefore we get

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3}\left(\frac{4}{3}\right)\cdot(2\pi) = \frac{1}{4\pi\varepsilon_0}\,\frac{2q^2 a^2}{3c^3} \qquad ...(9)$$

This is familiar *Larmor's formula* for the power radiated by a slowly moving (non-relativistic) accelerated charge.

12.7. Radiation from an Accelerated Charge at High Velocity

Relativistic Generalisation of Larmor's Formula.

Larmor's non-relativistic formula is

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2q^2a^2}{3c^3} \qquad ...(1)$$

This formula can be generalised for arbitrary velocities of charge by using the fact that radiated electromagnetic energy behaves under Lorentz transformation like the fourth component of a four-vector*. Since $P = dE_{rad}/dt$ *i.e.* $dE_{rad} = Pdt$ and time t also behaves under Lorentz transformations like the fourth component of *space-time* four-vector; this means that the power P is a Lorentz invariant quantity. If we can find a Lorentz invariant which reduces to non-relativistic Larmor's formula (1) for $\beta = v/c \ll 1$, then we have the required relativistic generalisation of Larmor's formula. There are, of course, many Lorentz invariants which reduce to desired for $\beta \to 0$. But from the general form of electromagnetic fields for a moving charge, it is evident that the general result must involve only $\overrightarrow{\beta}$ and $\dot{\overrightarrow{\beta}}$. With this restriction on the order of derivatives which can appear, we get unique result.

To find the appropriate generalisation of Larmor's formula let us use the fact that $\mathbf{a} = \mathbf{F}/m = \left(\frac{d\mathbf{p}/dt}{m}\right)$ and write Larmor's *non-relativistic formula* (1) in the form

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2q^2(\mathbf{a}\cdot\mathbf{a})}{3c^3} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{m^2c^2}\left(\frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}\right) \qquad ...(2)$$

where m is the mass of the charge and $\mathbf{p}$ its momentum. Equation (2) suggests that its Lorentz invariant generalisation must be

$$P = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{q^2}{m^2c^2}\left(\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau}\right) \qquad ...(3)$$

where p_μ is the charge *particle's momentum energy four vector* given by

$$p_\mu = \left(\mathbf{p}, \frac{iE}{c}\right) \qquad ...(4)$$

and $d\tau$ is the *proper time* interval given by

$$\left.\begin{aligned} dt &= \frac{d\tau}{\sqrt{(1-v^2/c^2)}} = \frac{d\tau}{\sqrt{(1-\beta^2)}} = \gamma\, d\tau \\ \text{where} \quad \gamma &= \frac{1}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad ...(5)$$

To check that equations (3) reduces properly to equation (2) as $\beta \to 0$, we evaluate the four-vector scalar product.

$$\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau} = \left(\frac{dp_1}{d\tau}\right)^2 + \left(\frac{dp_2}{d\tau}\right)^2 + \left(\frac{dp_3}{d\tau}\right)^2 + \left(\frac{dp_4}{d\tau}\right)^2$$

---

*If a pulse of electromagnetic radiatioin of finite spatial extent exists in charge and current free space, then total electromagnetic momentum and energy transform like a four-vector.

$$= \left(\frac{d\mathbf{p}}{d\tau}\right)^2 + \left\{\frac{d}{d\tau}\left(\frac{iE}{c}\right)\right\}^2 = \left(\frac{d\mathbf{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2$$

But $d\tau = \dfrac{dt}{\gamma}$

$$\therefore \qquad \frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau} = \gamma^2\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{\gamma^2}{c^2}\left(\frac{dE}{dt}\right)^2$$

Making this substitution in equation (3), we get

$$P = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{q^2\gamma^2}{m^2c^3}\left[\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right] \qquad ...(6)$$

Now in order to express above equation in terms of velocity and acceleration ; let us use relativistic formulae

$$E = \frac{mc^2}{\sqrt{(1-\beta^2)}} \text{ and } \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{(1-\beta^2)}} \qquad ...(7)$$

where m is the rest mass.

From (7), we have

$$\frac{dE}{dt} = \frac{d}{dt}\{mc^2(1-\beta^3)^{-1/2}\} = \frac{mc^2\,\vec{\beta}\cdot\dot{\vec{\beta}}}{(1-\beta^2)^{3/2}} = mc^2\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}} \qquad ...(8)$$

and

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left\{\frac{m\mathbf{v}}{\sqrt{(1-\beta^2)}}\right\} = \frac{d}{dt}\left\{\frac{mc\,\vec{\beta}}{\sqrt{(1-\beta^2)}}\right\}$$

$$= \frac{mc\,[\dot{\vec{\beta}}(1-\beta^2) + \dot{\vec{\beta}}(\beta\cdot\dot{\vec{\beta}})]}{(1-\beta^2)^{3/2}}$$

i.e.

$$\frac{d\mathbf{p}}{dt} = mc\gamma^3\,[\dot{\vec{\beta}}\cdot(1-\beta^2) + \vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})] \qquad ...(9)$$

Using equations (8) and (9), equation (6) becomes

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^3}{m^2c^3}\,[m^2c^2\gamma^6\{\dot{\vec{\beta}}(1-\beta^2) + \vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\}^2 - m^2c^2\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2]$$

on simplification, we get

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2}{c}\,\gamma^6\,[(\dot{\vec{\beta}})^2 - \beta^2\dot{\beta}^2 + (\vec{\beta}\cdot\dot{\vec{\beta}})^2]$$

But $(\vec{\beta}\times\dot{\vec{\beta}})^2 = (\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \beta^2\dot{\beta}^2$,

therefore

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^6}{c}\,[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2] \qquad ...(10)$$

This expression represents the *relativistic generalisation of Larmor's formula.*

Now let us consider the following particular cases ;

Case (i). If the motion of the particle is non-relativistic, then $\beta \to 0$ and $\gamma = \frac{1}{\sqrt{(1-\beta^2)}} \to 1$, therefore equation (10) takes the form

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c} (\vec{\dot{\beta}})^2$$

But $\vec{\dot{\beta}} = \mathbf{a}/c$; therefore we get

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2 \mathbf{a}^2}{c^3} \qquad ...(11)$$

This well known *non-relativistic Larmor's formula.* This verifies the relativistic generalisation of Larmor's formula given by (10).

Case (ii). If the motion of the particle is such that its velocity and acceleration are parallel (or collinear) ; then

$$|\vec{\beta} \times \vec{\dot{\beta}}| = \beta \dot{\beta} \sin 0 = 0$$

So that from equation (10)

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2\gamma^6}{c} \vec{\dot{\beta}}^2$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2q^2\gamma^6 \mathbf{a}^2}{3c^3} \qquad ...(12)$$

This principal example of this type of radiations is ***Bremsstrahlung radiations*** which are produced from accelerated electrons with the assumption that the direction of motion of electrons does not change.

Case. (iii). If the motion of the particle in such that its velocity and acceleration are perpendicular ; then

$$|\vec{\beta} \times \vec{\dot{\beta}}| = \beta \dot{\beta} \sin 90 = \beta \dot{\beta}$$

So that from equation (10), we get

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2\gamma^6}{c} (\dot{\beta}^2 - \beta^2 \dot{\beta}^2)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2\gamma^6}{c} \dot{\beta}^2 (1 - \beta^2)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2\gamma^4}{c} \dot{\beta}^2$$

i.e.

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2\gamma^4 a^2}{c^3} \qquad ...(13)$$

Example of this type of radiations occurs in circular accelerators like synchroton or cyclotron where energy loss is drastic.

0 and $\gamma = \frac{1}{\sqrt{(1-\beta^2)}} \to 1$,

...(11)

generalisation of Larmor's

celeration are parallel (or

...(12)

which are produced from
does not change.

ation are perpendicular

...(13)

cyclotron where energy

12.8. Angular Distribution of Radiation Emitted by an Accelerated Charge.

For relativistic motion of charged particle the acceleration field depends on the velocity as well as acceleration and is given by

$$\left.\begin{aligned} \mathbf{E}_a &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}\times(\mathbf{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c\,(R-\mathbf{R}\cdot\vec{\beta})^3}\right] \\ \text{and} \quad \mathbf{B}_a &= \frac{\mathbf{n}\times\mathbf{B}_a}{c} = \frac{\mathbf{R}\times\mathbf{E}_a}{Rc} \end{aligned}\right\} \quad \text{...(1)}$$

The *Poynting Vector is*

$$[\mathbf{S}_a] = [\mathbf{E}_a\times\mathbf{H}_a] = \left[\frac{\mathbf{E}_a\times\mathbf{B}_a}{\mu_0}\right]$$

$$= \left[\frac{\mathbf{E}_a\times(\mathbf{n}\times\mathbf{E}_a)}{\mu_0 c}\right] = \left[\frac{(\mathbf{E}_a\cdot\mathbf{E}_a)\,\mathbf{n}-(\mathbf{E}_a\cdot\mathbf{n})\,\mathbf{E}_a}{\mu_0 c}\right]$$

$$= \left[\frac{\mathbf{E}_a^2\mathbf{n}}{\mu_0 c}\right] \quad [\text{Since } \mathbf{E}_a\cdot\mathbf{n} = \mathbf{E}_a\cos 90^\circ = 0] \quad \text{...(2a)}$$

The *radial component of Poynting vector is*

$$[\mathbf{S}_a\cdot\mathbf{n}] = \left[\frac{\mathbf{E}_a^2}{\mu_0 c}\right] \quad \text{...(2b)}$$

Here $\mathbf{S}_a\cdot\mathbf{n}$ is the energy emitted per unit area per unit time detected at an observation point at time t due to radiation emitted by the charge at time $t' = t - R(t')/c$. If we want to calculate the energy radiated during a finite period, say from $t' = T_1$ to $t' = T_2$, we would write

$$W = \int_{t=T_1+[\{R(T_1)\}/c]}^{t=T_2+[\{R(T_2)\}/c]} [\mathbf{S}_a\cdot\mathbf{n}]\,dt$$

$$= \int_{t'=T_1}^{t'=T_2} (\mathbf{S}_a\cdot\mathbf{n})\,\frac{dt}{dt'}\,dt'$$

We thus note that the useful and meaningful quantity is $(\mathbf{S}_a\cdot\mathbf{n})\,(dt/dt')$, which represents the *power radiated per unit area is terms of charge's own time.*

The *power radiated per unit solid angle is defined as*

$$\frac{dP(t')}{d\Omega} = (\text{Power radiated per unit area}) \times \frac{4\pi R^2}{4\pi}$$

$$= R^2 \times (\text{Power radiated per unit area})$$

$$= R^2\,(\mathbf{S}_a\cdot\mathbf{n})\,\frac{dt}{dt'} \quad \text{...(3)}$$

As $t' = t - R/c$.

$$\therefore \quad \frac{dt'}{dt} = 1 - \frac{1}{c}\frac{dR}{dt} = 1 - \frac{1}{c}\frac{dR}{dt'}\frac{dt'}{dt}$$

i.e.

$$\left(1+\frac{1}{c}\frac{dR}{dt'}\right)\frac{dt'}{dt} = 1 \quad \text{or} \quad \frac{dt'}{dt} = \frac{1}{1+\frac{1}{c}\frac{dR}{dt'}}$$

But $$\frac{dR}{dt'} = -\frac{\mathbf{v}\cdot\mathbf{R}}{R} = -\mathbf{v}\cdot\mathbf{n} \text{ since } \mathbf{n} = \frac{\mathbf{R}}{R}$$

$$\therefore \quad \frac{dt'}{dt} = \frac{1}{1-(\mathbf{v}\cdot\mathbf{n}/c)} = \frac{1}{1-\mathbf{n}\cdot\vec{\beta}}$$

$$\therefore \quad \frac{dt}{dt'} = 1-\mathbf{n}\cdot\vec{\beta} = \kappa \qquad \text{...(4)}$$

Using (4), equation (3) gives

$$\frac{dP(t')}{d\Omega} = \kappa R^2 (\mathbf{S}_a\cdot\mathbf{n}) \qquad \text{...(5)}$$

$$= \kappa \mathbf{R}^2 \frac{E_a^2}{\mu_0 c} \text{ [using (2}b\text{)]}$$

$$= \frac{\kappa R^2}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \left[\frac{\mathbf{R}\times(\mathbf{R}-\mathbf{R}\vec{\beta})\times\vec{\dot{\beta}}}{c(R-R\cdot\vec{\beta})^3}\right]^2$$

$$= \frac{\kappa R^2}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \left[\frac{\mathbf{n}\times\{(\mathbf{n}-\vec{\beta})\times\vec{\dot{\beta}}\}}{c(1-\mathbf{n}\cdot\vec{\beta})^3}\right]^2$$

Using $\kappa = 1-\mathbf{n}\cdot\vec{\beta}$ and $\mu_0\varepsilon_0 = \frac{1}{c^2}$, we get

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{[\mathbf{n}\times\{(\mathbf{n}-\vec{\beta})\times\vec{\dot{\beta}}\}]^2}{c(1-\mathbf{n}\cdot\vec{\beta})^5} \qquad \text{...(6)}$$

This equation represents the general expression for the angular distribution of power radiated by an accelerated charge. Let us now consider the following particular cases.

Case (i). When velocity and acceleration are parallel, then

$$\vec{\beta}\times\vec{\dot{\beta}} = 0$$

Therefore equation (6) reduces to

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{[\mathbf{n}\times(\mathbf{n}\times\vec{\dot{\beta}})]^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{[(\mathbf{n}\cdot\vec{\dot{\beta}})\,\mathbf{n}-(\mathbf{n}\cdot\mathbf{n})\,\vec{\dot{\beta}}]^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{[\{(\mathbf{n}\cdot\vec{\dot{\beta}})\,\mathbf{n}-\vec{\dot{\beta}}\}]^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{[(\mathbf{n}\cdot\vec{\dot{\beta}})^2-2(\mathbf{n}\cdot\vec{\dot{\beta}})^2]+\vec{\dot{\beta}}^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

...(4)

...(5)

...(6)

of power radiated by an

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\vec{\beta}}^2-(\mathbf{n}\cdot\dot{\vec{\beta}})^2}{(1-\mathbf{n}\cdot\vec{\beta})^5} \qquad ...(7)$$

If θ is the angle of observation between **n** and common direction of $\dot{\vec{\beta}}$ and $\vec{\beta}$, then

$$\mathbf{n}\cdot\dot{\vec{\beta}} = \dot{\vec{\beta}}\cos\theta \quad \text{and} \quad \mathbf{n}\cdot\vec{\beta} = \beta\cos\theta$$

So that equation (7) reduces to

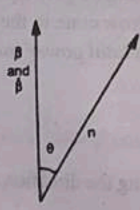


Fig. 12..4

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\beta}^2-\dot{\beta}^2\cos^2\theta}{(1-\beta\cos\theta)^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}$$

But $\dot{\vec{\beta}} = \frac{\mathbf{a}}{c}$

$$\therefore \qquad \frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad ...(8)$$

This is required relativistic expression.

For non-relativistic case $\beta \ll 1$, so that expression (8) takes the form

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\sin^2\theta \qquad ...(9)$$

Thus for an accelerated charge in non-relativistic motion the angular distribution shows a simple $\sin^2\theta$ behaviour. But as $\beta \to 1$, the angular distribution is tipped forward more and more and increase in magnitude as indicated schematically in fig. 11.5. Θ_{max} is the angle for which intensity is maximum.

The value of θ_{max} may be obtained by the condition

$$\frac{d}{d\theta}\left(\frac{dP}{d\Omega}\right) = 0 \quad \text{for} \quad \theta = \theta_{max}$$

$$\text{i.e.} \qquad \frac{d}{d\theta}\left[\frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right]_{\theta=\theta_{max}} = 0$$

$$\text{i.e.} \qquad \frac{d}{d\theta}\left\{\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right\}\Bigg|_{\theta=\theta_{max}} = 0$$

This gives $[3\beta\cos^2\theta + 2\cos\theta - 5\beta]_{\theta=\theta_{max}} = 0.$

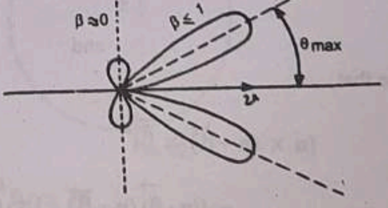


Fig. 12.5. Radiation pattern for charge accelerated along its direction of motion.

or

$$\cos\theta_{max} = \frac{\sqrt{(1+15\beta^2)}-1}{3\beta}$$

i.e.
$$\theta_{\max} = \cos^{-1}\left[\frac{\sqrt{(1+15\beta^2)}-1}{3\beta}\right] \qquad \text{...(10a)}$$

This in the limit β → 1 reduces to

$$\theta_{\max} = \tfrac{1}{2}\sqrt{(1-\beta^2)} = \frac{1}{2\gamma} \qquad \text{...(10b)}$$

For relativistic particles $\theta_{\max}$ is very small, thereby indicating that the angular distribution is confined to a very narrow cone in the direction of motion.

The *total power radiated* can be obtained by integrating equation (8) over all angles *i.e.*

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\int \frac{1-\sin^2\theta}{(1-\beta\cos\theta)^5}\, d\Omega$$

Choosing the direction of $\vec{\beta}$ (or $\vec{\dot{\beta}}$) along Z-axis, we get

$$\begin{aligned}
P &= \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\int_0^{\pi}\int_0^{2\pi}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\,\frac{(R\,d\theta)\,(R\sin\theta\,d\phi)}{R^2}\\
&= \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\int_0^{\pi}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\,d\theta\int_0^{2\pi}d\phi\\
&= \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\frac{4}{3(1-\beta^2)^3}\cdot(2\pi)\\
&= \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\frac{8\pi}{3(1-\beta^2)^3}
\end{aligned}$$

i.e.
$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2\gamma^6}{c^3} \qquad \text{...(11)}$$

This it exactly same result as equation (12) of section 12.7.

Case (ii). When velocity and acceleration are perpendicular, then we choose a coordinate system such that instantaneously β is along Z-axis and $\vec{\dot{\beta}}$ along X-axis. If (θ, ϕ) are polar coordinates defining the direction of observation, then

$$\left.\begin{aligned}
\mathbf{n}\cdot\vec{\beta} &= \beta\cos\theta\\
\text{and}\qquad \mathbf{n}\cdot\vec{\dot{\beta}} &= \dot{\beta}\,\sin\theta\cos\phi
\end{aligned}\right] \qquad \text{...(12)}$$

so that

$$\begin{aligned}
&[\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\vec{\dot{\beta}}]^2\\
&\quad= [(\mathbf{n}\cdot\vec{\dot{\beta}})(\mathbf{n}-\vec{\beta}) - \{\mathbf{n}\cdot(\mathbf{n}-\vec{\beta})\}\vec{\dot{\beta}}]^2\\
&\quad= (\mathbf{n}\cdot\vec{\dot{\beta}})^2(\mathbf{n}-\vec{\beta})^2 - 2(\mathbf{n}\cdot\vec{\dot{\beta}})(\mathbf{n}-\vec{\beta}).\\
&\qquad\quad \{\mathbf{n}\cdot(\mathbf{n}-\vec{\beta})\}\,\vec{\dot{\beta}} + \{\mathbf{n}\cdot(\mathbf{n}-\beta)\}^2\,\vec{\dot{\beta}}^2
\end{aligned}$$

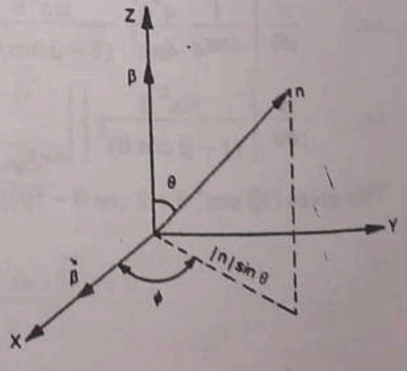


Fig. 12.6

$$= (\mathbf{n}\cdot\vec{\dot{\beta}})^2(1-2\beta\cdot\mathbf{n}+\beta^2) - 2(\mathbf{n}\cdot\vec{\dot{\beta}})(\mathbf{n}-\vec{\beta})\cdot(1-\mathbf{n}\cdot\vec{\beta})\vec{\dot{\beta}} + \dot{\beta}^2(1-\mathbf{n}\cdot\vec{\beta})$$

$$= (\mathbf{n}\cdot\vec{\dot{\beta}})^2(1-2\vec{\beta}\cdot\mathbf{n}+\beta^2) - 2(\mathbf{n}\cdot\vec{\dot{\beta}})^2(1-\mathbf{n}\cdot\vec{\beta}) + \dot{\beta}^2(1-\mathbf{n}\cdot\vec{\beta})^2$$

$$(\text{since } \beta\cdot\vec{\dot{\beta}} = \beta\,\vec{\dot{\beta}}\cos 90 = 0)$$

Using (12), we get

$$[\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\vec{\dot{\beta}}]^2 = (\dot{\beta}\sin\theta\cos\phi)^2(1-2\beta\cos\theta+\beta^2)$$
$$-2(\dot{\beta}\sin\theta\cos\phi)^2(1-\beta\cos\theta)+\dot{\beta}^2(1-\beta\cos\theta)^2$$
$$= (\dot{\beta}\sin\theta\cos\phi)^2[1-2\beta\cos\theta+\beta^2-2+2\beta\cos\theta]+\dot{\beta}^2(1-\beta\cos\theta)^2$$
$$= \dot{\beta}^2[(1-\beta\cos\theta)^2-\sin^2\theta\cos^2\phi(1-\beta^2)] \qquad \text{...(13)}$$

Making these substitutions in equation (6), we get

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\left[\frac{\dot{\beta}^2\{(1-\beta\cos\theta)^2-\sin^2\theta\cos^2\phi(1-\beta^2)\}}{(1-\beta\cos\theta)^5}\right]$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2\dot{\beta}^2}{4\pi c(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi(1-\beta^2)}{(1-\beta\cos\theta)^2}\right]$$

Using $\dot{\beta} = \frac{\mathbf{a}}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we obtain

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \qquad \text{...(14)}$$

This is required expression for the present case. The formula indicates that for large velocities the radiation is very much more intense in the forward direction than in any other direction as shown in fig. 12.7. As velocity v approaches c *i.e.* β approaches 1, then the radiation becomes a sharp forward ray.

The total power radiated can be found by integrating (14) over all angles

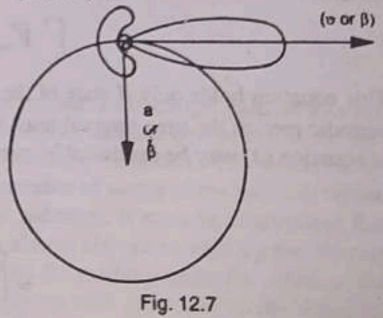


Fig. 12.7

i.e.
$$P(t') = \int \frac{dP}{d\Omega}\,d\Omega$$

i.e.
$$P = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^2}\int_0^{\pi}\int_0^{2\pi}\frac{1}{(1-\beta\cos\theta)^2}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]\sin\theta\,d\theta\,d\phi$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\int_0^{\pi}\int_0^{2\pi}\frac{1}{(1-\beta\cos\theta)^2}\left[1-\frac{(1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^2}\right]\sin\theta\,d\theta\,d\phi$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\cdot\frac{8\pi}{3(1-\beta^2)^2}$$

$$\therefore \qquad P(t') = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2a^2}{c^3}\,\gamma^4$$

which is result as equation (13) of section 12.7.

12.9. Radiation Damping. The Abraham-Lorentz Formula

According to the laws of classical electrodynamics an accelerated charge radiates electromagnetic energy, which comes at the expense of particle's kinetic energy. Consequently under the influence of a given force, a charged particle accelerates less than a neutral particle of the same mass. This implies that the radiation exerts a force back on the charge. This force is called the **radiation reaction** or **radiation damping.** It may be derived as follows :

If F_{ext} is the external force acting on a particle of mass m, then Newton's equation of motion is

$F_{ext} = ma$, where a is the acceleration.

If the particle under consideration a charged accelerated particle, it will emit radiation, which it turn will damp the motion of the particle. If F_{rad} is the damping force, then we can write

$$F_{ext} + F_{rad} = ma \qquad \text{...(1)}$$

According to Larmor's formula; the total power radiated by a non-relativisitic particle ($v << c$) is given by

$$P = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\,\frac{q^2a^2}{c^3} \qquad \text{...(2)}$$

i.e.

$$\vec{\mathbf{F}}_{rad}\cdot\vec{\mathbf{v}} = -\frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2a^2}{c^3} \qquad \text{...(3)}$$

For the conservation of energy, the energy lost by the particle in any interval of time must be equal to energy carried away by the radiation plus whatever extra energy has been supplied into the velocity fields. Now if we assume that at the end of the specified time interval the system returns to its initial state, then the net energy supplied into velocity fields is zero; so that equation (2), in the form of energy conservation becomes

$$\int_{t_1}^{t_2}\vec{\mathbf{F}}_{rad}\cdot\vec{\mathbf{v}}\,dt = -\frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2}{c^3}\int_{t_1}^{t_2} a^2\,dt \qquad \text{...(3)}$$

This equation holds only if state of the system is identical at instants t_1 and t_2. For example in the case of periodic motion the time interval must be equal to an integral number of full cycles. The integral on R.H.S. of equation (3) may be evaluated by parts *i.e.*

$$\int_{t_1}^{t_2} a^2\,dt = \int_{t_1}^{t_2}\frac{d\vec{\mathbf{v}}}{dt}\cdot\frac{d\vec{\mathbf{v}}}{dt}\,dt \qquad \left(\text{since } \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}\right)$$

$$= \left[\vec{\mathbf{v}}\cdot\frac{d\vec{\mathbf{v}}}{dt}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d^2\vec{\mathbf{v}}}{dt^2}\cdot\vec{\mathbf{v}}\,dt$$

First term is zero because velocities and accelerations at t_1 and t_2 are identical. So

$$\int_{t_1}^{t_2} a^2\,dt = -\int_{t_1}^{t_2}\frac{d^2\vec{\mathbf{v}}}{dt^2}\cdot\vec{\mathbf{v}}\,dt \qquad \text{...(4)}$$

Using this equation (3) takes the form

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$$\int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{v}\, dt = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c^3} \int_{t_1}^{t_2} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt$$

or

$$\int_{t_1}^{t_2} \left[\vec{F}_{rad} - \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c^3} \frac{d^2\vec{v}}{dt^2} \right] \cdot \vec{v}\, dt = 0$$

This equations is satisfied only if

$$\vec{F}_{rad} = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c^3} \frac{d^2\vec{v}}{dt^2}$$

Writing

$$\frac{d^2\vec{v}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{v}}{dt}\right) = \frac{d}{dt}\,\vec{a} = \dot{\vec{a}} = \ddot{\vec{v}}$$

∴

$$F_{rad} = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c^3}\, \dot{\vec{a}} \qquad ...(5)$$

This equation represents the radiation reaction or radiation damping force and is called the **Abraham-Lorentz Formula.**

$$F_{rad} = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c^3}\, \dot{a} = m\tau\dot{a}$$

where for convenience damping factor $\tau = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{mc^3}$

Now equation (1) may be expressed as

$$\mathbf{F}_{ext} + m\tau\dot{\mathbf{a}} = m\dot{\mathbf{v}}$$

$$\mathbf{a} = \dot{\mathbf{v}},\ \dot{\mathbf{a}} = \ddot{\mathbf{v}}$$

$$\mathbf{F}_{ext} + m\tau\ddot{\mathbf{v}} = m\dot{\mathbf{v}}$$

$$m(\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}}) = \mathbf{F}_{ext}$$

This equation is called **Abraham-Lorentz equation of motion.**

12.10. Cherenkov Radiation.

A charged particle in uniform motion in a straight line does not radiate. But charged particle moving with constant velocity in a material medium emits, in cetain circumstances an unusual type of radiation; first observed by P.A. Cherenkov. Hence such radiation is called *Cherenkov radiation.* The emission of Cherenkov radiation is a cooperative phenomenon involving a large number of atoms of the medium whose electrons are accelerated by fields of the passing particle and so emit radiation. It must be emphasised that Cherenkov radiation is entirely unrelated to bremssrahlung which is almost always emitted by the moving (accelerated) electron itself; while the Cherenkov radiation is emitted by the medium under the action of the passing particle. The distinction between the two types of radiation appears with particular clarity when the particle has a very large mass ; the bremsstrahlung disappears and the cherenkov radiation is unaffected.

The wave vector and frequency of an electromagnetic wave propagated in an isotropic, non-permeable medium ($K_m = 1$) are related by

$$k = \frac{\omega}{v} = \frac{n\omega}{c} \quad \text{where} \quad n = \frac{c}{v} = \sqrt{K_e} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

is the refractive index, which is real. We know that the frequency of the Fourier component of the field of a particle moving uniformly in the X-direction in a medium is related to the x-component of the wave vector by $\omega = k_x v$. If this component is a freely propagated wave, these two relations must be consistent. Since $k > k_x$, therefore it follows that we must have

$$v = \frac{\omega}{k_x} > \frac{\omega}{k}$$

i.e.

$$v > \frac{c}{n} \left(\text{Since } \frac{\omega}{k} = \frac{c}{n} \right) \qquad \text{...(A)}$$

where n is function of ω. Thus *Cherenkov radiation of frequency* ω *occurs if the velocity of the particle exceeds the phase velocity of waves of that frequency in the medium concerned.*

If θ is the angle between the direction of motion of the particle and the direction of emission. We have

$$k_x = k \cos\theta = \left(\frac{n\omega}{c} \right) \cos\theta \text{ and since } k_x = \frac{\omega}{v}$$

we find that

$$\frac{\omega}{v} = \left(\frac{n\omega}{c} \right) \cos\theta \text{ i.e. } \cos\theta = \frac{c}{nv} \qquad \text{...(B)}$$

This relation indicates that a definite value of angle θ corresponds to radiation of given frequency. That is the radiation of each frequency is emitted forwards and is distributed over the surface of a cone of vertical angle 2θ, where θ is given by equation (B) [fig. 12.8]. The distribution of radiation in angle and frequency are thus related in a definite manner.

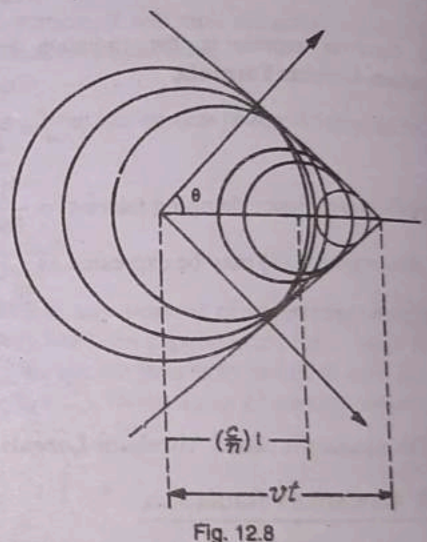

Fig. 12.8

Maxwell's equations are

$$\left. \begin{array}{ll} \text{div } \mathbf{D} = \rho, & \text{div } \mathbf{B} = 0 \\ \text{Curl } \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{Curl } \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} \end{array} \right] \qquad \text{...(1)}$$

These equations for isotropic non-permeable medium of permittivity ε (*i.e.* where $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$) take the form

$$\left. \begin{array}{rl} & \text{div } \mathbf{E} = \dfrac{\rho}{\varepsilon} \\ & \text{div } \mathbf{B} = 0 \\ & \text{curl } \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \text{and} & \text{curl } \mathbf{H} = \mu_0 \mathbf{J} + \mu_0 \varepsilon \dfrac{\partial \mathbf{E}}{\partial t} \end{array} \right\} \qquad \text{...(2)}$$

For a non-permeable medium the refractive index n is given by

$$n = \sqrt{(K_e K_m)} = \sqrt{\left(\frac{\varepsilon}{\varepsilon_0} \cdot \frac{\mu_0}{\mu_0} \right)} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = K_e$$

In the present case the extraneous charge and current distributions are given by

$$\rho = e\delta(\mathbf{r} - \mathbf{v}t) \text{ and } \mathbf{J} = e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t) \qquad \text{...(3a)}$$

If ϕ and $\mathbf{A}$ are scalar and vector potentials, then

$$\mathbf{B} = \text{curl}\,\mathbf{A},\ \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\,\phi. \qquad \text{...(3b)}$$

So that Maxwell's equations (2) are satisfied identically. The additional condition imposed on potentials **A** and ϕ is the Lorentz condition

$$\text{div}\,\mathbf{A} + \mu_0\,\varepsilon\,\frac{\partial\phi}{\partial t} = 0 \text{ or div}\,\mathbf{A} + \frac{n^2}{c^2}\frac{\partial\phi}{\partial t} = 0. \qquad \text{...(4)}$$

$$\left(\text{since } \mu_0\varepsilon_0 = 1/c^2 \text{ and } \frac{\varepsilon}{\varepsilon_0} = K_e = n^2\right)$$

Then substituting equation (3) in equation (2), we note that the Maxwell's equations are equivalent to the following two inhomogeneous equations for the potentials

$$\left.\begin{aligned} \nabla^2\mathbf{A} - \mu_0\,\varepsilon\,\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0\mathbf{J} \\ \text{and} \qquad \nabla^2\phi - \mu_0\,\varepsilon\,\frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\varepsilon} \end{aligned}\right\} \qquad \text{...(5)}$$

These equations may be equivalently written as

$$\left.\begin{aligned} \nabla^2\mathbf{A} - \frac{n^2}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0\,e\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t) \\ \nabla^2\phi - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -\frac{e}{\varepsilon}\,\delta(\mathbf{r} - \mathbf{v}t) \end{aligned}\right] \qquad \text{...(6)}$$

Expanding A and ϕ as Fourier space integral *i.e.*

$$\left.\begin{aligned} \mathbf{A} &= \int_{-\infty}^{+\infty} A_k\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \\ \phi &= \int_{-\infty}^{+\infty} \phi_k\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \end{aligned}\right] \qquad \text{...(7)}$$

Taking the Laplacian of these equations, we get

$$\left.\begin{aligned} \nabla^2\mathbf{A} &= -\int_{-\infty}^{+\infty} \mathbf{A}_k\, k^2\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \\ \text{and} \qquad \nabla^2\phi &= -\int_{-\infty}^{+\infty} \phi_k\, k^2\, e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned}\right\} \qquad \text{...(8)}$$

and the Fourier components of $\nabla^2\mathbf{A}$ and $\nabla^2\phi$ are

$$\left.\begin{aligned} (\nabla^2\mathbf{A})_k &= -k^2\,\mathbf{A}_k \\ (\nabla^2\phi)_k &= -k^2\,\phi_k \end{aligned}\right\} \qquad \text{...(9)}$$

Hence the Fourier components of equations (6) are

$$\begin{aligned} -\mathbf{A}^2\phi_k - \frac{n^2}{c^2}\frac{\partial^2\mathbf{A}_k}{\partial t^2} &= -\frac{1}{(2\pi)^3}\int \mu_0 e\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV \\ &= -\frac{1}{(2\pi)^3}\,\mu_0\, e\mathbf{v}\, e^{-i\mathbf{k}\cdot\mathbf{v}t} \end{aligned} \qquad \text{...(10a)}$$

and

$$-k^2\phi_k - \frac{n^2}{c^2}\frac{\partial^2\phi_k}{\partial t^2} = -\frac{1}{(2\pi)^3}\int \frac{\delta(\mathbf{r}-\mathbf{v}t)}{\varepsilon}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV$$

$$= \frac{1}{(2\pi)^3}\cdot\frac{e}{\varepsilon}\, e^{-i\mathbf{k}\cdot\mathbf{v}t} \qquad \text{...(10b)}$$

Equations (10) may be written as

$$k^2\mathbf{A}_k + \frac{n^2}{c^2}\frac{\partial^2\mathbf{A}_k}{\partial t^2} = \frac{\mu_0 e}{(2\pi)^3}\,\mathbf{v}\, e^{-i\mathbf{k}\cdot\mathbf{v}t} \qquad \text{...(11a)}$$

and

$$k^2\phi_k + \frac{n^2}{c^2}\frac{\partial^2\phi_k}{\partial t^2} = \frac{e}{(2\pi)^2\varepsilon}\, e^{-i\mathbf{k}\cdot\mathbf{v}t} \qquad \text{...(11b)}$$

This means that **A** and ϕ depend on time through the factor $e^{-i\mathbf{k}\cdot\mathbf{v}t}$. If we put $\omega = \mathbf{k}\cdot\mathbf{v} = k_x v$, with the assumption that the particle of charge e is moving along X-axis. Thus the time factor becomes $e^{-i\omega t}$, so that

$$\frac{\partial^2}{\partial t^2} \to -\omega^2$$

keeping this in mind and using $\varepsilon = K_e\varepsilon_0 = n^2\varepsilon_0$, we may write equations (11) as

$$\left.\begin{aligned} k^2\mathbf{A}_k - \frac{n^2\omega^2}{c^2}\mathbf{A}_k &= \frac{\mu_0 e}{(2\pi)^3}\, v e^{-i\omega t} \\ \text{and} \quad k^2\phi_k - \frac{n^2\omega^2}{c^2}\phi_k &= \frac{e}{(2\pi)^3\cdot n^2}\, e^{-i\omega t} \end{aligned}\right\} \qquad \text{...(12)}$$

Hence the Fourier components of **A** and ϕ are given by

$$\left.\begin{aligned} \mathbf{A}_k &= \frac{\mu_0}{4\pi}\frac{e}{2\pi^2}\frac{v}{k^2 - \frac{\omega^2 n^2}{c^2}}\, e^{-i\omega t} \\ \phi_k &= \frac{1}{4\pi\varepsilon_0}\frac{e}{2\pi^2 n^2}\frac{1}{\left(k^2 - \frac{\omega^2 n^2}{c^2}\right)}\, e^{-i\omega t} \end{aligned}\right\} \qquad \text{...(13)}$$

From equation (3), the Fourier component of electric field is given by

$$\mathbf{E}_k = i\omega\cdot\mathbf{A}_k - i\mathbf{k}\,\phi_k. \qquad \text{...(14)}$$

Substituting values of $\mathbf{A}_k$ and ϕ_k, we get

$$E_k = \frac{\mu_0}{2\pi}\frac{i\omega e}{2\pi^2}\frac{\mathbf{v}}{k^2 - \left(\frac{\omega^2 n^2}{c^2}\right)}\, e^{-i\omega t} - \frac{1}{4\pi\varepsilon_0}\frac{i\mathbf{k}e}{2\pi^2 n^2}\cdot\frac{1}{\left(k^2 - \frac{\omega^2 n^2}{c^2}\right)}\, e^{-i\omega t}$$

Writting $\mu_0\varepsilon_0 = \frac{1}{c^2}$ or $\mu_0 = \frac{1}{\varepsilon_0 c^2}$, we get

$$\mathbf{E}_k = \frac{1}{4\pi\varepsilon_0}\,\frac{ie\omega}{2\pi^2 n^2}\left[\frac{\dfrac{n^2}{c^2}\,\mathbf{v} - \dfrac{\mathbf{k}}{\omega}}{\left(k^2 - \dfrac{\omega^2 n^2}{c^2}\right)}\right] e^{-i\omega t} \qquad \text{...(15)}$$

The total electric field is obtained by inverting the Fourier transform

$$\mathbf{E} = \int_{-\infty}^{+\infty} \mathbf{E}_k\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \qquad \text{...(16a)}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{ie}{2\pi^2 n^2}\int_{-\infty}^{+\infty} \frac{\left(\dfrac{n^2}{c^2}\,\mathbf{v} - \dfrac{\mathbf{k}}{\omega}\right)\omega}{\left(k^2 - \dfrac{n^2\omega^2}{c^2}\right)}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega t}\, d\mathbf{k} \qquad \text{...(16b)}$$

Therefore the *force acting on a particle* is

$$\mathbf{F} = e\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\,\frac{ie^2}{2\pi^2 n^2}\int_{-\infty}^{+\infty} \frac{\left(\dfrac{n^2}{c^2}\,\mathbf{v} - \dfrac{\mathbf{k}}{\omega}\right)\omega}{\left(k^2 - \dfrac{n^2}{c^2}\,\omega^2\right)}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega t}\, d\mathbf{k}$$

Now let us find the value of the force at the point occupied by the particle namely $\mathbf{r} = \mathbf{v}t$. Then the factor $e^{i\mathbf{k}\cdot\mathbf{r}}$ in the integrand of (17) takes the form $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i\mathbf{k}\cdot\mathbf{v}t} = e^{i\omega t}$ and hence cancels with $e^{-i\omega t}$. Substituting $k_x v = \omega$, $q = \sqrt{(k_y^2 + k_z^2)}$; so that

$$k^2 = k_x^2 + k_y^2 + k_z^2 = (\omega^2/\mathbf{v}^2) + q^2$$

and replacing $dk_y\, dk_z = 2\pi q\, dq$, we can write the magnitude of $\mathbf{E}$ as

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{ie}{\pi}\int_{-\infty}^{+\infty}\int_0^{q_0} \frac{\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\omega\, q\, dq\, d\omega}{n^2\left[q^2 + \omega^2\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\right]}$$

The energy loss by the moving particle is just the work done by the force* $e\mathbf{E}$ exerted on the particle by the field which it produces. Therefore the magnitude of force acting at the point occupied by the particle *i.e.* $\mathbf{r} = \mathbf{v}t$ is

$$F = eE = \frac{1}{4\pi\varepsilon_0}\,\frac{ie^2}{\pi}\int_{-\infty}^{+\infty}\int_0^{q_0} \frac{\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\omega\, q\, dq\, d\omega}{n^2\left[q^2 + \omega^2\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\right]} \qquad \text{...(19)}$$

The *energy loss per unit length* is the work done by the force over that distance which is just $U = F$; it is called the *stopping power* of the substance with respect to the particle. The *energy loss in the frequency interval* $d\omega$ *is*

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*The magnetic force $q\mathbf{v} \times \mathbf{B}$ is seen by symmetry to be zero and in any case is perpendicular to velocity of the particle and hence does no work on it.

$$dU = dF = -d\omega \cdot \frac{1}{4\pi\varepsilon_0} \frac{ie^2}{\pi} \Sigma\omega \left(\frac{1}{c^2} - \frac{1}{n^2 v^2}\right) \int \frac{q\,dq}{q^2 - \omega^2 \left(\frac{n^2}{c^2} - \frac{1}{v^2}\right)}$$

where the summation is over terms with ± $\omega = |\omega|$. Let us introduce a new variable

$$p = q^2 - \omega^2 \left(\frac{n^2}{c^2} - \frac{1}{v^2}\right).$$

Then

$$dU = dF = -d\omega \left(\frac{1}{c^2} - \frac{1}{n^2 v^2}\right) \int \frac{dp}{p}.$$

The integration is performed by the method of contour integration. The singular point is given by $p = 0$ (for which $q^2 + k_x^2 = k^2$). In integration along the real p-axis we must pass, through the singular point $p = 0$ in some manner which is determined by the fact that, although we suppose that $n^2(\omega)$ is real, the medium being transparent, it actually has a small imaginary part, which is positive for $\omega > 0$ and negative for $\omega < 0$. Accordingly p has a small negative or positive imaginary part, and the path of integration ought to pass below or above the real axis respectively. This means that, when the path of integration is displaced to the real axis we must pass below or above the singular point respectively. This gives a contribution to dU and the real parts cancel in the sum. Identing the path of integration with infinitesimal semicircles, we find

$$\Sigma\,\omega \int \frac{dp}{p} = 2\pi i\omega.$$

Thus the final expression for the energy loss per unit length in the frequency range $d\omega$ is

$$dU = dF = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{c^2} \left(1 - \frac{c^2}{n^2 v^2}\right) \omega\, d\omega. \qquad \text{...(20)}$$

This expression represents for the intensity of radiation in a freqeuncy range $d\omega$.

From expression (20) it is obvious that Cherenkov radiation is emitted only when

$$1 > \frac{c^2}{n^2 v^2} \text{ or } v > \frac{c}{n} \qquad \text{...(21)}$$

which is in agreement with equation (A).

According to equation (B), the radiation is emitted in an angle interval

$$d\theta = \frac{c}{vn^2 \sin\theta} \frac{dn}{d\omega}\, d\omega.$$

The total intensity of Cherenkov radiation is obtained by integrating (20) over all frequencies for which the medium is transparent.

12.11. Radiation due to an Oscillating Electric Dipole

An electric dipole consists of two equal and opposite charges separated by a finite distance. If q is the magnitude of each charge and $d\mathbf{l}$ is the separation. then *dipole moment* **p** is defined as

$$\mathbf{p} = q\, d\mathbf{l} \qquad \text{...(1)}$$

where direction of **p** or $d\mathbf{l}$ is from $-q$ to $+q$.

If the charge varies sinusoidally, the dipole moment will vary accordingly, *i.e.* if $q = q_0 e^{-i\omega t}$ then

$$\mathbf{p} = (q_0 e^{-i\omega t})\, d\mathbf{l}$$

i.e.
$$\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}, \quad \text{where } \mathbf{p}_0 = q_0\, dl \qquad ...(2)$$

A dipole whose moment varies sinusoidally is known as *oscillating dipole. An oscillating dipole may be assumed to be accelerated charge equivalent to distribution* whose dipole-moment is equal to the dipole moment of oscillating dipole

$$\mathbf{p} = \int \rho \mathbf{r}\, dV'. \qquad ...(3)$$

The time rate of change of *dipole moment is given by*

$$\dot{\mathbf{p}} = \frac{dp}{dt} = \frac{d}{dt}(q\,dl) = \frac{dq}{dt}\,dl = I\,dl \qquad ...(4)$$

This shows that the *time rate of change of dipole moment of an oscillating dipole is equal to a current element.* As oscillating charges of electric dipole are undergoing acceleration* and an accelerated charge always radiates energy, *therefore an electric dipole radiates energy.*

Now in order to calculate the radiation emitted by an electric dipole, let us assume that the dipole consists of two charged spheres located at $z = \pm\, dl/2$ connected by a wire of negligible capacitance as shown in fig. 12.9.

Let us further assume that the length of the dipole $d\mathbf{l}$ is small compared to wavelength λ of radiation emitted due to periodic oscillations of charges so that all source points have the same retarded time and no phase difference occurs at observation point P between waves originating from different parts of the source.

Let polar coordinates of the observation point P be (r, θ) only due to axial symmetry. Now let us calculate the following parameters of interest.

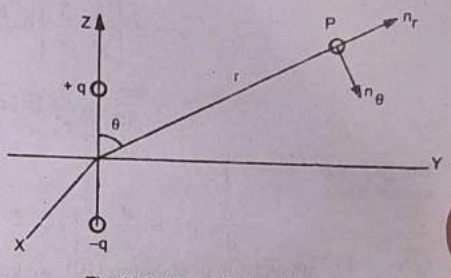

Fig. 12.9. An oscillating dipole.

(i) Vector Potential A. The vector potential due to current distribution is given by the relation

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}]}{r}\, dV' = \frac{\mu_0}{4\pi r}[I]\, dl \qquad \left[\text{since } \int \mathbf{J}\, dV' = I\, dl\right].$$

Using equation (4), we get

$$\mathbf{A} = \frac{\mu_0}{4\pi r}[\dot{\mathbf{p}}]. \qquad ...(5)$$

From this equation it is clear that

(i) The vector potential **A** depends only on r and varies as r^{-1}.

(ii) The direction of **A** is always that of $\dot{\mathbf{p}}$; this indicates that the vector potential is always parallel to $d\mathbf{l}$ (*i.e.* z-axis or polar axis).

(iii) If **n** is a unit vector along the polar axis, then we may write

$$\mathbf{n} = \mathbf{n}_r \cos\theta - \mathbf{n}_\theta \sin\theta \qquad ...(6)$$

then equation (5) may be expressed as

$$\mathbf{A} = \frac{\mu_0}{4\pi r}[\dot{\mathbf{p}}](\cos\theta\, \mathbf{n}_r - \sin\theta\, \mathbf{n}_\theta)$$

or

$$(A_r\mathbf{n}_r + A_\theta\mathbf{n}_\theta + A_\phi\mathbf{n}_\phi) = \frac{\mu_0}{4\pi r}[\dot{\mathbf{p}}](\cos\theta\, \mathbf{n}_r - \sin\theta\, \mathbf{n}_\theta) \qquad ...(7)$$

Therefore the components of **A** may be expressed as

---

*This equivalence is used to discuss the radiation due to antenna (oscillating current elements).

$$\left.\begin{aligned} A_r &= \frac{\mu_0}{4\pi}\,[\dot{\mathbf{p}}]\,\frac{\cos\theta}{r} \\ A_\theta &= -\frac{\mu_0}{4\pi}\,[\dot{\mathbf{p}}]\,\frac{\sin\theta}{r} \\ A_\phi &= 0. \end{aligned}\right\} \qquad \text{...(8)}$$

(ii) Scalar Potential ϕ. The Lorentz condition foı free space is

$$\operatorname{div}\mathbf{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t} = 0 \text{ or } \operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0. \qquad \text{...(9)}$$

i.e.

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= -c^2\operatorname{div}\mathbf{A} \\ &= -c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\ &= -c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\mu_0[\dot{p}]\cos\theta}{4\pi r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-\mu_0[\dot{p}]\sin^2\theta}{4\pi r}\right)\right] \\ &= -\frac{\mu_0c^2}{4\pi}\left[\left(\frac{1}{r^2}[\dot{p}]\cos\theta + \frac{r\cos\theta}{r^2}\frac{\partial}{\partial r}[\dot{p}]\right) - \left(\frac{2\sin\theta\cos\theta}{r^2\sin\theta}[\dot{p}]\right)\right] \qquad \text{...(10)}\end{aligned}$$

But $[\dot{p}] = \dot{p}\left(t - \frac{r}{c}\right)$

$$\therefore \qquad \frac{\partial}{\partial r}[\dot{p}] = \frac{\partial}{\partial r}\dot{p}\left(t - \frac{r}{c}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\dot{p}\left(t - \frac{r}{c}\right) = -\frac{1}{c}[\ddot{p}]$$

Making this substitution in equation (10), we get

$$\frac{\partial\phi}{\partial t} = -\frac{\mu_0c^2}{4\pi}\left[-\frac{\cos\theta}{rc}[\ddot{p}] - \frac{\cos\theta}{r^2}[\dot{p}]\right]$$

Using

$$\mu_0\varepsilon_0 = \frac{1}{c^2}.$$

and

$$\frac{\partial}{\partial t}\left\{\frac{[\dot{p}]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right\} = +\left\{\frac{\cos\theta}{r^2}[\dot{p}] + \frac{\cos\theta[\ddot{p}]}{rc}\right\}$$

We get

$$\frac{\partial\phi}{\partial t} = \frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\left\{\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right]$$

This gives

$$\phi = \frac{1}{4\pi\varepsilon_0}\left\{\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right\} \qquad \text{...(11)}$$

From equation (11) it is clear that :

(i) The scalar potential varies as $\cos\theta$. This indicates that ϕ is zero in the equatroial plane where the potentials due to two charges cancel exactly.

(ii) The scalar potential consists of two parts, one varying as r^{-2} and the other varying as r^{-1}. The first term dominates for small values of r and hence this term is responsible for induction field in the nearer zone, while the second term dominantes for large values of r and hence this term is responsible for radiation field in the distant zone.

From the knowledge of **A** and ϕ we may calculate field vectors **B** and **E** using the relations.

$$\mathbf{B} = \text{curl } \mathbf{A} \text{ and } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi. \qquad \text{...(12)}$$

(iii) Magnetic field Induction. B. We have $\mathbf{B} = \text{curl } \mathbf{A}$.

$$= \frac{1}{r \sin\theta}\left\{\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta - \frac{\partial A_\theta}{\partial\phi}\right)\right\}\mathbf{n}_r + \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right\}\mathbf{n}_\theta + \frac{1}{r}\left\{\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\mathbf{n}_\phi$$

Here $A_\phi = 0$ and A_θ, A_r are independent of co-ordinate ϕ, hence we may write

$$\mathbf{B} = \frac{1}{r}\left\{\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\mathbf{n}_\phi$$

Substituting values of A_r and A_θ, we obtain

$$\mathbf{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left\{-\frac{\mu_0 r}{4\pi r}[\dot{p}]\sin\theta\right\} - \frac{\partial}{\partial\theta}\left\{\frac{\mu_0[\dot{p}]\cos\theta}{4\pi r}\right\}\right)\mathbf{n}_\phi$$

$$= \frac{1}{r}\left(-\frac{\mu_0}{4\pi}\sin\theta\left(-\frac{\ddot{p}}{c}\right) + \frac{\mu_0}{4\pi r}[\dot{p}]\sin\theta\right)\mathbf{n}_\phi$$

This gives

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left\{\frac{[\dot{p}]\sin\theta}{r^2} + \frac{[\ddot{p}]\sin\theta}{rc}\right\}\mathbf{n}_\phi$$

From equation (13) it is clear that :

(i) The magnetic induction varies as $\sin\theta$. This indicates that the magnetic induction **B** is minimum along the axis of the dipole and maximum along the equatorial plane.

(ii) The magnetic induction **B** contains two terms, one varying as r^{-2} and the other varying as r^{-1}. The first term dominates for small values of r and hence this term is responsible for induction field; while the second term dominates for large values of r and hence this term is responsible for radiation field.

thus

$$\mathbf{B}_{\text{induction}} = \frac{\mu_0}{4\pi}\frac{[\dot{p}]\sin\theta}{r}\mathbf{n}_\phi = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}\mathbf{n}_\phi = \frac{\mu_0}{4\pi}\frac{I\,dl \times \mathbf{r}}{r^3} \qquad \text{...(14)}$$

which is exactly *Biot-Savart law*

and

$$\mathbf{B}_{\text{radiation}} = \frac{\mu_0}{4\pi}\frac{[\ddot{p}]\sin\theta}{rc}\mathbf{n}_\phi \qquad \text{...(15)}$$

(iv) Electric field strength E. We know that electric field strength is given by

$$\mathbf{E} = -\text{grad } \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$= -\left(\frac{\partial\Phi}{\partial r}\mathbf{n}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{n}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{n}_\phi\right) - \frac{\partial}{\partial t}[A_r\mathbf{n}_r + A_\theta\mathbf{n}_\theta + A_\phi\mathbf{n}_\phi]$$

Using equations (8) and (11), we get

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\left\{\frac{\partial}{\partial r}\left(\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right)\right\}\mathbf{n}_r + \frac{1}{r}\left\{\frac{\partial}{\partial\theta}\left(\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right)\right\}\mathbf{n}_\theta\right]$$

$$-\frac{\mu_0}{4\pi}\left[\frac{\partial}{\partial t}\left\{\frac{[\dot{p}]\cos\theta}{r}\mathbf{n}_r - \frac{[\dot{p}]\sin\theta}{r}\mathbf{n}_\theta\right\}\right]$$

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\left\{-\frac{2[p]\cos\theta}{r^3} + \frac{\cos\theta}{r^2}\frac{\partial[p]}{\partial r} - \frac{[\dot{p}]\cos\theta}{r^2 c} + \frac{\cos\theta}{rc}\frac{\partial[\dot{p}]}{\partial r}\right\}\mathbf{n}_r + \left\{-\frac{[p]\sin\theta}{r^3} - \frac{[\dot{p}]\sin\theta}{r^2 c}\right\}\mathbf{n}_\theta\right] - \frac{\mu_0}{4\pi}\left[\frac{[\ddot{p}]\cos\theta}{r}\mathbf{n}_r - \frac{[\ddot{p}]\sin\theta}{r}\mathbf{n}_\theta\right] \quad \text{...(16)}$$

But

$$\frac{\partial}{\partial r}[\dot{p}] = \frac{\partial}{\partial r}p\left(t - \frac{r}{c}\right) = -\frac{1}{c}\frac{\partial}{\partial t}p\left(t - \frac{r}{c}\right) = -\frac{1}{c}[\dot{p}]$$

and

$$\frac{\partial}{\partial r}[\dot{p}] = -\frac{1}{c}[\ddot{p}].$$

Also

$$\mu_0\varepsilon_0 = \frac{1}{c^2} \text{ or } \mu_0 = \frac{1}{c^2\varepsilon_0}.$$

Making these substitution in equation (16), we get

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\left\{-\frac{2[p]\cos\theta}{r^3} - \frac{[\dot{p}]\cos\theta}{r^2 c} - \frac{[\dot{p}]\cos\theta}{r^2 c} - \frac{[\ddot{p}]\cos\theta}{rc^2}\right\}\mathbf{n}_r + \left[-\frac{[p]\sin\theta}{r^3} - \frac{[\dot{p}]\sin\theta}{r^2 c}\right\}\mathbf{n}_\theta\right] - \frac{1}{4\pi\varepsilon_0 c^2}\,[\ddot{p}]\frac{\cos\theta}{r}\mathbf{n}_r - \frac{[\ddot{p}]\sin\theta}{r}\mathbf{n}_\theta\right]$$

Now collecting terms containing $[p]$, $[\dot{p}]$ and $[\ddot{p}]$, we get

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2[p]\cos\theta}{r^3}\mathbf{n}_r + \frac{[p]\sin\theta}{r^3}\mathbf{n}_\theta\right] + \frac{1}{4\pi\varepsilon_0}\left[\frac{2[\dot{p}]\cos\theta}{r^2 c}\mathbf{n}_r + \frac{[\dot{p}]\sin\theta}{r^2 c.}\mathbf{n}_\theta\right] + \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{c^2 r}\mathbf{n}_r. \quad \text{...(17)}$$

It is clear from equation (17) that the electric field consists of three parts :

(i) The first part falls off as r^{-3} and dominates over other parts only at small distances. This part represents only the static field of an electric dipole. Thus

$$\mathbf{E}_{\text{static}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2[p]\cos\theta}{r^3}\mathbf{n}_r + \frac{[p]\sin\theta}{r^3}\mathbf{n}_\theta\right] \quad \text{...(17a)}$$

(ii) The second part depends on $[\dot{p}]$ or current I and arises due to steady currents. This part represents the *induced field* and falls of as r^{-2}. Thus

$$\mathbf{E}_{\text{induction}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2[\dot{p}]\cos\theta}{r^2 c}\mathbf{n}_r + \frac{[\dot{p}]\sin\theta}{r^2 c}\mathbf{n}_\theta\right]. \quad \text{...(17b)}$$

(iii) The third part depends on $[\ddot{p}]$ and falls off r^{-1}. This part dominates at large distances and represents the *radiation field.* Thus

$$\mathbf{E}_{\text{radiation}} = \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{rc^2}\mathbf{n}_\theta. \quad \text{...(17c)}$$

(5) **Poynting Vector and Radiated Power.** The energy flow takes place due to radiation fields given by equations (15) and (17c). Therefore the *Poynting vector* will be given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E}_{\text{radiation}} \times \left(\frac{\mathbf{B}_{\text{radiation}}}{\mu_0} \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{[\ddot{p}] \sin\theta}{c^2 r} \mathbf{n}_\theta \times \frac{\mu_0}{4\pi} \frac{[\ddot{p}] \sin\theta}{rc^2 \mu_0} \mathbf{n}_\phi \text{ Using (15) and (17c)}$$

$$= \frac{[\ddot{p}]^2 \sin^2\theta}{16\pi^2 \varepsilon_0 c^3 r^2} (\mathbf{n}_\theta \times \mathbf{n}_\phi) = \frac{[\ddot{p}]^2 \sin^2\theta}{16\pi^2 \varepsilon_0 c^3 r^2} \mathbf{n}_r \qquad \text{...(18)}$$

But $[p] = p_0 \cos\omega(t - r/c)$, we get

$$[\dot{p}] = -p_0\omega \sin\omega\left(t - \frac{r}{c}\right) \text{ and } [\ddot{p}] = -p_0\omega^2 \cos\omega\left(t - \frac{r}{c}\right)$$

$$[\ddot{p}]^2 = p_0^2 \omega^4 \cos^2\omega(t - r/c).$$

Making this substituting in equation (18), we get

$$\mathbf{S} = \frac{p_0^2 \omega^4 \cos^2\omega(t - r/c) \sin^2\theta}{16\pi^2 \varepsilon_0 c^3 r^2} \mathbf{n}_r \qquad \text{...(18a)}$$

∴ The time averaged poynting vector is given by

$$<\mathbf{S}> = \frac{p_0^2 \omega^4 < \cos^2\omega(t - r/c) > \sin^2\theta}{16\pi^2 \varepsilon_0 c^3 r^2} \mathbf{n}_r$$

$$= \frac{p_0^2 \omega^4 \sin^2\theta}{32\pi^2 \varepsilon_0 c^3 r^2} \mathbf{n}_r . \qquad \text{...(19)}$$

[*Since average value of* $\cos^2\omega(t - r/c)$ *over a complete cycle is* $\frac{1}{2}$].

This expression represents the average energy radiated per second per unit area in a normal direction by an oscillating dipole.

From equation (19) it is clear that the *time averaged* poynting vector varies as $\sin^2\theta$. This means that the flow of energy is zero along the dipole axis and maximum along the equitorial axis as shown in fig. 12·10.

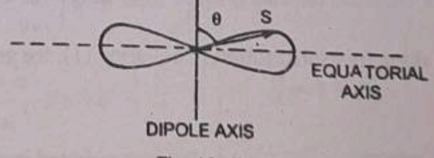

Fig. 12.10

∴ *The averaged power radiated per unit solid angle is given by*

$$\frac{dP}{d\Omega} = |<\mathbf{S}>| \frac{4\pi r^2}{4\pi} = <S> r^2 = \frac{p_0^2 \omega^4 \sin^2\theta}{32\pi^2 \varepsilon_0 c^3} \qquad \text{...(20)}$$

The *total power radiated* may be obtained by integrating equation (20) for all angles *i.e.*

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{p_0^2 \omega^4}{32\pi^2 \varepsilon_0 c^3} \int_0^\pi \int_0^{2\pi} \sin^2\theta \, (d\theta) \, (\sin\theta \, d\phi)$$

$$= \frac{p_0^2\omega^4}{32\pi^2\varepsilon_0 c^3}\int_0^{\pi}\sin^3\theta\, d\theta\int_0^{2\pi} d\phi = \frac{p_0^2\omega^4}{32\pi^2\varepsilon_0 c^3}\cdot\left(\frac{4}{3}\right)\cdot 2\pi$$

$$= \frac{p_0^2\omega^4}{12\pi\varepsilon_0 c^3} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{p_0^2\omega^4}{3c^3}. \qquad ...(21)$$

From this equation it is clear that the total power radiated by an oscillating dipole is proportional to the fourth power of the frequency. The examples of electric dipole radiators are the emission of radiation by an excited atom and electromagnetic waves transmitted by an antenna.

Ex. 1. *Treating electric dipole to be equivalent to an accelerated charge, calculate :*

(i) *The dipole moment amplitude* p_0 *in terms of charge* **q** *and acceleration* **a** *of accelerated charge.*

(ii) *Instantaneous rate of radiation from the charge.*

Solution. (i) Equation of motion of accelerated charge in sinusoidally varying electric field **E** is given by

$$m\mathbf{a} = q\mathbf{E} \qquad \text{or} \qquad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{q\mathbf{E}}{m}.$$

As $\qquad \mathbf{E} = \mathbf{E}_0 e^{-i\omega t} \qquad \therefore \frac{d^2\mathbf{r}}{dr^2} = \frac{q\mathbf{E}}{m}e^{-i\omega t}.$

This gives $\qquad \mathbf{r} = -\frac{q\mathbf{E}_0}{m\omega^2}e^{-i\omega t}.$

But dipole moment $\mathbf{p} = q\,\mathbf{r} = -\frac{q_0\mathbf{E}_0}{m\omega^2}e^{-i\omega t}$

or $\qquad |\mathbf{p}| = \left|-\frac{q^2\mathbf{E}_0}{m\omega^2}\right|e^{-i\omega t} \qquad$ But $\qquad p = p_0 e^{-i\omega t}.$

This gives

$$p_0 = \frac{q^2\mathbf{E}_0}{n\omega^2}. \qquad ...(1)$$

If **a** is the acceleration of the charged particle under an electric field of frequency ω at $t = 0$, then

$$m\,\mathbf{a} = q\,\mathbf{E}_0 e^{-i\omega t} = q\mathbf{E}_0\Big|_{t=0} \qquad \text{or} \qquad \mathbf{a} = \frac{q\,\mathbf{E}_0}{m}\Big|_{t=0}$$

Making this substitution in equation (1), we get

$$p_0 = \frac{qa}{\omega^2}. \qquad ...(2)$$

(ii) The time averaged power radiated is given by

$$<P> = \frac{1}{4\pi\varepsilon_0}\frac{(q^2a^2/\omega^4)\,\omega^4}{3c^3} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{3c^3} \qquad ...(3)$$

If P is the instantaneous power, then

$$<P> = <P\cos^2\omega(t - r/c)> = P<\cos^2\omega(t - r/c)> = P/2$$

$$P = 2<P> = \frac{1}{4\pi\varepsilon_0}\frac{2q^2a^2}{3c^3}, \qquad ...(4)$$

which is exactly Larmor's non-relativistic formula.

12.12. Electric Quadrupole Radiation

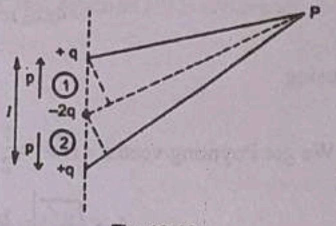

Fig. 12.10

A quadrupole is a combination of two electric dipoles. Consider a quadrupole containing two electric dipoles, each of dipole moment p, aligned along a given straight line (fig.). This system possesses quadrupole moment which varies with time and hence it will emit radiation, called quadrupole radiation. The quadrupole field may be derived by considering it to be the superposition of two dipole fields. The electric field due to a single dipole located at the origin, at any point P in the radiation zone is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{rc^2}\,\mathbf{n}_\theta \qquad \text{...(1)}$$

But $\qquad [p] = p_0\, e^{-i\omega\left(t-\frac{r}{c}\right)}$, we get $[\dot{p}] = -p_0\, i\omega\, e^{-i\omega\left(t-\frac{r}{c}\right)}$

and
$$[\ddot{p}] = -p_0\,(i\omega)^2 e^{-i\omega\left(t-\frac{r}{c}\right)} = p_0\omega^2 e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$\therefore$
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2\sin\theta}{rc^2}\, e^{-i\omega\left(t-\frac{r}{c}\right)}\mathbf{n}_\theta$$

i.e.
$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2\sin\theta}{rc^2}\, e^{-i\omega\left(t-\frac{r}{c}\right)} \qquad \text{...(3)}$$

The centres of dipoles are at distances $\frac{l}{4}$ and $-\frac{l}{4}$ from the origin. From fig it is clear that the dipole (1) is at a distance $\frac{l}{4}$ cone nearer to point P than from its distance from origin and (2) is farther by the same amount. The phases of two dipoles relative to origin are therefore $\frac{kl}{4}\cos\theta$ and $-\frac{kl}{4}\cos\theta - \pi$

where $k = \frac{\omega}{c}$ and $-\pi$ is introduced because two dipoles are oscillating out of phase. Therefore the electric field component due to quadrupole is

$$E_\theta = +\frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2\sin\theta}{rc^2}\, e^{-i\left\{\omega\left(t-\frac{r}{c}\right)-\frac{kl}{4}\cos\theta\right\}} + \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\, e^{-i\left\{\omega\left(t-\frac{r}{c}\right)-\left(-\frac{kl}{4}\cos\theta-\pi\right)\right\}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\left\{e^{\frac{ikl}{4}\cos\theta} + e^{-i\pi}e^{-\frac{ikl}{4}\cos\theta}\right\}e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\left\{e^{\frac{ikl}{4}\cos\theta} - e^{-\frac{ikl}{4}\cos\theta}\right\}e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\; 2\,i\sin\left(\frac{kl}{4}\cos\theta\right)e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$$= \frac{2i\, p_0 \omega^2 \sin\theta}{4\pi\varepsilon_0\, rc^2} \sin\left(\frac{kl}{4}\cos\theta\right) e^{-i\omega\left(t-\frac{r}{c}\right)} \qquad \text{...(4)}$$

using
$$H_\phi = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_\theta \qquad \text{...(5)}$$

We get Poynting vector $<\vec{S}> = \frac{1}{2}\,\mathrm{Re}\,(\vec{E} \times \vec{H}^*) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, |E_\theta|^2\, \vec{n}_r$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \left| \frac{2i\, p_0\omega^2}{4\pi\,\varepsilon_0\, rc^2} \sin\theta \sin\left(\frac{kl}{4}\cos\theta\right) e^{-i\omega\left(t-\frac{r}{c}\right)} \right|^2 \vec{n}_r$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{4p_0^2\,\omega^4 \sin^2\theta}{(4\pi\varepsilon_0)^2\, r^2 c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right)\vec{n}_r$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{p_0^2\omega^4 \sin^2\theta}{8\pi^2\, \varepsilon_0^2\, r^2\, c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right)\vec{n}_r$$

$$= \frac{1}{\sqrt{\mu_0\varepsilon_0}}\, \frac{p_0^2\omega^4 \sin^2\theta}{8\pi^2\, \varepsilon_0\, r^2 c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right)\vec{n}_r \qquad \left(\because\ k = \frac{\omega}{c}\right)$$

$$= \frac{c\, p_0^2\omega^4 \sin^2\theta}{8\pi^2\, \varepsilon_0\, r^2 c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right)\vec{n}_r$$

$$= \frac{p_0^2\, \omega^4 \sin^2\theta}{8\pi^2\, \varepsilon_0\, c^3 r^2} \sin^2\left(\frac{kl}{4}\cos\theta\right)\vec{n}_r \qquad \text{...(6)}$$

Average power radiated per unit solid angle $<\frac{dP}{d\Omega}> = <S> r^2\, \vec{n}_r$

$$= \frac{p_0^2\omega^4 \sin^2\theta}{8\pi^2\, \varepsilon_0 c^3} \sin^2\left(\frac{kl}{4}\cos\theta\right)\vec{n}_r \qquad \text{...(7)}$$

Special cases

case (i) If $l = \lambda$ *i.e.* $kl = 2\pi$;

then $$<\frac{dP}{d\Omega}> = \frac{\omega^4 p_0^2}{8\pi^2\, \varepsilon_0\, c^3} \sin^2\theta \sin^2\left(\frac{\pi}{2}\cos\theta\right)$$

The quadrupole moment

$$Q = 2p_0 l$$

$\therefore$
$$<\frac{dP}{d\Omega}> = \frac{\omega^4 Q^2}{8\pi^2\, \varepsilon_0\, c^3\, 4l^2} \sin^2 \sin^2\left(\frac{\pi}{2}\cos\theta\right)$$

using
$$kl = 2\pi \Rightarrow l = \frac{2\pi}{k} = \frac{2\pi\, c}{\omega},$$

we get
$$<\frac{dP}{d\Omega}> = \frac{\omega^6\, Q^2}{128\,\pi^2\, \varepsilon_0\, c^5} \sin^2\theta \sin^2\left(\frac{\pi}{2}\cos\theta\right)$$

case (ii) If $l \to 0$; then as p_0 increases; Q remains constant; In this case the dipole pair reduces to a point quadrupole located at the origin; then $\sin\left(\frac{kl}{4}\cos\theta\right) = \frac{kl}{4}\cos\theta$, so that equation (2) is written as

$$E_\theta = \frac{ik^2 p_0}{2\pi\,\varepsilon_0 r}\sin\theta\cdot\left(\frac{kl}{4}\cos\theta\right)e^{i\omega t}$$

Hence poynting vector $<\vec{\mathbf{S}}> = \frac{1}{2}\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}^{*}\right) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,|E_\theta|^2\,\vec{\mathbf{n}}_r$

$$= \frac{\omega^6 p_0^2 l^2}{128\,\pi^2\varepsilon_0 c^5 r^2}\sin^2\theta\cos^2\theta \text{ and } <\frac{dP}{d\Omega}> = \frac{\omega^6 p_0^2 l^2}{128\,\pi^2\varepsilon_0 c^5}\sin^2\theta\cos^2\theta$$

It is obvious from this expression that the maximum power is radiated at $\theta = 45^\circ$.

12.13. Radiation due to a Small Current Element

An oscillating dipole may be treated as a small current element since dipole moment $\mathbf{p} = q\,d\mathbf{l}$.

$$\therefore \dot{\mathbf{p}} = I\,d\mathbf{l} = \dot{q}\,d\mathbf{l} \;(\text{for } I = \dot{q} = dq/dt),$$

i.e. the time rate of variation of dipole moment is equal to small current element.

Also $$\ddot{\mathbf{p}} = \frac{dI}{dt}\,d\mathbf{l}.$$

Now taking current variations as sinusoidal *i.e.*

$$I = I_0 e^{-i\omega t}, \text{ then } \frac{dI}{dt} = -i\omega I_0 e^{-i\omega t}$$

So that $$\dot{\mathbf{p}} = I\,d\mathbf{l} = I_0 e^{i\omega t}\,d\mathbf{l}$$

$$\ddot{\mathbf{p}} = \frac{dI}{dt}\,d\mathbf{l} = -i\omega I_0 e^{-i\omega t}\,d\mathbf{l} \qquad \text{...(1)}$$

This implies that radiation field equations developed for a dipole may be applied to the case of small linear current element. The electric and magnetic radiation fields for an oscillating dipole are given by

$$\mathbf{E}_{\text{radiation}} = \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{c^2 r}\,\mathbf{n}_\theta \qquad \text{...(2)}$$

and $$\mathbf{B}_{\text{radiation}} = \frac{\mu_0}{4\pi}\frac{[\ddot{p}]\sin\theta}{cr}\,\mathbf{n}_\phi$$

$\therefore$ Therefore in this case *i.e. for a small current element*, using (1)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{[-i\omega I_0 e^{-i\omega t}\,dl\,]\sin\theta}{c^2 r}\,\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0\,dl\cdot\sin\theta}{c^2 r}\,e^{-i\omega(t-r/c)}\,\mathbf{n}_\theta \qquad \text{...(3)}$$

and $$\mathbf{B} = \frac{\mu_0}{4\pi}\cdot\frac{[-i\omega I_0 e^{-i\omega t}\,dl\sin\theta]}{cr}\,\mathbf{n}_\phi = -\frac{\mu_0}{4\pi}\frac{i\omega I_0\,dl\sin\theta}{cr}\,e^{-i\omega(t-r/c)}\,\mathbf{n}_\phi.$$

or

$$\mathbf{B} = \frac{1}{4\pi\varepsilon_0} \frac{i\omega I_0\, dl \sin\theta}{c^3 r} e^{-i\omega(t-r/c)}\, \mathbf{n}_\phi \qquad \text{...(4)}$$

Comparing (3) and (4), we note that $\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi$ ...(5)

where $|\mathbf{E}|$ is the modulus of electric field given by (3).

The Poynting vector for complex fields is given by

$$\mathbf{S} = \frac{1}{2}\text{Re}\,[\mathbf{E}\times\mathbf{H}^*] = \frac{1}{2}\text{Re}\left(\frac{\mathbf{E}\times\mathbf{B}^*}{\mu_0}\right)$$

Making substitutions from (3) and (4), we get

$$\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\text{Re}\left[\left\{-\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0\, dl\sin\theta}{c^2 r}e^{-i\omega(t-r/c)}\,\mathbf{n}_\theta\right\} \times\left\{-\frac{\mu_0}{4\pi}\frac{(-i\omega I_0\, dl\sin\theta)}{cr}e^{-i\omega(t-r/c)}\,\mathbf{n}_\phi\right\}\right]$$

$$= \frac{1}{2}\frac{\omega^2 I_0^2\, dl^2\sin^2\theta}{16\pi^2\varepsilon_0 r^2 c^3}\,\mathbf{n}_\theta\times\mathbf{n}_\phi .$$

or

$$\langle\mathbf{S}\rangle = \frac{\omega^2 I_0^2\, dl^2\sin^2\theta}{32\pi^2\varepsilon_0 r^2 c^3}\,\mathbf{n}_r \qquad \text{...(6)}$$

The **averaged power radiated per unit solid angle** is given by

$$\frac{dP}{d\Omega} = \langle\mathbf{S}\rangle\cdot\frac{4\pi r^2}{4\pi} = \langle\mathbf{S}\rangle r^2 = \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\sin^2\theta \qquad \text{...(7)}$$

The total power radiated may be obtained by integrating (7) over all angles. Thus

$$P = \int\frac{dP}{d\Omega}\,d\Omega = \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\int_0^\pi\int_0^{2\pi}\sin^2\theta\, d\theta\,(\sin\theta\, d\phi)$$

$$= \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\int_0^\pi\sin^3\theta\, d\theta\int_0^{2\pi}d\phi = \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\left(\frac{4}{3}\right)\cdot(2\pi)$$

$$\left[\text{since } \int_0^\pi\sin^3\theta\, d\theta = \tfrac{4}{3} \text{ and } \int_0^{2\pi}d\phi = 2\pi\right]$$

i.e.

$$P = \frac{\omega^2 I_0^2 dl^2}{12\pi\varepsilon_0 c^2}. \qquad \text{...(8)}$$

A more conventional form of above equation is obtained by introducing

$$\lambda \doteq \frac{c}{\nu} = \frac{2\pi c}{2\pi\nu} = \frac{2\pi c}{\omega} \quad i.e.\quad \omega = \left(\frac{2\pi c}{\lambda}\right);\; I_{rms} = \frac{I_0}{\sqrt{2}}.$$

The result is

$$P = \frac{\left(\frac{2\pi c}{\lambda}\right)^2(\sqrt{2}\,I_{rms})^2\, dl^2}{12\pi\,\varepsilon_0 c^3}$$

$$= \frac{2}{3}\frac{\pi}{\varepsilon_0 c}\left(\frac{dl}{\lambda}\right)^2 I_{rms}^2 \qquad (\text{since } \mu_0\varepsilon_0 = 1/c^2 \;\; i.e.\;\; 1/\varepsilon_0 c = \mu_0 c)$$

Making substitution for μ_0 and c, we get

$$P = \frac{2}{3}(4\pi \times 10^{-7} \times 3 \times 10^{8})\,\pi \left(\frac{dl}{\lambda}\right)^2 I_{rms}^2 = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 I_{rms}^2 \qquad \text{...(9)}$$

This is the required expression for power radiated by a short linear current element.

The power loss in a resistance R carrying current I_{rms} is given by

$$P = I_{rms}^2 R \qquad \text{...(10)}$$

Comparing equations (9) and (10), we see that it is sensible to define the *radiation resistance* of a dipole or a small current *element in free space** by

$$R_r = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 = 787 \left(\frac{dl}{\lambda}\right)^2 \textit{ohms.} \qquad \text{...(11)}$$

The radiation resistance may be defined as the resistance in which the power dissipated in the form of heat is equal to that (power) radiated by the current element carrying the same current, in the form of electromagnetic waves.

A centre-fed short linear antenna for which $dl \ll \lambda$ (shown in fig. 12.12) is an example of *linear current element radiator*. From equation (11) it is clear that radiation resistance for such an antenna is quite low. For example if $dl/d\lambda = 0{\cdot}01$, then radiation resistance is only $\{787 \times (0{\cdot}01)^2\} = 0{\cdot}0787$ ohms, a value much smaller than the ohmic resistance of such a conductor. Thus such an antenna dissipates most of the input power in the form of heat and radiates a very little power in the form of electromagnetic energy. Therefore a short linear antenna is, in general, an inefficient radiator. For efficient radiator the radiation resistance must be high and consequently the length (dl) of the antenna must be comparable with the wavelength λ, for which the dipole approximation does not hold. A half wave antenna (*i.e.* $dl = \lambda/2$) is commonly used to radiate electromagnetic energy into space.

12.14. Radiation From Linear Antenna

The small current element is not a practical antenna. The *practical antenna* is a centre-fed thin linear antenna having a length comparable with free space wavelength. The current amplitude of such an antenna decreases uniformly from a maximum at the centre to minimum at the ends. The antenna of length 'd' is assumed to be oriented along Z-axis with its gap at the origin as shown in fig. 12.12.

If damping due to emission of radiation is neglected, the current along the antenna may be taken sinusoidal in time. Let $I = \text{Re}\,(I_0 e^{i\omega t}) = I_0 \cos \omega t$ be the current fed to the centre-fed linear antenna by means of a suitable electronic circuit [Here Re denotes "real part of"]. It is obvious that at $t = 0$, the amplitude of current at the centre is I_0. If dl is an element of antenna at a distance l from the centre, then amplitude of current-element dl will be

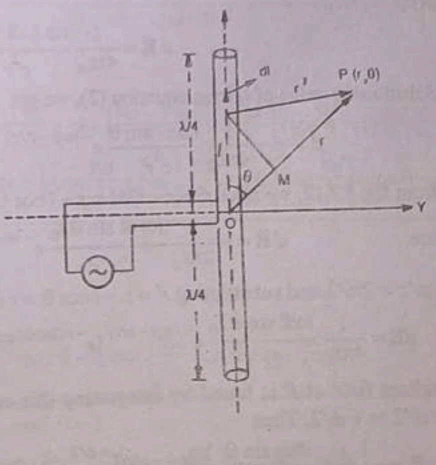

Fig. 12.12. Centre-fed linear antenna

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* The radiation resistance in a medium of permittivity and permeabilityis defined as $B = \frac{2\pi}{3c}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{dl}{\lambda}\right)^2 I_{rms}^2$.

$$I_0 \cos \omega\left(\frac{l}{c}\right) = I_0 \cos \frac{2\pi l}{\lambda}\left[\text{since at } dl,\ t = \frac{l}{c} \text{ and } \frac{\omega}{c} = \frac{2\pi}{\lambda}\right]$$

Therefore *instantaneous current at dl* will be $I = \left(I_0 \cos \frac{2\pi l}{\lambda}\right) \cos \omega t$

This may be expressed as $I = \frac{I_0}{2}\left[\cos\left(\omega t - \frac{2\pi l}{\lambda}\right) + \cos\left(\omega t + \frac{2\pi l}{\lambda}\right)\right]$

In exponential form, this may be expressed as

$$I = \text{Re}\left[\frac{I_0}{2}\left\{e^{-i(\omega t - 2\pi l/\lambda)} + e^{-i(\omega t + 2\pi l/\lambda)}\right\}\right] \qquad ...(1a)$$

i.e.

$$I = \frac{I_0}{2} e^{-i\omega t}\{e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)}\} \qquad ...(1b)$$

where the real part of complex exponentials is implied.

∴ The current amplitude at d :

$$I_l = \frac{I_0}{2}\{e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)}\} \qquad ...(2)$$

1. Electric field Strength : From preceding section we know that the electric and magnetic fields due to a small current element dl carrying current amplitude I_0 at a distance r from the element is given by

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0\, dl \sin\theta}{c^2 r} e^{-i\omega(t - r/c)}\, \mathbf{n}_\theta$$

and

$$\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi \qquad ...(3)$$

In this case the field point P lies at a distance r' from the element dl and the current amplitude at a distance l from the centre is I_l given by (2) : therefore the electric field at point P due to small element dl is given by

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{i\omega I_l\, dl \sin\theta}{c^2 r'} e^{-i\omega(t - r'/c)}\, \mathbf{n}_\theta \qquad ...(4)$$

Substituting value of I_l from equation (2), we get

$$d\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{i\omega\, dl \sin\theta}{c^2 r'} e^{-i\omega(t - r'/c)}.\left[\frac{I_0}{2}\left\{e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)}\right\}\right]\mathbf{n}_\theta$$

Now from fig. 12.12, we have $r' = r - OM = r - l\cos\theta$

therefore

$$d\mathbf{E} = -\frac{1}{2\pi\varepsilon_0}\frac{i\omega\, dl \sin\theta\, I_0}{2c^2 r'} e^{-i\omega(t - r/c)} e^{-i(\omega/c)\cos\theta}\left\{e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)}\right\}\mathbf{n}_\theta$$

Using $\omega/c = 2\pi/\lambda$ and substituting $r' = r - l\cos\theta \approx r$ in the denominator, we get

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\omega dl \sin\theta\, I_0}{2c^2 r} e^{-i\omega(t - r/c)}\left[e^{-i(2\pi l/\lambda)(\cos\theta - 1)} + e^{-i(2\pi l/\lambda)(\cos\theta + 1)}\right]\mathbf{n}_\theta$$

The resultant field at P is found by integrating this equation for the whole length of antenna *i.e.* between limits $-d/2$ to $+d/2$. Thus

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0 \sin\theta}{2c^2 r} e^{-i\omega(t - r/c)}\int_{-d/2}^{+d/2}\left[e^{-i(2\pi l/\lambda)(\cos\theta - 1)} + e^{-i(2\pi l/\lambda)(\cos\theta + 1)}\right] dl\, \mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0 \sin\theta}{2c^2 r} e^{-i\omega(t-r/c)} \times \left[\frac{e^{-i(2\pi l/\lambda)(\cos\theta-1)}}{\left(-\frac{2\pi i}{\lambda}(\cos\theta-1)\right)} + \frac{e^{-i(2\pi l/\lambda)(\cos\theta+1)}}{\left(-\frac{2\pi i}{\lambda}(\cos\theta+1)\right)}\right]_{-d/2}^{+d/2} \mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{2cr}\left(\frac{\omega}{c}\right)\left(\frac{\lambda}{\pi}\right) e^{-i\omega(t-r/c)} \left(\frac{e^{-i(\pi d/\lambda)(\cos\theta-1)} - e^{i(\pi d/\lambda)(\cos\theta-1)}}{-2i(\cos\theta-1)}\right.$$

$$\left. + \left(\frac{e^{-i(\pi d/\lambda)(\cos\theta+1)} - e^{i(\pi d/\lambda)\cos\theta+1}}{-2i(\cos\theta+1)}\right)\right]\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{2cr}\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{\pi}\right) e^{-i\omega(t-r/c)} \times \left[\frac{\sin\left\{\frac{\pi d}{\lambda}(\cos\theta-1)\right\}}{(\cos\theta-1)} + \frac{\sin\left(\frac{\pi d}{\lambda}\right)(\cos\theta+1)}{(\cos\theta+1)}\right]\mathbf{n}_\theta$$

Now using trigonometric relations

$$\sin\left\{\frac{\pi d}{\lambda}(\cos\theta-1)\right\} = \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right) - \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}$$

and

$$\sin\left\{\frac{\pi d}{\lambda}(\cos\theta+1)\right\} = \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda} + \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}$$

we get $\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{cr} e^{-i\omega(t-r/c)}$

$$\left[\frac{\sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda} - \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}}{\cos\theta-1}\right.$$

$$\left. + \frac{\sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda} + \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}}{\cos\theta+1}\right]\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{cr} e^{-i\omega(t-r/c)}$$

$$\left[\frac{\left[\sin\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\cos\frac{\pi d}{\lambda} - \cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}\right](\cos\theta+1) + \left[\sin\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\cos\frac{\pi d}{\lambda} + \cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}\right](\cos\theta-1)}{(\cos^2\theta-1)}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{cr} e^{-i\omega(t-r/c)}$$

$$\frac{\left[2\sin\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\cos\frac{\pi d}{\lambda}\cos\theta - 2\cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}\right]}{\cos^2\theta-1}$$

$$= -i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{cr} e^{-i(t-r/c)} \frac{\left[\cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda} - \sin\left\{\cos\theta\frac{\pi d}{\lambda}\right\}\cos\frac{\pi d}{\lambda}\cos\theta\right]}{\sin\theta} \qquad \text{...(5)}$$

This equation represents the required expression for electric field.

2. Magnetic field Induction B. We have $\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi$

where $|\mathbf{E}|$ is the numerical value of $\mathbf{E}$.

Substituting $|\mathbf{E}|$ from equation (5), we get

$$\mathbf{B} = -i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{c^2 r}e^{-i\omega(t-r/c)}\left[\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}\cdot\cos\theta}{\sin\theta}\right]\mathbf{n}_\phi$$

$$= -i\frac{\mu_0}{4\pi}\frac{2I_0}{r}e^{-i\omega(t-r/c)}\cdot\left[\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}\cdot\cos\theta}{\sin\theta}\right]\mathbf{n}_\theta \quad \text{...(6)}$$

This is the required expression for *magnetic field* induction.

8. Poynting Vector and Radiated Power. The Poynting vector *i.e.* the flow of energy per unit area per second, for complex fields is

$$\mathbf{S} = \tfrac{1}{2}\,\mathrm{Re}\,(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2\mu_0}\,\mathrm{Re}\,(\mathbf{E}\times\mathbf{B}^*)$$

Substituting values of $\mathbf{E}$ and $\mathbf{B}$ from (5) and (6), we get

$$\mathbf{S} = \frac{1}{2\mu_0}Re\left[\left\{-i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{cr}e^{-i\omega(t-r/c)}\cdot\left[\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}-\cos\theta\right)\cos\frac{\pi d}{\lambda}\cos\theta}{\sin\theta}\right]\mathbf{n}_\theta\right\}\times\left[\frac{i\mu_0}{4\pi}\frac{2I_0}{r}e^{i\omega(t-r/c)}\left(\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cdot\cos\theta}{\sin\theta}\right)\mathbf{n}_\phi\right]\right]$$

$$= \frac{1}{2\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\mu_0}{4\pi}\right)\left(\frac{2I_0}{cr}\right)\left\{\frac{2I_0}{r}\right\}\left|\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cdot\cos\theta}{\sin\theta}\right|^2\mathbf{n}_r$$

Again using $\mu_0\varepsilon_0 = 1/c^2$, we get

$$\mathbf{S} = \frac{\mu_0 c}{8\pi^2}\frac{I_0^2}{r^2}\left|\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cos\theta}{\sin\theta}\right|^2\mathbf{n}_r \quad \text{...(7a)}$$

or

$$\mathbf{S} = \frac{\mu_0 c}{4\pi^2}\frac{I_{rms}^2}{r^2}\cdot\left|\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cos\theta}{\sin\theta}\right|^2\mathbf{n}_r \quad \text{...(7b)}$$

The power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \mathbf{n}_r \frac{4\pi r^2}{4\pi} = (\mathbf{S} \cdot \mathbf{n}_r r^2)$$

$$= \frac{\mu_0 c}{4\pi^2} I_{rms}^2 \left| \frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cos\theta}{\sin\theta} \right|^2 \qquad \text{...(8)}$$

The total power radiated may be obtained by integrating above expression over all angles. As the integration of expression (8) is complicated, therefore we shall consider only the particular case of half wave antenna which is of common use for radio broadcast.

12.15. Radiation from a Linear Half-wave Antenna

A linear half-wave antenna is simply a straight conductor of length equal to half a free space wavelength (fig. 12.13a). In many cases the purpose of a half wave antenna is served by a linear conductor of length equal to a quarter free space wavelength set vertically on the ground, which acts as a mirror and so an additional quarter wavelength is provided by ground reflection (fig. 12.13b). To achieve good conductivity ground in the neighbourhood of antenna may be covered with a conducting screen. Such antennas are commonly used for radio broadcast purposes. The *electric and magnetic fields* may be obtained directly as in the preceding section by substituting $\lambda/2$ for d. Then we get

$$\text{Electric field. } \mathbf{E} = -i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{cr}e^{-i\omega(t-r/c)} \times \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\theta. \qquad \text{...(1)}$$

From equation (1) it is clear that : (i) The electric field strength due to a half-wave antenna for a given current $I = I_0 e^{-i\omega t}$ fed at the centre is independent of frequency. (ii) At $\theta = 0$. **E** becomes indeterminate; but in the limit $\theta \to 0$. **B** becomes zero which may be seen as follows :

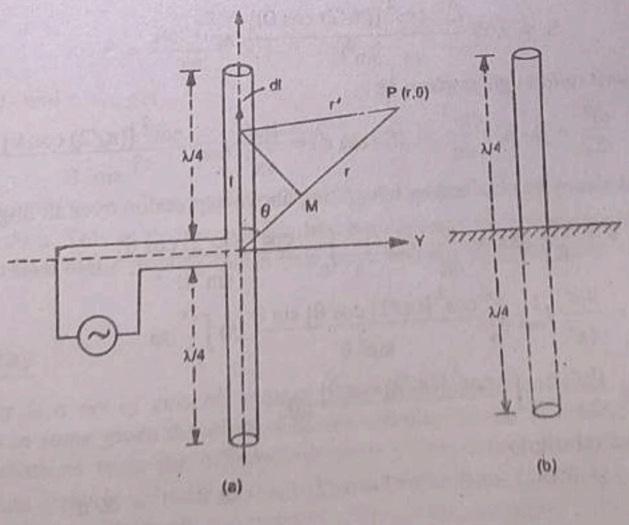


Fig. 12.13

$$\lim_{\theta \to 0} \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta} = \lim_{\theta \to 0} \frac{\sin[(\pi/2)\cos\theta]\,[(\pi/2)\sin\theta]}{\cos\theta} = 0$$

and magnetic field $\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi = -\frac{i}{4\pi\varepsilon_0}\frac{2I_0}{c^2 r}e^{-i\omega(t-r/c)}\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\phi$

Using $\mu_0\varepsilon_0 = 1/c^2$ or $\varepsilon_0 c^2 = 1/\mu_0$, we get

$$\mathbf{B} = -i\frac{\mu_0}{4\pi}\frac{2I_0}{r}e^{-i\omega(t-r/c)}\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\phi. \qquad ...(2)$$

Poynting Vector and Radiated Power

The Poynting vector, *i.e.* flow of energy per unit area per second, for complex field is

$$\mathbf{S} = \tfrac{1}{2}\,\text{Re}\,(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2\mu_0}\,\text{Re}\,(\mathbf{E}\times\mathbf{B}^*)$$

Substituting values of **E** and **B*** from (1) and (2), we get

$$\mathbf{S} = \frac{1}{2\mu_0}\,\text{Re}\left[\left\{-\frac{i}{4\pi\varepsilon_0}\frac{2I_0}{cr}e^{-i\omega(t-r/c)}\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\theta\right\}\right.$$
$$\left.\times\left\{\frac{i\mu_0}{4\pi}\frac{2I_0}{cr}e^{i\omega(t-r/c)}\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\phi\right\}\right]$$

Using $\mu_0\varepsilon_0 = 1/c^2$ and rearranging, we get $\mathbf{S} = \frac{\mu_0 c}{8\pi^2}\frac{I_0^2}{r^2}\frac{\cos^2(\pi/2)\cos\theta}{\sin^2\theta}\mathbf{n_r}$ ...(3a)

or

$$\mathbf{S} = \frac{\mu_0 c}{4\pi^2}\frac{I_{rms}^2}{r^2}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}\mathbf{n_r} \qquad ...(3b)$$

Substituting values of μ_0 and c, we get

$$\mathbf{S} = \frac{(4\pi\times10^{-7})(3\times10^8)}{4\pi^2}\frac{I_{rms}^2}{r^2}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}\mathbf{n_r}$$

$$\mathbf{S} = 9.55\,\frac{I_{rms}^2\cos^2[(\pi/2)\cos\theta)]}{\sin^2\theta}\,n_r \qquad ...(3c)$$

The power flow per unit solid angle is given by

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\mathbf{n}_r\,\frac{4\pi r^2}{4\pi} = (\mathbf{S}\cdot\mathbf{n}_r)\,r^2 = \frac{\mu_0 c}{4\pi^2}\,I_{rms}^2\,\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta} \qquad ...(4)$$

The total power radiated may be obtained by integrating above expression over all angles *i.e.*

$$P = \int\frac{dP}{d\Omega}\,d\Omega = \frac{\mu_0 c}{4\pi^2}\,I_{rms}^2\int_0^\pi\int_0^{2\pi}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}\,(\sin\theta\,d\theta\,d\phi)$$

$$= \frac{\mu_0 c}{4\pi^2}\,I_{rms}^2\int_0^\pi\frac{\cos^2[(\pi/2)\cos\theta]\sin\theta}{\sin^2\theta}\,d\theta\int_0^{2\pi}d\phi$$

$$= \frac{\mu_0 c}{2\pi}\,I_{rms}^2\int_0^\pi\frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}\,d\theta. \qquad ...(5)$$

To solve the integral, we substitute

$$(\pi/2)\cos\theta = \pi/2 - \alpha/2 \quad i.e.\ \ \cos\theta - 1 = \alpha/\pi.$$

So that
$$\cos^2[(\pi/2)\cos\theta] = \cos^2(\pi/2-\alpha/2) = \sin^2(\alpha/2) = \frac{1-\cos\alpha}{2}$$

and
$$-\sin\theta\, d\theta = -\frac{d\alpha}{\pi} \quad i.e.\quad d\theta = \frac{d\alpha}{\pi\sin\theta}.$$

Also
$$\sin^2\theta = 1-\cos^2\theta = 1-\left(1-\frac{\alpha}{\pi}\right)^2 = \left(\frac{2\alpha}{\pi}-\frac{\alpha^2}{\pi^2}\right)$$

Making these substitutions in equation (5), we get

$$P = \frac{\mu_0 c I_{rms}^2}{2\pi}\int_0^{\pi}\frac{(1-\cos\alpha)/2}{\sin\theta}\cdot\frac{d\alpha}{\pi\sin\theta}$$
$$= \frac{\mu_0 c I_{rms}^2}{2\pi}\int_0^{2\pi}\frac{(1-\cos\alpha)}{2\pi\left(\dfrac{2\alpha}{\pi}-\dfrac{\alpha^2}{\pi^2}\right)}d\alpha$$
$$= \frac{\mu_0 c I_{rms}^2}{2\pi}\int_0^{2\pi}\frac{(1-\cos\alpha)}{(4\pi\alpha-2\alpha^2)}\,d\alpha$$

Using $\dfrac{1}{4\pi\alpha-2\alpha^2} = \dfrac{1}{4\pi}\left[\dfrac{1}{\alpha}+\dfrac{1}{(2\pi-\alpha)}\right]$, we get

$$P = \frac{\mu_0 c\, I_{rms}^2}{8\pi}\left[\int_0^{2\pi}\frac{1-\cos\alpha}{\alpha}\,d\alpha + \int_0^{2\pi}\frac{(1-\cos\alpha)}{(2\pi-\alpha)}\,d\alpha\right] \qquad \text{...(6)}$$

From tables of cosine integral, we get

$$\int_0^{2\pi}\frac{1-\cos\alpha}{\alpha}\,d\alpha = \int_0^{2\pi}\frac{(1-\cos\alpha)}{(2\pi-\alpha)}\,d\alpha = 2{\cdot}44 \qquad \text{...(7)}$$

Thus we get

$$P = \frac{\mu_0 c\, I_{rms}^2}{8\pi}\times 2\times 2{\cdot}44 = \frac{4{\cdot}88}{8\pi}\,\mu_0 c\, I^2_{rms} \qquad \text{...(8)}$$

Substituting value of μ_0 and c, we get

$$P = \frac{4{\cdot}88}{8\pi}(4\pi\times 10^{-7})(3\times 10^{8})\,I_{rms}^2 = 73{\cdot}2\, I_{rms}^2 \qquad \text{...(9)}$$

Comparing this with ohmic power loss equation $P = I_{rms}^2 R$, we note that the *radiation resistance for half wave antenna is* 73·2 *ohms*. This value is much greater as compared with dipole antenna for which radiation resistance R_r = 0·08 ohms. This implies that a half wave antenna is much more efficient than a dipole antenna.

12.16. Antenna Array

An antenna array is a set of suitably spaced, similarly oriented antennas (or aerials) to radiate energy predominently in some given direction. Antenna arrays make use of interference phenomenon that occur between the radiations from the different elements of the array. Accordingly the resultant field developed by an antenna array at a distant point will be the superposition of field contributions from each consituent antenna of the array. There are two types of common antenna arrays :

1. Broad side array. It exhibits the marked directional properties in the horizontal plane and in a direction normal to the line of the array (*i.e.* for which $\phi = \pi/2$ and $\theta = \pi/2$).

2. End fire array. It exhibits the marked directional properties in the horizontal plane and in a direction along the line of the array. (*i.e.* for which $\phi = 0$ and $\theta = \pi/2$).

Consider a linear* antenna array consisting of n antennas equally spaced along a straight line with common spacing d between two consecutive antennas fig. 12.14. If $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \ldots\ldots$ are the fields due to antennas 1, 2, 3..., respectively, then the resultant field at P will be given by

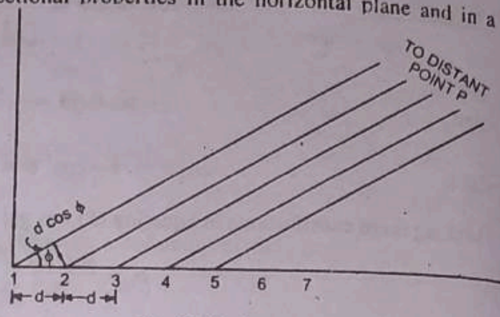


Fig. 12.14. Antenna array

$$\mathbf{E}_r = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \ldots + \mathbf{E}_n = \sum_{i=1}^{n} \mathbf{E}_i$$

If ψ is the phase angle between radiations from two consecutive antennas, and if $E_i = E$, then we may write

$$E_2 = Ee^{i\psi},\ E_3 = Ee^{2i\psi},\ E_4 = Ee^{3i\psi} \ldots\ldots, E_n = Ee^{i(n-1)\psi}$$

$$E_r = E + Ee^{i\psi} + Ee^{2i\psi} + Ee^{3i\psi} + \ldots + Ee^{i(n-1)\psi}$$

or

$$E_r = E[1 + e^{i\psi} + e^{2i\psi} + \ldots + e^{i(n-1)\psi}] \qquad \ldots(1)$$

Multiplying equation (1) throughout by $e^{i\psi}$, we get

$$E_r e^{i\psi} = E[e^{i\psi} + e^{2i\psi} + e^{3i\psi} + \ldots + e^{in\psi}] \qquad \ldots(2)$$

Subtracting equation (2) from equation (1), we get

$$(1 - e^{i\psi}) E_r = E(1 - e^{in\psi})$$

$$\therefore \quad E_r = E\left[\frac{1 - e^{in\psi}}{1 - e^{i\psi}}\right] \qquad \ldots(3)$$

$$= \frac{e^{in\psi/2}[e^{in\psi/2} - e^{-in\psi/2}]}{e^{i\psi/2}[e^{i\psi/2} - e^{-i\psi/2}]}$$

$$= Ee^{i(n-1)\psi/2}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]$$

$$= E\,\frac{\sin n\psi/2}{\sin\psi/2}\,e^{i\xi} \qquad \ldots(4)$$

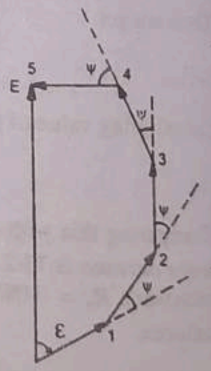


Fig. 12.14

where value of ξ is given by

$$\xi = \frac{(n-1)\psi}{2} \qquad \ldots(5)$$

and refers to the field from antenna-1.

If the phase is referred to the central point of the array, then $\xi = 0$ and equation (4) takes the form

$$E_r = E\,\frac{\sin n\psi/2}{\sin\psi/2} \qquad \ldots(6)$$

When $\psi = 0$, equation (6) becomes indeterminate, so that for this case E_r must be obtained in the limit $\psi = 0$; then

---

*An antenna array is said to the linear when the elements of the array are spaced equally along a straight line.

$$E_r = \lim_{\psi \to 0} E \frac{\left(\cos \frac{n\psi}{2}\right) \frac{n}{2}}{\left(\cos \frac{\psi}{2}\right) \frac{1}{2}} = nE \qquad \text{...(7)}$$

This is the maximum value of the field which E can attain and this happens when radiations from n antennas reach P in the same phase. Thus

$$E_{\max} = nE \qquad \text{...(8)}$$

Using equation (8) equation (6) may be expressed as

$$E_r = E_{\max} \frac{\sin n\psi/2}{n \sin \psi/2} \qquad \text{...(9)}$$

From fig. 12.14 it is clear that the path difference between radiations from the consecutive antennas at distant point P is $d \cos \phi$. Therefore the phase difference between fields at P due to this path difference is

$$\delta = \frac{2\pi}{\lambda} (d \cos \phi)$$

If α is the progressive phase shift between elements (*i.e.* α is) net angle by which the current in any element leads the current in the preceding element), then the resultant phase difference ψ between fields from two consecutive antennas will be given by

$$\psi = \frac{2\pi}{\lambda} (d \cos d) + \alpha \qquad \text{...(10)}$$

We know that the power radiated is proportional to the square of field vector therefore it will be maximum where field E_r is maximum and minimum where field E_r is minimum. Hence the conditions of principal maxima and minima may be defined by :

(a) Principal Maxima. The directions of principal maxima are given by

$$\frac{E_r}{E_{\max}} = \frac{1}{n} \frac{\sin (n\psi/2)}{\sin (\psi/2)} = 0 \qquad \text{...(11a)}$$

The principal maximum of the array occurs at $\psi \to 0$.

Since
$$\psi = \frac{2\pi}{\lambda} \cdot (d \cos \phi + \alpha),$$

therefore principal maximum occurs when

$$\frac{2\pi}{\lambda} (d \cos \phi + \alpha) = 0 \text{ i.e. } \cos \phi = -\frac{\alpha\lambda}{2\pi d} \qquad \text{...(11b)}$$

(a) Null or Minima. The directions of minima are given by

$$\frac{E_r}{E_{\max}} = \frac{1}{n} \frac{\sin \left(\frac{n\psi}{2}\right)}{\sin \left(\frac{\psi}{2}\right)} = 0 \qquad \text{...(12)}$$

i.e. when $\frac{n\psi}{2} = \pm p\pi \,;\ p = 1, 2, 3 \ldots$

(b) Secondary Maxima. The directions for secondary maxima are given by

$$\frac{d}{d\phi}\left(\frac{E_r}{E_{\max}}\right) = \frac{d}{d\phi}\left\{\frac{1}{n} \frac{\sin (n\psi/2)}{(\sin \psi/2)}\right\} = 0 \qquad \text{...(13)}$$

where ψ is given by equation (10).

Alternatively we may say that the secondary maxima occur when the numerator of equation (9) is a maximum,* that is when

$$\frac{n\psi}{2} = \pm (2m + 1) \frac{\pi}{2}, \ m = 1, 2, 3, \ldots$$

---

*It may be noted that $n\psi/2 = \pi/2$ does not give a maximum.

These secondary maxima occur approximately midway between the minima.

(d) The width of principal maxima measured between first minima is twice the angle between the principal maximum and first minimum.

The latter angle $\Delta\phi = \psi_1$ is given by

$$\frac{n\psi_1}{2} = \pi \quad i.e. \quad \Delta\phi = \pm\frac{2\pi}{n} \qquad \text{...(14a)}$$

Therefore width of *principal maximum* (or *principal lobe*)

$$2\,\Delta\phi = 2\psi_1 = \frac{4\pi}{n} \qquad \text{...(14b)}$$

A wide variety of radiation patterns may be obtained by different geometrical shaping and phase relationships between constitutent antennas. Let us now consider the particular cases of interest already specified.

(i) Broad side array. For *broad side array* the maximum radiations occurs normal to the line of the array at $\phi = \pi/2$ and so from equation (11b), we get

$$\cos\frac{\pi}{2} = -\frac{\alpha\lambda}{2\pi d}$$

This gives $\alpha = 0$.

Hence *for broad side array the current in all the antennas must be fed in the same phase.*

Hence in this case ϕ is given by

$$\psi = \frac{2\pi}{\lambda}\,d\cos\phi \quad i.e. \quad \cos\phi = \frac{\lambda\psi}{2\pi d} \qquad \text{...(15)}$$

Since in this case the principal maximum occurs at $\pi/2$, therefore let us say that the *first null* occurs at an angle $(\pi/2 + \Delta\phi)$; where

$$\cos\left(\frac{\pi}{2} + \Delta\phi\right) = \frac{\lambda\psi_1}{2\pi d}$$

But from (14a), $\psi_1 = \frac{2\pi}{n}$; therefore

$$\cos\left(\frac{\pi}{2} + \Delta\phi\right) = \frac{(2\pi/n)\,\lambda}{2\pi d} = \frac{\lambda}{nd}$$

If $\Delta\phi$ is small, it is approximately given by

$$\Delta\phi = \frac{\lambda}{nd}$$

and the width of principal lobe is

$$2\,\Delta\phi = \frac{2\lambda}{nd} \qquad \text{...(16)}$$

Hence ***for a uniform broad side array the width of the principal lobe (in radians) is approximately twice the reciprocal of array length in wavelengths.***

Case (ii). End-fire array. For an end-fire array the maximum radiation is along the line of the array at $\phi = 0$, so the condition

$$\cos\phi = -\frac{\alpha\lambda}{2\pi d} \text{ gives}$$

$$\cos 0 = -\frac{\alpha\lambda}{2\pi d} \quad \text{or} \quad \alpha = -\frac{2\pi d}{\lambda} \qquad \text{...(17)}$$

This indicates that *for an end-fire array the phase between consecutive antennas must be retarded progressively by the same amount as the spacing between elements in radians.* Thus the condition of phase difference for this case takes the form

$$\psi = \frac{2\pi}{\lambda}(d\cos\phi) - \frac{2\pi d}{\lambda}$$

i.e.

$$\psi = \frac{2\pi d}{\lambda}(\cos\phi - 1) \qquad \text{...(18)}$$

The principal maximum is at $\phi = 0$ and the first null is at $\phi_1 = \Delta\phi$ where

$$\psi_1 = \frac{2\pi d}{\lambda}(\cos\phi_1 - 1) = -\frac{2\pi}{n} \qquad \text{using (14a)}$$

or

$$(\cos\Delta\phi - 1) = \frac{\lambda}{nd}$$

For $\Delta\phi$ small, the above equation gives approximately

$$\frac{(\Delta\phi)^2}{2} = \frac{\lambda}{nd}$$

∴ *The width of principal lobe,*

$$2\,\Delta\phi = 2\sqrt{\frac{2\lambda}{nd}} \qquad \text{...(19)}$$

i.e. ***the width of the principal lobe of a uniform end-fire array, as given approximately by (19), is greater than that for a uniform broad side array of the same length.***

QUESTIONS AND PROBLEMS

1. (a) Derive expression for the inhomogeneous wave equations of scalar and vector potentials.
   (b) Using Fourier analysis, derive the solutions of scalar and vector potentials in terms of charge and current distributions respectively. **(Agra 1972)**
2. (a) What is the significance of retarded potentials ? Derive expressions for Lienard-Wiechert potentials.
   (b) Show that the total power radiated by an accelerated charge at low velocity is

$$P = \frac{e^2a^2}{6\pi\varepsilon_0 c^3}$$

   where symbols have their usual meanings. **(Lucknow 1998; Agra 96; Rohilkhand 77)**
3. Discuss electromagnetic scalar and vector potentials. Obtain an expression of the field radiated by an accelerated charge. **(Rohilkhand 1984)**
4. What are Lienard-Wiechert potentials ? Calculate the electric and magnetic field vectors for a uniformly moving electron using *L.W.* potentials. Discuss the results obtained. **(Rohilkhand 1986; Meerut 81)**
5. What are Lienard-Wiechart potentials ? Calculate the electric and magnetic field vector for a uniformly moving point charge using these potentials. Discuss the results obtained. **(Rohilkhand 1987)**
6. Write down the Lienard Wiechert potentials produced by a moving point charge. Obtain the corresponding electric and magnetic fields and identify their respective induction and radiation parts. Show that there is no radiation of charge if its acceleration is zero. **(Rohilkhand 1986)**
7. Starting from the expression for the retarded potentials find the electromagnetic field of a point charge in arbitrary motion. **(Meerut 1989, 83)**
8. Derive the expression for the rate of radiatioin of energy from an accelerated charge at low velocity. **(Rohilkhand 1980)**
9. (a) Obtain expressions for the Lienard Wiechert potentials of a moving point charge. Discuss what do you mean by the retarded time. **(Meerut 1998, 71)**
   (b) Calculate the electric and magnetic fields of a point charge moving in a straight line with constant velocity. **(Meerut 1991)**
10. Explain the term retarded potential and obtain the electromagnetic field of a moving point charge in space. Discuss the result when the distances are large. **(Kanpur 1986)**
11. Derive expressions for the field radiated by an accelerated charge at low velocity. What are Bremsstrahung and Cherenkov radiations. **(Rohilkhand 1993, Agra 92)**
12. Find the power radiated by an accelerated charged particle in the non-relativistic motion and describe its angular distribution. **(Meerut 1988)**

13. Obtain the power radiated and angular distribution of radiation emitted by an accelerated charged particle in non-relativistic motion.

    Does a charge moving with uniform velocity radiate ? **(Meerut 1997)**
14. Discuss the effect of radiation damping on the motion of an accelerated charge. Give a simple account of the theory of line width. **(Meerut 1998, 97)**
15. Derive an expression for the radiated energy from a high velocity electron moving with an acceleration parallel to velocity. How do you explain the constancy of the energy spectrum throughout the frequency scale. **(Meerut 1976)**
16. Obtain the electric and magnetic fields from an oscillating electric dipole and calculate the time averaged power radiated per unit solid angle and the total power radiated. **(Meerut 1980, 71)**
17. Find expression for the radiation field of an occillating electric dipole, Calculate the power radiated. Enumerate other multiple sources of electromagnetic radiation. **(Meerut 1989, 87)**
18. Deduce expressions for electric and magnetic fields of an oscillating electric dipole. **(Rohilkhand 1983)**
19. Give an account of Lienard and Wiechert potentials and find an expression for the field of a charge in uniform motion. **(Rohilkhand 1993, 83)**
20. Give expression for the field produced by an accelerated charge in space. **(Meerut 1992)**
21. Find the power radiated by an oscillating electric dipole and describe its angular distribution. **(Meerut 1982; Rohilkhand 92, 89, 85, 81)**
22. Find the Poynting vector and the power radiated by electric quadrupole. **(Delhi 1996)**
23. Calculate the radiation resistance of centre fed half wave antenna. Hence explain why it is efficient radiator while dipole antenna is not ?
24. What is an antenna array ? Discuss its directional properties analytically.
25. An antenna of length L carries alternating current of angular frequency ω. Treating it as an oscillating dipole, find the total power radiated. Show that the equivalent resitance to produce the same power loss (the radiation resistance) is

$$R = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{L\omega}{2\pi c}\right)^2$$

(Meerut 1975)

26. Discuss the principle of working of an antenna array. Derive an expression for the resultant field in an direction making an angle ϕ with the line of an array consisting of N equally spaced antennas carrying currents of equal magnitude and having successive phase difference. **(Meerut 1991, 89, 81)**
27. Define antenna array. What is meant by gain of an antenna array ? Show that the resultant field in a direction making an angle ϕ with the line of an array consisting of N aerials equally spaced d apart and carrying equal aerial currents and phase in given by

$$F = F_0 \frac{\sin\left(\frac{N\phi}{2}\right)}{N \sin\frac{\phi}{2}}$$

where F_0 is the value of maximum field and

$$\phi = \frac{2\pi}{\lambda}(d\cos\theta)$$

(Meerut 1980, 79)

28. Discuss the phenomenon of electric multipole radiation and give the complete analysis for the dipole radiation. **(Agra 1983)**
29. (a) Discuss physically the basis of production of electromagnetic radiation by moving charges.

    (b) Give the theory of radiation emitted by an accelerated charge. **(Agra 1981)**
30. Write short notes on
    - (i) Retarded potentials or Lienard and Wiechert potentials, **(Meerut97; Rohilkhand 1996, 92)**
    - (ii) Virtual photon, **(Rohilkhand 1983; Agra 72)**
    - (iii) Cherenkov radiation, **(Meerut 97; Rohilkhand 1984; Agra 82)**
    - (iv) Bremsstrahlung;

9

Applications of Electromagnetic Waves I : Reflection and Refraction

[Including Wave Guides and Resonant Cavities]

9.1 Introduction

So far we have discussed the behaviour of a plane electromagnetic wave in free space and in a homogeneous medium and showed that it is propagated with speed of light $c = 1/\sqrt{(\mu_0\varepsilon_0)}$ in free space and with a velocity $u = c/n$ (speed of light divided by refractive index) in a homogeneous medium. Now we shall consider the macroscopic behaviour(particularly optical) of a plane electromagnetic wave when it is incident an infinite plane boundary between two semi-infinite media of different indices of refraction,such for instance as free space and a metallic conductor. We shall be led to the familiar laws of reflection and refraction, laws that were established, before the advent of electromagnetic theory merely from general wave theory. In order to discuss the behaviour of electromagnetic waves at the boundary, we must find the boundary conditions holding at a surface of discontinuity between two media

9.2 Boundary Conditions at the Surface of Discontinuity

In each of the two media, let us assume that the solution of Maxwell's equations that we desire is a plane wave, just as in an infinite media. At the boundary beetween the two-media, however, certain boundary conditions are to be met and these demand that there be a definite relations between the boundary conditions which must be satisfied by field vectors **D, B, E** and **H** at the interface between media.

(i) Boundary condition for electric displacement vector D : According to Maxwell's first equation

$$\text{div}\,\mathbf{D} = \rho \qquad \text{...(1)}$$

At any interface between two media, let us construct a pillbox-like surface S composed of S_1, S_2, S_3 and S_4 shown in fig. 9.1 Integrating equation (1) over the pill-box-shaped volume V, we obtain

$$\int_V \text{div}\,\mathbf{D}\,dv = \int_V \rho\,dv \qquad \text{...(2)}$$

Changing volume integral on L.H.S of above equation into surface integral by usual Gauss's divergence theorem, we get

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\,dv$$

i.e.

$$\int_{S_1} \mathbf{D}_1.\mathbf{n}_1\,dS + \int_{S_2} \mathbf{D}_2.\mathbf{n}_2\,dS + \int_{S_3} \mathbf{D'}_1.\mathbf{n'}_1\,dS + \int_{S_4} \mathbf{D'}_2.\mathbf{n'}_2\,dS = \int_V \rho\,dV \qquad \text{...(3)}$$

where **n**'s represent unit vectors normal to concerned surface.

In above equation the third and fourth terms given the contribution to the surface integral from the walls of the pillbox. If **D** is bounded, letting the height of the pillbox h, approach zero, then the third and fourth terms vanish and S_1 approaches S geometrically and the entire surface takes the form of A as shown in fig. 9.1.

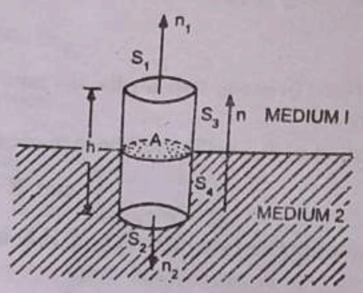


Fig. 9.1

Thus in the limit $h \to 0$ equation (3) takes the form

$$\lim_{h \to 0}\left[\int_S \mathbf{D}_1.\mathbf{n}_1\, dS + \int_S \mathbf{D}_2.\mathbf{n}_2\ dS\right] = \lim_{h \to 0}\int_V \rho\, dV.$$

If σ is the surface charge density at the interface, then above equation takes the form

$$\mathbf{D}_1.\mathbf{n}_1 S_1 + \mathbf{D}_2.\mathbf{n}_2 S_2 = \sigma A$$

i.e.

$$(\mathbf{D}_1 \cdot \mathbf{n}_1 + \mathbf{D}_2 \cdot \mathbf{n}_2)\, A = \sigma A \qquad (\text{since } S_1 = S_2 = A).$$

Since area is arbitrary and $\mathbf{n}_1 = \mathbf{n}$, $\mathbf{n}_2 = -\mathbf{n}$; we get

$$\mathbf{D}_1.n - \mathbf{D}_2.\mathbf{n} = \sigma$$

i.e.

$$\mathbf{D}_{1n} - \mathbf{D}_{2n} = \sigma \qquad ...(4)$$

where D_{1n} and D_{2n} are the normal components of electric displacement vector in the two media, the normal being chosen along **n**. Thus we conclude that ***the normal component of electric displacement is not continuous at the interface but changes by an amount equal to the free surface charge density of charge at the interface.***

(ii) Boundary conditions for magnetic induction B :

Maxwell's second equation is

$$\text{div}\,\mathbf{B} = 0.$$

Taking volume integral of above equation over the entire volume of pillbox, we get

$$\int_V \text{div}\,\mathbf{B}\, dv = 0. \qquad ...(5)$$

Using Gauss's divergence theorem to change volume integral into surface integral

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

i.e.

$$\int_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1\, dS + \int_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2\, dS + \int_{S_3} \mathbf{B}'_1 \cdot \mathbf{n}_1\, dS + \int_{S_4} \mathbf{B}'_2 \cdot \mathbf{n}_2\, dS = 0.$$

Letting $h \to 0$ as in part (i), we note that third and fourth terms vanish while S_1 and S_2 approach each other ; so that in the limit $h \to 0$, above equation takes the form

$$\int_A (\mathbf{B}_1 \cdot \mathbf{n}_1 + \mathbf{B}_2 \cdot \mathbf{n}_2)\, dS = 0 \qquad (\text{since } S_1 = S_2 = A)$$

Since surface is arbitrary, we get

$$\mathbf{B}_1 \cdot \mathbf{n}_1 + \mathbf{B}_2 \cdot \mathbf{n}_2 = 0.$$

Now taking into account the opposite direction of $\mathbf{n}_1$ and $\mathbf{n}_2$, *viz* $\mathbf{n}_1 = \mathbf{n}$ and $\mathbf{n}_2 = -\mathbf{n}$, we get

$$\mathbf{B}_1 \cdot \mathbf{n} - \mathbf{B}_2 \cdot \mathbf{n} = 0 \quad i.e. \quad \mathbf{B}_{1n} = \mathbf{B}_{2n}, \qquad ...(6)$$

i.e. the normal component of magnetic induction (B) is continuous across the interface.

(iii) Boundary condition for electric intensity vector E :

Maxwell's third equation is

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad \text{...(7)}$$

At the interface between two media, we now construct a rectangular loop *abcd* bounding a surface *S* as shown in fig 9.2.

Integrating equation (7) over the loop *abcd*, we get

$$\int_S \text{curl } \mathbf{E} \cdot \mathbf{n}\, dS = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS.$$

Now transforming surface integral on L.H.S. into line integral over the path *abcd* by using Stroke's theorem we get

$$\oint_{abcd} \mathbf{E} \cdot dl = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS. \quad \text{...(8a)}$$

or

$$\int_{ab} \mathbf{E}_1 \cdot dl + \int_{cd} \mathbf{E}_2 \cdot dl + \int_{bc \text{ and } da} \mathbf{E} \cdot dl = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot n\, dS. \quad \text{...(8b)}$$

If the loop is now shrunk by letting $h \to 0$, then the contribution to the integral from sides *bc* and *da* will vanish

i.e.

$$\lim_{h \to 0} \int_{bc \text{ and } da} \mathbf{E} \cdot dl \to 0$$

and also the surface integral on R.H.S. tends to zero provided $(\partial \mathbf{B}/\partial t)$ is everywhere finite. Thus in the limit $h \to 0$, equation (8b) takes the form

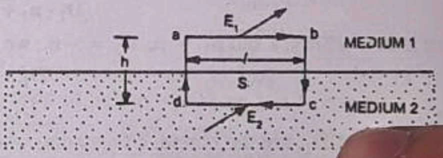


Fig. 9.2

$$\left[\int_{ab} \mathbf{E}_1 \cdot dl + \int_{cd} \mathbf{E}_2 \cdot dl\right] = 0$$

or

$$E_{1t}\, l - E_{2t}\, l = 0 \quad i.e. \quad E_{2t} = E_{1t}, \quad \text{...(9)}$$

where E_{1t} and E_{2t} are the tangential components of the electric field in the two media. Equation (9) represents that *tangential components of* **E** *must be continuous across the interface.*

(iv) Boundary condition for magnetic field intensity H :

Maxwell's fourth equation is

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(10)}$$

Again taking the surface integral over rectangular loop *abcd*, we get

$$\int_S \text{curl } \mathbf{H} \cdot \mathbf{n}\, dS = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{n}\, dS. \quad \text{...(11)}$$

Using Stoke's theorem to change surface integral on L.H.S. into line integral over the path *abcd*, we get

$$\oint_{abcd} \mathbf{H} \cdot dl = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{n}\, dS$$

or

$$\int_{ab} \mathbf{H}_1 \cdot dl + \int_{cd} \mathbf{H}_2 \cdot dl + \int_{bc \text{ and } da} \mathbf{H} \cdot dl = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{n}\, dS \quad \text{...(12)}$$

If the loop is shrunk again in the limit $h \to 0$, we note that the contribution to line integral along sides *bc* and *da* vanishes

i.e.

$$\lim_{h \to 0} \int_{bc+da} \mathbf{H} \cdot dl \to 0 \qquad \text{...(13a)}$$

$$\lim_{h \to 0} \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n}\, dS \to 0$$

i.e.

$$\frac{\partial \mathbf{D}}{\partial t} \text{ is bounded everywhere} \qquad \text{....(13b)}$$

and

$$\lim_{h \to 0} \int_S \mathbf{J} \cdot d\,S \to \mathbf{J}_{S\perp}\, l \qquad \text{...(13c)}$$

where $J_{S\perp}$ represents the components of surface current density perpendicular to the direction of H-component which is being matched. The idea of surface current density is closely analogous to that of a surface charge density-it represents a finite current in an infinitesimal layer. Then in the limit $h \to 0$, equation (12) takes the form

$$\int_{ab} \mathbf{H} \cdot dl + \int_{cd} \mathbf{H} \cdot dl = \mathbf{J}_{S\perp}\, l$$

or

$$H_{1t} l - H_{2t} l = J_{S\perp}\, l$$

i.e.

$$H_{1t} - H_{2t} = J_{S\perp}\, l. \qquad \text{...(14)}$$

Thus the *tangential component of magnetic field intensity is not continuous at the interface* ; but changes by an amount equal to the component of the surface curent density perpendicular to tangential component of H.

The surface current density is zero unless the conductivity is infinite ; hence for finite conductivity $J_S = 0$; so

$$H_{1t} - H_{2t} = 0 \text{ i.e. } H_{1t} = H_{2t}.$$

That is unless one medium has infinite conductivity the tangential component of magnetic field intensity is continuous.

All the above boundary conditions may be summarised as :

1. The normal component of electric displacement (D) is discontinuous across the interface by an amount equal to free surface charge density of charge at the interface.

i.e.

$$D_{1n} - D_{2n} = \sigma. \qquad \text{...(8.1)}$$

2. The normal component of magnetic induction **B** is continuous across the interface

i.e.

$$B_{1n} = B_{2n}. \qquad \text{...(8.2)}$$

3. The tangential component of electric field intensity **E** is continuous across the interface,

i.e.

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \qquad \text{...(8.3)}$$

4. (a) If either of the two media has infinite conductivity, the tangential component of magnetic field intensity is discontinuous at the interface by an amount equal to surface current density at the interface.

i.e.

$$H_{1t} = H_{3t} = J_{S\perp}. \qquad \text{...(7.4a)}$$

(b) If both media have finite conductivity, the tangential component of magnetic field intensity is continuous across at the interface

i.e.

$$H_{1t} = H_{2t}. \qquad \text{...(7.4b)}$$

9.2. Reflection and Refraction of Electromagnetic Waves at the Interface of Non-conducting Media.

The reflection and refraction of light at a plane surface between two media of different dielectric properties are familiar phenomenon. The various aspects of the phenomenon divide themselves into two categories :

1. Kinematic Properties.

(i) Law of reflection. The angle of reflection is equal to the angle of incidence *i.e.* $\theta_r' = \theta_i$ where θ_i is angle of incidence and θ'_r is angle of reflection.

(ii) Snell's law of refraction states.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1}$$

where θ_i and θ_r are the angles of incidence and refraction ; while n_1 and n_2 are corresponding indices of refraction.

(iii) Law of frequency. The incident, reflected and refracted waves all have the same frequency.

2. Dynamic Properties. These properties are concerned with :

(i) intensities of reflected and refracted waves.

(ii) phase changes and polarisation.

The kinematic properties follow immediately from the wave nature of the phenomenon and the fact that there are certain boundary conditions imposed on field vectors to be satisfied. But they do not depend on the nature of waves or the boundary conditions. On the other hand, the dynamic properties depend entirely on the specific nature of electromagnetic field and their boundary conditions.

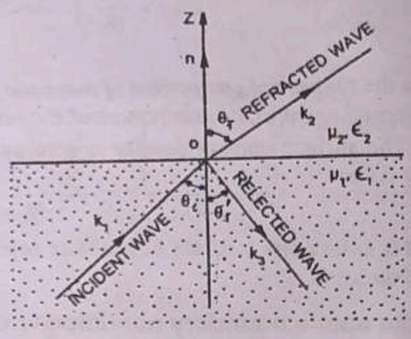


Fig. 9.3 Reflection and refraction

Let us consider a plane interface at $Z = 0$ separating two homogeneous, chargefree and non-conducting isotropic media, characterised by permittivities ε_1 and ε_2 ; permeabilities μ_1 and μ_2 respectively as shown in fig. 9.3. Let a plane wave with wave vector $\mathbf{k}_1$ and frequency ω_1 be incident from medium 'I' at point O on the interface. This wave is partly reflected and partly transmitted (or refracted). Let the reflected and refracted waves have wave vectors $\mathbf{k}'_1$ and $\mathbf{k}_2$, frequencies ω'_1 and ω_2 respectively. Also let $\mathbf{n}$ be the unit vector normal to the interface and directed from medium '1' into medium '2'. The field vectors for incident reflected and refracted waves may be expressed as

$$\left.\begin{aligned} \textit{For incident wave}\quad \mathbf{E}_1 &= \mathbf{E}_{01}\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1 t} \\ \mathbf{B}_1 &= \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\omega_1} \text{ or } \mathbf{H}_1 = \frac{\mathbf{k}_1 \times \mathbf{E}}{\mu_1\omega_1} \end{aligned}\right\} \quad ...(1)$$

$$\left.\begin{aligned} \textit{For reflected wave}\quad \mathbf{E}'_1 &= \mathbf{E}'_{01}\, e^{i\mathbf{k}_1'\cdot\mathbf{r} - i\omega t} \\ \mathbf{B}'_1 &= \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\omega_1'} \text{ or } \mathbf{H}_1' = \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\mu_1\omega_1'} \end{aligned}\right\} \quad ...(2)$$

$$\left.\begin{aligned} \textit{For refracted wave}\quad \mathbf{E}_2 &= \mathbf{E}_{02}\, e^{i\mathbf{k}_2\cdot\mathbf{r} - i\omega t} \\ \mathbf{B}_2 &= \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\omega_2} \text{ or } \mathbf{H}_2 = \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2\omega_2} \end{aligned}\right\} \quad ...(3)$$

Note. The cause of notation used above may be understood as : Incident and reflected wave are in medium 'I', hence incident wave has been characterised by symbol I (unprimed), while reflected wave has been represented by symbol I (primed). The refracted wave is in medium '2', hence it has been represented by symbol 2.

We can apply the boundary condition (iii) derived in preceding section namely that the tangential component of the electric field is continuous across the interface between the media (*i.e.* at $z = 0$). In this case at every point in the interface

$$(E_1)_{tangential} + (E_1)'_{tangential} = (E_2)_{tangential} \qquad \text{...(4)}$$

or

$$(E_{01})_{tangential}\, e^{i\,\mathbf{k}_1\cdot\mathbf{r} - i\omega t} + (E_{01}')_{tangential}\, e^{i\,\mathbf{k}_1'\cdot\mathbf{r} - i\omega_1' t}$$
$$= (E_{02})_{tangential}\, e^{i\,\mathbf{k}_2\cdot\mathbf{r}}\, e^{-i\omega_2 t}$$

i.e.

$$(E_{01})_{tangential}\, e^{i\,\mathbf{k}_1\cdot\mathbf{r}}\, e^{-i\omega t} + (E_{01}')_{tangential}\, e^{i\,\mathbf{k}_1\cdot\mathbf{r}}\, e^{i\omega_1' t}$$
$$= (E_{02})_{tangential}\, e^{i\,\mathbf{k}_2\cdot\mathbf{r}}\, e^{-i\omega_2 t} \qquad \text{...(5)}$$

Since this equality is independent of time, it immediately follows that

$$\omega_1 = \omega_1' = \omega_2 = \omega\,(\text{say}).$$

That is the incident, reflected and refracted waves all have the same frequency.

Since equation (5) holds for all points of the interface ($z = 0$), we msut have

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_1'\cdot\mathbf{r})_{z=0} = (\mathbf{k}_2\cdot\mathbf{r})_{z=0}\,. \qquad \text{...(7)}$$

This equation is independent of the nature of the boundary conditions and contain the kinematic aspects of reflection and refraction.

Writting equation (7) is somewhat expanded from *i.e.*

$$k_{1x}x + k_{1y}y = k'_{1x}x + k'_{1y}y = k_{2x}x + k_{2y}y \qquad \text{...(8)}$$

we get

$$\left.\begin{aligned} k_{1x} &= k'_{1x} = k_{2x} && \text{...}(a)\\ k_{1y} &= k'_{1y} = k_{2y} && \text{...}(b) \end{aligned}\right\} \qquad \text{...(9)}$$

Since incident beam is in *XZ* plane $k_{1y} = 0$, therefore equation (9) implies $k'_{1y} = k_{2y} = 0$, that is both $\mathbf{k}_1'$ and $\mathbf{k}_2$ also lie in *XZ* plane. As normal **n** is along Z axis, thus we conclude that all the ***three wave vectors and normal to the interface* n *all lie in the same plane.*** In other words ***the incident, reflected; refracted waves and the normal to the interface all lie in the same plane.***

Furthermore fig. 9.3, we get

$$\left.\begin{aligned} \mathbf{k}_1\cdot\mathbf{r} &= k_1\,(x\sin\theta_i + z\cos\theta_i) && \text{...(a)}\\ \mathbf{k}_1'\cdot\mathbf{r} &= k_1'\,(x\sin\theta_r' - z\cos\theta_r') && \text{...(b)}\\ \mathbf{k}_2\cdot\mathbf{r} &= k_1\,(x\sin\theta_r - z\cos\theta_r) && \text{...(c)} \end{aligned}\right\} \qquad \text{...(10)}$$

Substitution values from (10a) and (10b) in equation (7) viz.

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_1'\cdot\mathbf{r})_{z=0}$$

we get

$$k_1 x\sin\theta_i = k_1'\,x\sin\theta_r'$$

or

$$k_1\sin\theta_i = k_1'\sin\theta_r' \qquad \text{...(11)}$$

Since wave vector $\mathbf{k}_1$ and $\mathbf{k}_1'$ lie in the same medium, hence $k_1 = k_1' = \omega\sqrt{(\mu_1\varepsilon_1)} = (\omega/v_1)\,v_1$, being phase velocity of electromagnetic wave in medium 1. Therefore equation (11) gives

$$\sin\theta_i = \sin\theta_r' \;\text{ or }\; \theta_i = \theta_r' \qquad \text{...(12)}$$

i.e. the angle of incident (θ_i), *is equal to the angle of reflection* (θ_r'),

Now substituting values of $\mathbf{k}_1\cdot\mathbf{r}$ and $\mathbf{k}_2\cdot\mathbf{r}$ from equation (10) in (7) viz

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_2\cdot\mathbf{r})_{z=0}$$

we get

$$k_1 x\sin\theta_i = k\,x\sin\theta_r$$

$$k_1 \sin \theta_i = k_2 \sin \theta_r$$

i.e.

or

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{k_2}{k_1} = \frac{\omega_2 \sqrt{(\mu_2 \varepsilon_2)}}{\omega_1 \sqrt{(\mu_1 \varepsilon_1)}} = \frac{\sqrt{(\mu_2 \varepsilon_2)}}{\sqrt{(\mu_1 \varepsilon_1)}} \quad (\text{since } \omega_1 = \omega_2 = \omega)$$

This gives

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{n_2}{n_1}$$

(since refractive index, $n = \sqrt{(\mu\, \varepsilon)} = (c/v)$)

This is well known *Snell's law of refraction.*

9.4. Fresnel's Equations (Dynamic Properties of Reflection and Refraction).

The equations relating the amplitudes of the reflected and transmitted waves with that of incident wave are known as Fresnel's equations (or Fresnel's formulae).

We know that field vectors **E** and **B** in a plane electromagnetic wave are mutually perpendicular and also they are perpendicular to the direction of propagation of the wave. Though the field vector **E** is always at right angles to the direction of propagation, but that does not fix the direction uniquely, and is said that the wave is polarised in a particular direction of its electric (or according to an alternative convention, its magnetic vector) points in that direction.

It is convenient to consider two extreme cases (i) in which incident wave is polarised such that electric field vector **E** is normal to the plane of incidence, and (ii) in which field vector **E** is parallel to the plane of incidence. The general result may be obtained by a suitable linear combination of these two extreme results.

Case (i) When E-vector is perpendicular to the plane of incidence : Thc electric and magnetic field vector $\mathbf{E}_1$ and $\mathbf{B}_1$ of the incident wave are perpendicular to direction of propagation $\mathbf{k}_1$ as shown in fig. 9.4. The electric field vector $\mathbf{E}_1$ is perpendicular to the plane of incidence. Since the media are isotropic, the electric field vectors of the reflected and refracted waves are also perpendicular to the plane of incidence. All the electric field vectors are shown directed away from the observer. The orientations of the B-vectors are chosen to give a positive flow of energy in the direction of wave vectors. The field vectors of incident, reflected and refracted waves are given by equations (1), (2) and (3) of section 9.3.

Since electric field vectors are parallel to the boundary surface, the *continuity of the tangential component of electric field vectors* **E** *at the interface* requires that

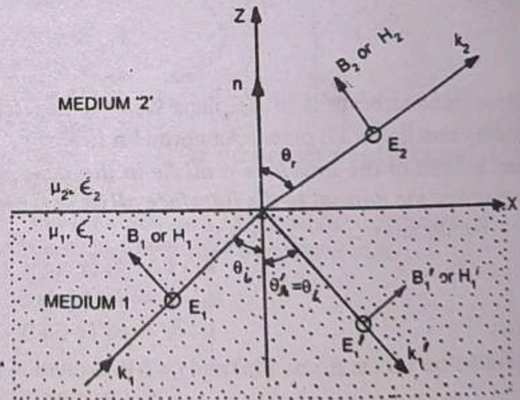


Fig. 9.4 Reflection and refraction with polarization perpendicular to plane of incidence,

$$E_{01} + E'_{01} = E_{02}, \qquad \text{..(1)}$$

Similarly, *the continuity of the tangential component of the magnetic field at the interface* requires that

$$H_{1t} + H'_{2t} = H_{2t}.$$

i.e.

$$-H_{01} \cos \theta_i + H'_{01} \cos \theta'_r = -H_{02} \cos \theta_r$$

i.e.

$$(H_{01} - H'_{01}) \cos \theta_i = H_{02} \cos \theta_r \qquad \text{...(2)}$$

(since from law of reflection $\theta_i = \theta'_r$)

But $\mathbf{B}_1 = \dfrac{\mathbf{k}_1 \times \mathbf{E}_1}{\omega_1}$; therefore $\mathbf{H}_1 = \dfrac{\mathbf{k}_1 \times \mathbf{E}_1}{\mu_1 \omega_1}$ (since $\mathbf{B}_1 = \mu_1 \mathbf{H}_1$)

or

$$\mathbf{H}_1 = \frac{k_1 n_1 \times \mathbf{E}_1}{\mu_1 \omega_1} \qquad \text{(where } n \text{ is vector along } \mathbf{k}_1\text{)}$$

$$= \frac{\sqrt{(\mu_1 \varepsilon_1)}\, n_1 \times \mathbf{E}_1}{\mu_1}$$

$$\left(\text{since } k_1 = \frac{\omega_1}{v_1} = \omega_1 \sqrt{(\mu_1 \varepsilon_1)} \text{ i.e. } \frac{k_1}{\omega_1} = \mu_{1-1}\right)$$

This gives,

$$\left.\begin{aligned} H_0 &= \sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{01} \\ H'_{01} &= \sqrt{\frac{\varepsilon_1}{\mu_1}}\, E'_{01} \\ H_{02} &= \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \end{aligned}\right\} \qquad \text{...(3)}$$

Using these results equation (2), gives

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_{01} - E'_{01}) \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \cos \theta_r. \qquad \text{...(4)}$$

Eliminating E_{02} from equations (1) and (2), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_{01} - E'_{01}) \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,(E_{01} + E'_{01}) \cos \theta_r.$$

On simplification, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \frac{\sqrt{\dfrac{\varepsilon_1}{\mu_1}} \cos \theta_i - \sqrt{\dfrac{\varepsilon_2}{\mu_2}} \cos \theta_r}{\sqrt{\dfrac{\varepsilon_1}{\mu_1}} \cos \theta_i + \sqrt{\dfrac{\varepsilon_2}{\mu_2}} \cos \theta_r} \qquad \text{...(5)}$$

where the symbol ⊥ denotes that in the case under consideration electric field is perpendicular to the plane of incidence. Equation (5), gives *the ratio of amplitudes of electric field vectors in reflected and incident waves i.e. relative electric field amplitudes of reflected wave with respect to incident wave.*

Similarly, eliminating E'_{01} from equations (1) and (4), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,\{(E_{01} - (E_{02} - E_{01}))\} \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \cos \theta_r$$

or

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(2E_{01} - E_{02}) \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \cos \theta_r.$$

On simplification, we get

$$\left(\frac{E_{02}}{E_{01}}\right)_\perp = \frac{2\sqrt{(\varepsilon_1/\mu_1)} \cos \theta_r}{\sqrt{(\varepsilon_1/\mu_1)} \cos \theta_i + \sqrt{(\varepsilon_2/\mu_2)} \cos \theta_r}. \qquad \text{...(6)}$$

This equation gives *relative electric field amplitude of refracted wave with respected to that of incident wave*. Equations (5), and (6) are known as *Fresnel's equations.* Now let us find the form of these equations in the particular case of non-conducting medium, where $\mu_1 = \mu_2 = \mu_0$, so that

$$n_1 = \sqrt{(K_{1m}K_{1e})} = \sqrt{\left(\frac{\mu_1}{\varepsilon_0}\frac{\varepsilon_1}{\varepsilon_0}\right)} = \sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$$

$$n_2 = \sqrt{(K_{2m}K_{2e})} = \sqrt{\left(\frac{\mu_2}{\mu_0}\frac{\varepsilon_2}{\varepsilon_0}\right)} = \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}$$

so that

$$\frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2/\varepsilon_0}{\varepsilon_1/\varepsilon_0}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{...(7)}$$

In this case equation (5) becomes

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \frac{\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_r}{\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_r}$$

$$= \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_3}{\varepsilon_1}}\cos\theta_r}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_r}$$

$$= \frac{\cos\theta_i - \sqrt{\frac{n_2}{n_1}}\cos\theta_r}{\cos\theta_i + \sqrt{\frac{n_2}{n_1}}\cos\theta_r} \qquad \text{...(8a)}$$

$$\left(\text{since } \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

Now using Snell's law $\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$, we obtain

$$\left(\frac{E'_1}{E_{01}}\right)_\perp = \frac{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_r}\cos\theta_r}{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_r}\cos\theta_r}$$

$$= \frac{\sin\theta_r\cos\theta_i - \sin\theta_i\cos\theta_r}{\sin\theta_r\cos\theta_i + \sin\theta_i\cos\theta_r}$$

i.e.

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} \qquad \text{...(8b)}$$

Similarly, equation (6) becomes

e with respected to that of incid
us find the form of these equations
hat

$\left(\text{since } \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \right)$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i}{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_r}$$

$$= \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_r}$$

$$= \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{n_2}{n_1}}\cos\theta_r} \qquad \text{...(9a)}$$

Using Snell's law

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_r}\cos\theta_r} = \frac{2\sin\theta_r\cos\theta_i}{\sin\theta_r\cos\theta_i + \sin\theta_i\cos\theta_r}$$

i.e.

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i + \theta_r)} \qquad \text{...(9b)}$$

Equations (8) and (9) represents *Fresnel's equations* for nonconducting media when electric field vector **E** is perpendicular to the plane of incidence.

Fresnel's equations (8) and (2) lead to following results :

(a) When $(n_1/n_2) < 1$ *i.e.* $n_1 < n_2$ or in words *when an electromagnetic wave in incident on the interface of two dielectrics from a rare medium,* then from Snell's law $\sin\theta_i/\sin\theta_r = n_2/n_1 > 1$; $\sin\theta_i > \sin\theta_r$ or $\theta_i > \theta_r$; *i.e.* refracted rays are deviated towards the normal hence from equation (8b), $\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}$ is negative thereby indicating that *the reflected and the incident waves are in a opposite phases at the interface.* In other words, *when an electromagnetic wave is reflected from a denser medium , it suffers a phase change of π radians.*

(b) When $(n_1/n_2) > 1$ *i.e.* $n_1 > n_2$ or in words, when an electromagnetic wave is incident on the interface of two dielectric from a denser medium, then from Snell's law

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} < 1 \text{ or } \theta_i < \theta_r$$

i.e. the refracted ray is deviated away from the normal and hence from equation (8a), $\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}$ is positive, thereby indicating that reflected and incident waves are in same phase.

(c) In both the above cases (a) and (b), $\left[\frac{E_{02}}{E_{01}}\right]_{\perp}$ in equation (9b) is always positive; there by indicating that the refracted or transmitted wave does not suffer any phase change.

Case (ii). When E vector is parallel to the plane of incidence : In this case electric field vectors of ail in the three waves are parallel to the-plane of incidence as shown fig. 9.5.

The directions of **B** (or **H**) vectors are chosen so as to give a position flow of energy in the direction of wave vectors. In this situation the magnetic field vectors **H** are all parallel to the boundary surface, hence the continuity of the tangential component of the magnetic field vectors **H** at the interface requires that

$$H_{01} + H'_{01} = H_{02}. \qquad ...(10)$$

Similarly, the continuity of the tangential component of electric field vector **E** requires that

$$E_{01} \cos \theta_i - E'_{01} \cos \theta'_i = E_{02} \cos \theta_r$$

or

$$(E_{01} - E'_{01}) \cos \theta_i = E_{02} \cos \theta_r$$

$$(\text{since } \theta'_r = \theta_i) \qquad ...(11)$$

Now using equation (3), equation (10) takes the form

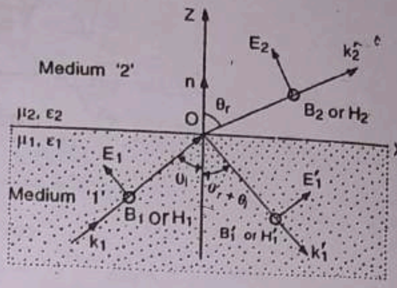


Fig. 9.5. Reflection and refraction with polarization parallel to the plane of incidence.

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{01} + \sqrt{\frac{\varepsilon_1}{\mu_1}}\, E'_{01} = \sqrt{\frac{\varepsilon_0}{\mu_2}}\, E_{01}. \qquad ...(12)$$

Substituting E_{02} from (11) in eqn. (12), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_{01} + E'_{01}) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,(E_{01} - E'_{01}) \frac{\cos \theta_i}{\cos \theta_r}$$

on simplification, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{-\left(\frac{\varepsilon_1}{\mu_1}\right) + \sqrt{\frac{\varepsilon_2}{\mu_2}}\,\frac{\cos \theta_i}{\cos \theta_r}}{\left(\frac{\varepsilon_1}{\mu_1}\right) + \left(\frac{\varepsilon_2}{\mu_2}\right)\frac{\cos \theta_i}{\cos \theta_r}} = \frac{\sqrt{\frac{\varepsilon_2}{\mu_2}} \cos \theta_i - \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_r}{\sqrt{\frac{\varepsilon_1}{\mu_2}} \cos \theta_i + \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_r} \qquad ...(13)$$

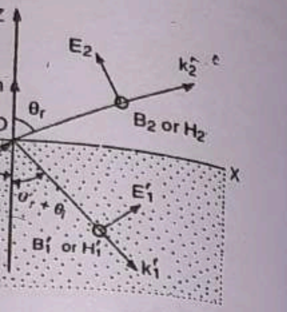
where the symbol parallel (‖) denotes that in the case under consideration electric field vector is parallel to the plane of incidence. Equation (13) gives the *ratio of amplitudes of reflected and incident waves.*

Now substituting E'_{01} from (11) in (12), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{01} + \sqrt{\frac{\varepsilon_1}{\mu_1}} \left(E_{01} - E_{02} \frac{\cos \theta_r}{\cos \theta_i}\right) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02}.$$

On simplification, we get

$$\sqrt{\left(\frac{E_{02}}{E_{01}}\right)_{\parallel}} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_1}}}{\sqrt{\frac{\varepsilon_2}{\mu_2}} + \sqrt{\frac{\varepsilon_1}{\mu_1}}\,\frac{\cos \theta_r}{\cos \theta_i}} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_i}{\sqrt{\frac{\varepsilon_2}{\mu_2}} \cos \theta_i + \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_r}. \qquad ...(14)$$



ction with polarization parallel to of incidence.

...(12)

cos θr

...(13)

cos θr

field vector is parallel to incident waves.

...(14)

cos θr

This equation gives the *ratio of amplitudes of refracted and incident waves.* Equations (13) and (14) represent. *Fresnel's equations* for the case when **E** vector is parallel to the plane of incidence.

For non-conducting media $\mu_1 = \mu_2 = \mu_0$ so

$$n_1 = \sqrt{(\mu_1\varepsilon_1/\mu_0\varepsilon_0)} = \sqrt{(\varepsilon_1/\varepsilon_0)}\ ;\ n_2 = \sqrt{(\mu_2\varepsilon_2/\mu_0\varepsilon_0)} = \sqrt{(\varepsilon_2/\varepsilon_0)}$$

and

$$\frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2/\varepsilon_0}{\varepsilon_1/\varepsilon_0}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{...(15)}$$

In this case equation (13) takes the form

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\sqrt{(\varepsilon_2/\mu_0)}\cos\theta_i - \sqrt{(\varepsilon_1/\mu_0)}\cos\theta_r}{\sqrt{(\varepsilon_2/\mu_0)}\cos\theta_i + \sqrt{(\varepsilon_2/\mu_0)}\cos\theta_r} = \frac{\sqrt{(\varepsilon_2/\varepsilon_1)}\cos\theta_i - \cos\theta_r}{\sqrt{(\varepsilon_2/\varepsilon_1)}\cos\theta_i + \cos\theta_r}$$

$$= \frac{\frac{n_2}{n_1}\cos\theta_i - \cos\theta_r}{\frac{n_2}{n_1}\cos\theta_r + \cos\theta_r}. \qquad \text{...(16 a)}$$

Using Snell's law $\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\left(\frac{\sin\theta_i}{\sin\theta_r}\right)\cos\theta_i - \cos\rho_r}{\left(\frac{\sin\theta_i}{\cos\theta_r}\right)\cos\theta_i + \cos\theta_r}$$

$$= \frac{\sin\theta_i\cos\theta_i - \sin\theta_r\cos\theta_r}{\sin\theta_i\cos\theta_i + \sin\theta_i\cos\theta_r} = \frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)} \qquad \text{...(16 b)}$$

Similarly equation (14) becomes

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i}{\sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_i + \sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_r} = \frac{2\cos\theta_1}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i + \cos\theta_r}$$

$$= \frac{2\cos\theta_i}{\frac{n_2}{n_1}\cos\theta_i + \cos\theta_r}. \qquad \text{...(17a)}$$

Using Snell's law, we get

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} \frac{2\cos\theta_i}{\left(\frac{\sin\theta_i}{\sin\theta_r}\right)\cos\theta_i + \cos\theta_r} = \frac{2\cos\theta_i\sin\theta_r}{\sin\theta_i\cos\theta_i + \sin\theta_r\cos\theta_r}$$

$$= \frac{2\cos\theta_i\sin\theta_r}{\frac{1}{2}(\sin 2\theta_i + \sin 2\theta_r)} = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)}. \qquad \text{...(17 b)}$$

Equations (16) and (17) represents *Fresnel's equations for nonconducting media when* **E** *vector is parallel to the plane of incidence.*

It is obvious that $(E_{02}/E_{01})_{\parallel}$ is always positive as in case (i), hence *the incidence and refracted beams are always in the same phase at the interface.* In this case there is an additional interesting feature if $\theta_i + \theta_r = 90°$, a perfectly possible situation, the denominator of (16b) is infinite, so that the reflection coefficient is zero. This angle of incidence $\theta_i = (90^\circ - \theta_r)$ is called the *Brewster's angle or polarising angle* and is denoted by θ_p. This if an unpolarised radiation consisting of a mixture of both types of radiation falls on the interface at this angle, only the radiation with its electric vector at right angles to the plane of incidence will be reflected, and the reflected radiation will be polarised.

From equation (16b) it is clear that the ratio $(E'_{01}/E_{01})_{\parallel}$ can be either positive or negative depending upon the value of $\frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$. This is positive only if either $\theta_i > \theta_r$ and $\theta_i + \theta_r < \frac{\pi}{2}$ or $\theta_i < \theta_r$ and $\theta_i + \theta_r > \frac{\pi}{2}$. In all other cases $\frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$ is negative. When $\frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$ is positive, then $\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel}$ is positive *i.e.* the reflected and incident waves will be in the phase at the boundary and when $\frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$ is netgative $\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel}$ is negative *i.e.* the reflected wave will suffer a phase change of π radian. Thus in this case the phase of reflected wave does not depend the ratio (n_2/n_1), but on relative values of θ_i and θ_r and their sum.

Fig. 9.6 shows the amplitudes of reflected and refracted disturbances as function of incident amplitudes for $n_2/n_1 = 1{\cdot}5$ and corresponds to light incident from air to glass.

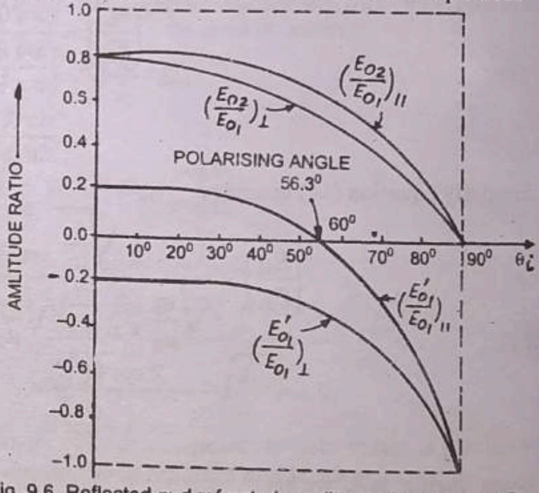


Fig. 9.6. Reflected and refracted amplitudes for $n_2/n_1 = 1{\cdot}5$.

For normal incidence $\theta_i = 0^\circ, \theta_r = 0^\circ$ Fresnel's equations for non-conducting media for the two cases becomes identical as is expected from fig. since then the plane of incidence is indeterminate

$$\left(\frac{E'_{01}}{E_{01}}\right) = \frac{n_2 - n_1}{n_2 + n_1} \quad \text{...(18)}$$

$$\left(\frac{E_{02}}{E_{01}}\right) = \frac{2n_1}{n_2 + n_1}. \quad \text{...(19)}$$

Note. The amplitudes of electric field vector **E** for incident. reflected and transmitted (refracted) waves are also denoted by E_i, E_r and E_t respectively. In these notation.

$$E_{01} \to E_i,\ E'_{01} \to E_r \text{ and } E_{02} \to E_t.$$

Summary. Fresnel's equations for non conducting media (where $\mu_1 = \mu_2 = \mu_0$) relating amplitudes of reflected and refracted waves to that of incident wave are :

Case. (i) E perpendicular to the plane of incidence.

$$\left.\begin{aligned} \left(\frac{E'_{01}}{E_{01}}\right)_{\perp} &= \left(\frac{E_r}{E_i}\right)_{\perp} = -\frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} \quad &\text{...}(a) \\ \left(\frac{E_{02}}{E_{01}}\right)_{\perp} &= \left(\frac{E_t}{E_i}\right)_{\perp} = \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)} \quad &\text{...}(b) \end{aligned}\right] \quad \text{...(8.5)}$$

Case (ii) E parallel to plane of incidence.

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \left(\frac{E_r}{E_i}\right)_{\perp} = -\frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)} \quad \text{...}(a)$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \left(\frac{E_t}{E_i}\right)_{\perp} = \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)} \quad \text{...}(b) \qquad \text{...(8.6)}$$

At normal incidence $(\theta_i = 0 , \theta_r = 0)$, both above equations reduce to

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \left(\frac{E_r}{E_i}\right)_{\perp} = \frac{n_2 - n_1}{n_2 + n_1} \quad \text{...}(a)$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \left(\frac{E_t}{E_i}\right)_{\perp} = \frac{2n_1}{(n_2 + n_1)} \quad \text{...}(b)$$

Discussion

(i) The two components $E_{\perp}$ and $E_{\parallel}$ are not transmitted and reflected equally. For case (i) when electric vector is perpendicular to the plane of incidence, there is reflection unless $\theta_i = \theta_r$ *i.e.* unless the two media have the same optical properties.

(ii) The reflected amplitude in case (ii) *i.e.* when E is parallel to the plane of incidence vanishes if $\theta_i + \theta_r = 90^\circ$. The angle of incidence θ_i satisfying this condition is called Brewster's angle θ_p.

(iii) In case (i). the reflected ray suffers a phase change of π at the interface.

(iv) In case (ii) the reflected ray suffers a phase change of π radian only when angle of incidence is greater than Brewster's angle. For $\theta_i < \theta_p$, the reflected and incident waves are in the same phase at the interface.

(v) The refracted ray never suffers a phase change at the interface.

(vi) At grazing incidence $\theta_i = 90^\circ$, the reflection coefficient is equal to unity. The formula then gives a monotonic increase of amplitude as the angle of incidence increases.

9.5. Experimental Verification of Fresnel's Equations.

The angle between plane of incidence and the plane of vibration of **E** in the linearly polarised light is called *azimuth of the wave.* If β_1, β'_1 and β_2 are the azimuths of incident, reflected and refracted electromagnetic waves, then we have

$$\tan\beta_1 = \frac{(E_{01})_{\perp}}{(E_{01})_{\parallel}} ; \tan\beta_1' = \frac{(E_{01})_{\perp}}{(E_{01}')_{\parallel}} \text{ and } \tan\beta_2 = \frac{(E_{01})_{\perp}}{(E_{02})_{\parallel}} \qquad \text{...(1)}$$

Now by the help of Fresnel's equations, it is possible to express the azimuth's of reflected and refracted waves in terms of incident azimuth as follows :

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = -\frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} \quad \text{...(a)}$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)} \quad \text{...(b)}$$

$$\left(\frac{E_{01}}{E_{01}}\right)_{\parallel} = \frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)} \quad \text{...(c)} \qquad \text{...(2)}$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i + \theta_r)} \quad \text{...(d)}$$

$$\tan \beta'_1 = \left(\frac{(E'_{01})_\perp}{(E_{01})_\parallel}\right) = \frac{\left(\dfrac{E'_{01}}{E_{01}}\right)_\perp (E_{01})_\perp}{\left(\dfrac{E'_{01}}{E_{01}}\right)_\perp ({}^0E_1)_\parallel} = \frac{\left(\dfrac{E'_{01}}{E_{01}}\right)_\perp}{\left(\dfrac{E'_{01}}{E_{01}}\right)_\perp} \tan \beta_1$$

Now using (2a) and (2c), we get

$$\tan \beta'_1 = -\frac{[\sin(\theta_i - \theta_r)/\sin(\theta_i + \theta_r)]}{[\tan(\theta_i - \theta_r)/\tan(\theta_i + \theta_r)]} \tan \beta_1 .$$

$$\therefore \qquad \tan \beta_1' = -\frac{\cos(\theta_i - \theta_r)}{\cos(\theta_i + \theta_r)} \tan \beta_1$$

Similarly

$$\tan \beta_2 = \frac{(E_{02})_\perp}{(E_{02})_\parallel} = \frac{(E_{02}/E_{01})_\perp (E_{01})_\perp}{(E_{02}/E_{01})_\parallel (E_{01})_\parallel} .$$

Using equations (1), (2b) and (2d), we get

$$\tan \beta_2 = \frac{[2\cos\theta_i \sin\theta_r/\sin(\theta_i + \theta_r)]}{[2\cos\theta_i \sin\theta_r/\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)]} \tan \beta_1$$

$$\therefore \qquad \tan \beta_2 = \cos(\theta_i + \theta_r) \tan \beta_1 . \qquad \text{...(4)}$$

Obviously equations (2) and (4) are equivalent to Fresnel's equations (2).

For experimental verification of Fresnel's equations the polarising nicol is set with its principal section at 45° to the plane of incidence. Thus equation (3) takes the form

$$\tan \beta_1' = -\frac{\cos(\theta_i - \theta_r)}{\cos(\theta_i + \theta_r)} \qquad \text{...(5)}$$

The angle β_1 between the direction of vibration of reflected light and plane of incidence can be determined experimentally for different values of θ_i.

The theoretical values of β_1' can be obtained by calculating θ_r from Snell's law

$$\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$$

i.e.

$$\sin\theta_r = \frac{\sin\theta_i}{(n_2/n_1)} = \frac{\sin\theta_i}{{}_1n_2}$$

if ${}_1n_2$ is known initially be some other experiment.

The experimental and theoretical values of β_1' are found to be in agreement; thus verifying Fresnel;'s equations.

Experimental Procedure : The refractive index n_2 of the material of the prism is determined for monochromatic light ; say sodium light by means of an adjusted spectrometer. Now polarising angle ϕ_p is calculated from Brewster's law

$${}_1n_2 = \tan\phi_p \quad \text{or} \quad \phi_p = \tan^{-1}({}_1n_2) .$$

Now the telescope is focussed to a position to receive the direct image of the slit and then rotated through $(180° - 2\phi_p)$ and clamped. A cleaned prism face is placed on the prism table such that the cleaned face is along the axis of spectrometer and also parallel to the collimater axis. The prism table rotated to a position to receive the light reflected from this face into the clamped telescope. In this position light is incident on the prism face at the polarising angle ϕ_p. Hence the reflected light is plane polarised with its vibration perpendicular to the plane of incidence (horizontal).

Now a polaroid is fitted on the telescope objective and rotated without disturbing the prism in any way to get a minimum intensity of image of slit. This is known as *tint of passage*. This reading of polaroid pointer is noted.

The prism is now removed, the telescope is again brought into the initial position (*i.e.* in the line with the collimator) and *another polaroid* is mounted on the collimator lens. This polaroid is rotated till the tint of passage reappears. From this setting the collimator polaroid is rotated through 45°. This setting makes vibration of plane of incident light inclined at 45° to the plane of incidence. This condition is necessary to make $\tan \beta_1 = 1$.

The telescope is now rotated from the direct setting through a known small angle α, say about 10°, and clamped. The prism is remounted in its original position and then rotated by rotating the prism table to a position to receive the light reflected from the prism face into position to receive the light reflected from the prism face into the telescope. The angle of incidence is then given by

$$\theta_i = \frac{180 - \alpha}{2}$$

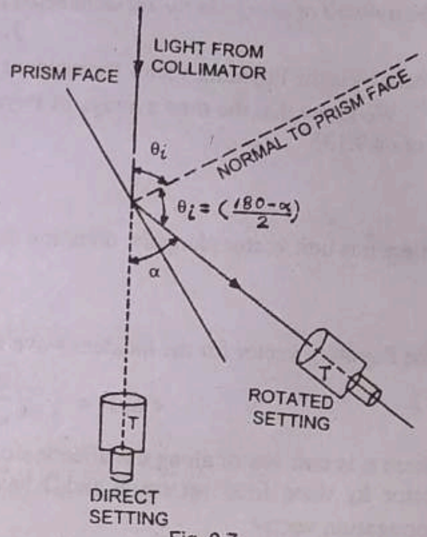


Fig. 9.7

The analysing (or telescope) polaroid is rotated from its initial setting through an angle to restore the tint of passage. This angle of rotation is β_1', then the angle between plane of vibration of reflected light and the plane of incidence is $\beta_1' = 90° - \beta_1$. The angle θ_i is varied in steps of 10° (by turning the telescope) and the angle β_1' is measured for all values of θ_i.

Finally a graph is plotted between $\tan \beta_1'$ and

$$\frac{\cos(\theta_i - \theta_r)}{\cos(\theta_i + \theta_r)}.$$

which comes out to be a straight line passing through the origin and inclined at an angle of 45° with the axes. This verifies the *Fresnel's equations for reflection.*

9.6 Reflection and Transmission Coefficients at the Interface Between two Non-conducting Media

Reflection coefficient or reflectivity *at the interface between on two non-conducting media is defined as the fraction of incident energy flux reflected per unit area per second at the interface*

$$i.e.\ \text{Reflection coefficient, } R = \frac{\begin{array}{c}\text{Magnitude of reflected energy flux}\\ \text{per unit area per second at interface}\end{array}}{\begin{array}{c}\text{Magnitude of incident energy flux}\\ \text{per unit area per second at interface}\end{array}}$$

Transmission coefficient or *transmittivity at the interface between two media is defined as the fraction of incident energy flux transmitted (or refracted) per unit area per second at the interface*

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* The face of the prism is thoroughly cleaned with chromic acid.

$$i.e.\ \text{Transmission coefficient, } T = \frac{\text{Refracted (or transmitted) energy flux per unit area per second at interface}}{\text{Incident energy flux per unit area per second at interface}} \qquad \text{...(2)}$$

At oblique incidence at the interface the cross sectional areas of incident and reflected beams are different from that of the transmitted beam. Therefore, the transmitted intensity is not complementary to the reflected one; but the total energy of these beams is complementary. Suppose a polarised beam of finite cross-section is incident on unit area of the boundary. Then the cross-section area of the incident. Reflected and transmitted beams are respectively.

$$a' = \cos\theta_i,\ a_1' = \cos\theta_r' \text{ and } a_2 = \cos\theta_r \qquad \text{...(3)}$$

The *amount of energy in the incident beam per second* is

$$\mathbf{J}_1 = \mathbf{S}_1 \cdot \mathbf{a}_1 = \mathbf{S}_1 \cos\theta_i \qquad \text{...(4)}$$

where $\mathbf{S}_1$ is the Poynting vector for incident beam.

We know that the time average of Poynting vector in a non-conducting medium is [refer equation (25) section 9.13]

$$\langle \mathbf{S} \rangle = \frac{E_0^2}{2Z}\,\hat{n} \qquad \text{...(5a)}$$

where $\hat{n}$ is unit vector along the direction of energy flow and **Z** is *wave impedance* given by $Z = \sqrt{(\mu/\varepsilon)}$

$$\therefore \qquad \langle \mathbf{S} \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,E_0\,\hat{n} \qquad \text{...(5b)}$$

The Poynting vector for the incident wave at the interface is

$$\langle \mathbf{S}_1 \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{01}^2\,\hat{\mathbf{n}} \quad \text{or} \quad \langle \mathbf{S}_1 \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{01}^2 \qquad \text{...(6)}$$

where **n** is unit vector along the direction of energy flow in the incident beam of along the direction of wave vector $\mathbf{k}_1$ since field vectors **E** and **B** have been chosen to give the energy flow along the direction of propagation vector.

Therefore the incident energy flux per unit area of the boundary per second is

$$J_1 = \langle S_1 \rangle \cos\theta_i = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{01}^2\cos\theta_i \qquad \text{...(7)}$$

Similarly for reflected and transmitted wave, we have

$$J_1' = \langle S_1' \rangle \cos\theta_i' = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{01}^2\cos\theta_i \qquad \text{...(8)}$$

(since $\theta_r' = \theta_i$)

and

$$J_2 = \langle S_2 \rangle \cos\theta_r = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{02}^2\cos\theta_r \qquad \text{...(9)}$$

$\because$ *Reflection coefficient,*

$$R = \frac{J_1'}{J_1} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{02}^2\cos\theta_i}{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\,E_{01}^2\cos\theta_i}$$

$$= \frac{E_{01}^2}{E_{01}^2} = \left(\frac{E_{01}'}{E_{01}}\right)^2 \qquad ...(10)$$

and *transmission coefficient,*

$$T = \frac{J_2}{J_1} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_2}{\mu_0}}\, E_{02}^2 \cos\theta_r}{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2 \cos\theta_i}$$

$$= \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}\, \frac{E_{02}^2}{E_{01}^2} \cdot \frac{\cos\theta_r}{\cos\theta_i} = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left\{\frac{E_{02}}{E_{01}}\right\}^2 \qquad ...(11)$$

Now from Fresnel s equations

$$\left.\begin{aligned} \left(\frac{E_{01}'}{E_{01}}\right)_{\parallel} &= \frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)} && ...(a) \\ \left(\frac{E_{02}}{E_{01}}\right)_{\parallel} &= \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)} && ...(b) \end{aligned}\right] \qquad ...(12$$

$$\left.\begin{aligned} \left(\frac{E_{01}'}{E_{01}}\right)_{\perp} &= -\frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)}. && ...(a) \\ \left(\frac{E_{02}}{E_{01}}\right)_{\perp} &= \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)} && ...(b) \end{aligned}\right] \qquad ...(13)$$

When **E** vector is parallel to the plane of incidence, the coefficients are given by

$$R_{\parallel} = \left(\frac{E_{01}'}{E_{01}}\right)_{\parallel}^2 = \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)} \qquad ...(14)$$

and

$$T_{\parallel} = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left(\frac{E_{02}}{E_{01}}\right)_{\parallel}^2 = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left[\frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i + \theta_r)}\right]^2$$

Using Snell's law,

$$\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$$

$$T_{\parallel} = \frac{\sin\theta_i \cos\theta_r}{\sin\theta_r \cos\theta_i} \cdot \frac{4\cos^2\theta_i \sin^2\theta_r}{\sin^2(\theta_i + \theta_r)\cos^2(\theta_i - \theta_r)}$$

$$= \frac{\sin 2\theta_i \sin 2\theta_r}{\sin^2(\theta_i + \theta_r)\cos^2(\theta_i - \theta_r)} \qquad ...(15)$$

When **E** vector is perpendicular to the plane of incidence, coefficients are

$$R_{\perp} = \left(\frac{E'_{01}}{E_{01}}\right)_{\perp}^2 = \frac{\sin^2(\theta_i - \theta_r)}{\sin^2(\theta_i + \theta_r)} \qquad ...(16)$$

$$T_{\perp} = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left\{\frac{E_{02}}{E_{01}}\right\}_{\perp}^2 = \frac{\sin\theta_i \cos\theta_r}{\sin\theta_r \cos\theta_i}\left[\frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)}\right]^2$$

$$= \frac{\sin 2\theta_i \sin 2\theta_r}{\sin^2(\theta_i + \theta_r)} \qquad ...(17)$$

For normal incidence, we have

$$\left.\begin{aligned}\frac{E_{01}'}{E_{01}} &= \frac{n_2-n_1}{n_2+n_1} \qquad \ldots(a)\\ \text{and} \qquad \frac{E_{02}}{E_{01}} &= \frac{2n_1}{n_2+n_1} \qquad \ldots(b)\end{aligned}\right\} \qquad \ldots(18)$$

∴ Reflection and transmission coefficients for normal incidence $(\theta_i+\theta_r = 0)$ are given by

$$R = \left(\frac{E_{01}'}{E_{01}}\right)^2 = \frac{(n_2-n_1)^2}{(n_2+n_1)^2} \qquad \ldots(19)$$

$$\begin{aligned}T &= \frac{n_2}{n_1}\left(\frac{E_{02}}{E_{01}}\right)^2 \frac{\cos\theta_r}{\cos\theta_i}\\ &= \frac{n_2}{n_1}\left(\frac{2n_1}{n_2+n_1}\right)^2 = \frac{4n_1n_2}{(n_2+n_1)^2} \qquad \ldots(20)\end{aligned}$$

It is obvious that

$$R+T = 1, \qquad \ldots(21)$$

in accordance with the law of conservation of energy.

9.7 Brewster's Law and Degree of Polarisation.

Fresnel equations are

$$\left.\begin{aligned}\left(\frac{E_{01}'}{E_{01}}\right)_{\parallel} &= \frac{\tan(\theta_i-\theta_r)}{\tan(\theta_i+\theta_r)} \qquad \ldots(a)\\ \left(\frac{E_{02}}{E_{01}}\right)_{\parallel} &= \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)\cos(\theta_i-\theta_r)} \qquad \ldots(b)\end{aligned}\right\} \qquad \ldots(1)$$

$$\left.\begin{aligned}\left(\frac{E_{01}'}{E_{01}}\right)_{\perp} &= -\frac{\sin(\theta_i-\theta_r)}{\sin(\theta_i+\theta_r)} \qquad \ldots(a)\\ \left(\frac{E_{02}}{E_{01}}\right)_{\perp} &= -\frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)} \qquad \ldots(b)\end{aligned}\right\} \qquad \ldots(2)$$

It is obvious from equation [1 (a)] that when $(\theta_i+\theta_r) = \pi/2$, $(E_{01}'/E_{01})_{\parallel} = 0$; so that the reflection coefficient is zero. Thus when the reflected and refracted waves are perpendicular to each other, there is no energy carried by the reflected ray. This angle of incidence $\theta_i = (\pi/2-\theta_r)$ is known as the *Brewster's angle* or *polarising angle* and is denoted by θ_B or θ_p.

Now from Snell's law

$$\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$$

or

$$\frac{n_2}{n_1} = \frac{\sin\theta_p}{\sin(\pi/2-\theta_p)} \qquad \left\{\text{because } \theta_i+\theta_r = \frac{\pi}{2}\right\}$$

i.e.

$$\tan\theta_p = \frac{n_2}{n_1} \quad \text{or} \quad \theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}({}_1n_2)$$

This is well known *Brewster's law*. Thus if a plane wave of mixed polarisation (or an unpolarised wave) is incident on a plane interface at the *Brewster's angle*, then the only portion of wave with electric vector perpendicular to the plane of incidence will be reflected. Thus, the reflected radiation is completely plane polarised with polarisation vector perpendicular to the plane of incidence. Hence, the *Brewster's* *angle is also known as polarising angle.* For a wave entering glass ($n_2 = 1{\cdot}5$) from air ($n_1 = 1$), $\theta_p = 56^\circ$ and so $\theta_r = (90^\circ - 56^\circ) = 34^\circ$. Thus, if we allow unpolarised light to strike glass at angle of incidence 56°, the reflected light will be plane polarised with polarisation vector perpendicular to the plane of incidence and transmitted light will also be plane polarised with polarisation vector parallel to the plane of incidence as shown in fig. 9.8 (a). Hence again Maxwell's equations predict an effect long known in optics.

For radio waves incidence on water ($n_1 = 9$), refractive index of water radio frequencies from ($n_1 = 1$), $\theta_p = 84^\circ$ and so $\theta_r = 6^\circ$. Therefore, if radio waves are incident on water at angle of incidence 84°, water will not effect radio waves which are polarised with polarisation vector in the plane of incidence and will not transmit radio wave initially polarised with polarisation vector perpendicular to the plane of incidence as shown in fig. 9.8 (b) and 9.8 (c).

Even if the unpolarised wave is reflected at angles other than the Brewster's angle, there is a tendency for the reflected wave to be predominately polarised perpendicular to the plane of incidence. The success of dark glasses which selectively transmits only one direction of polarisation depends on this fact. In such a case the *degree of polarisation* is defined as

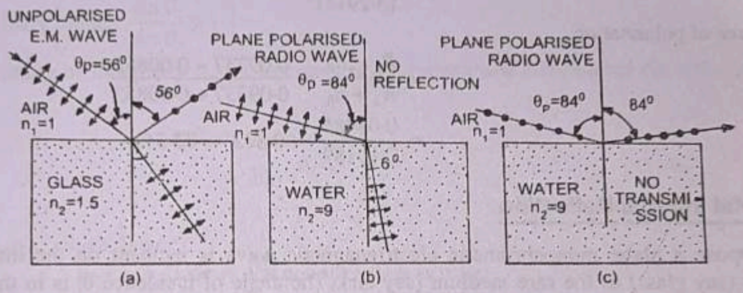

Fig. 9.8

$$\text{degree of polarisation } P = \frac{R_\perp - R_\parallel}{R_\perp + R_\parallel}$$

where $R_\perp$ and $R_\parallel$ are reflection coefficient for perpendicular and parallel components of reflected light are given by

where
$$R_\perp = \frac{\sin^2(\theta_i - \theta_r)}{\sin^2(\theta_i + \theta_r)} \quad \text{and} \quad R_\parallel = \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)}$$

Ex. 1. *Calculate the degree of polarisation for ordinary light reflected from glass (refractive index 1.5) at an angle of incidence 45°.* **(Meerut 1970)**

Solution. From Snell's law

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} = {}_1n_2 = n \text{ (say)}$$

Here $\theta_i = 45^\circ$ and $n = 1{\cdot}5$.

$$\therefore \qquad \sin\theta_r = \frac{\sin\theta_i}{n} = \frac{\sin 45^\circ}{1{\cdot}5} = \frac{1}{\sqrt{2} \times 1{\cdot}50}$$

or
$$\theta_r = \sin^{-1}\left(\frac{1}{1{\cdot}5\sqrt{2}}\right)$$
$$= \sin^{-1}\left(\frac{2}{3\sqrt{2}}\right) = \sin^{-1}\left(\frac{2\sqrt{2}}{6}\right) = \sin^{-1}(0{\cdot}4713) = 28{\cdot}1^\circ$$

$$\therefore \quad R_\perp = \frac{\sin^2(\theta_i - \theta_r)}{\sin^2(\theta_i + \theta_r)} = \frac{\sin^2(45^\circ - 28{\cdot}1^\circ)}{\sin^2(45^\circ + 28{\cdot}1^\circ)}$$

$$= \frac{\sin^2 16{\cdot}9^\circ}{\sin^2 73{\cdot}1^\circ} = \frac{(0{\cdot}2294)^2}{(0{\cdot}9568)^2}$$

$$= 0{\cdot}09337.$$

and

$$R_\parallel = \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)} = \frac{\tan^2 16{\cdot}9^\circ}{\tan^2 73{\cdot}1^\circ}$$

$$= \frac{(0{\cdot}3038)^2}{(3{\cdot}2914)^2} = 0{\cdot}00852.$$

∴ Degree of polarisation

$$= \frac{R_\perp - R_\parallel}{R_\perp + R_\parallel} = \frac{0{\cdot}09337 - 0{\cdot}00852}{0{\cdot}09337 + 0{\cdot}00852}$$

$$= \frac{0{\cdot}08485}{0{\cdot}10189} = \mathbf{0{\cdot}833 = 83{\cdot}5\%}.$$

9.8 Total Internal Reflection

Suppose a plane monochromatic electromagnetic wave is incident on the interface from a denser medium (say glass) to the rare medium (say air). The angle of incidence θ_i is in the denser medium and angle of refraction θ is the rarer medium. The word internal implies that the incident and reflected waves are in denser medium (*i.e.* medium of larger refracted index than the refracted wave ($n_1 > n_2$).

Now from Snell's law is

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} \qquad \text{...(1)}$$

Since $\frac{n_2}{n_1} < 1$; this implies $\frac{\sin\theta_i}{\sin\theta_r} < 1$ or $\theta_i < \theta_r$.

Thus for a particular value of angle of incidence θ_i, the angle of refraction θ_r will becomes 90°. This value of angle of incidence for which θ_r becomes 90° is known as *critical angle* and is denoted by θ_c. Substituting $\theta_i = \theta_c$ and $\theta_r = 90$ in (1), we get

$$\frac{\sin\theta_c}{\sin 90^\circ} = \frac{n_2}{n_1}$$

i.e.

$$\sin\theta_c = \frac{n_2}{n_1} \text{ or } \theta_c = \left(\sin^{-1}\frac{n_2}{n_1}\right) \qquad \text{...(2)}$$

Thus for wave incident at $\theta_i = \theta_c$, the refracted wave is propagated parallel to the surface (since $\theta_r = 90^\circ$) as shown in fig. 9.9. For glass and air

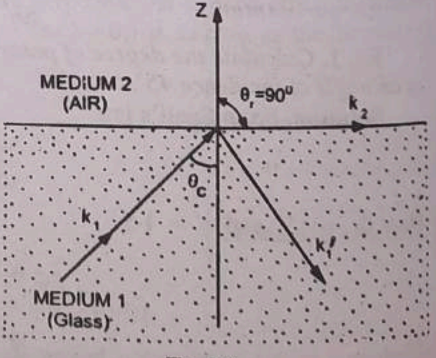

Fig. 9.9

$$\frac{n_2}{n_1} = \frac{1}{1{\cdot}5} = \frac{2}{3};$$

$$\therefore \quad \theta_c = \sin^{-1}\tfrac{2}{3} = \sin^{-1}(0{\cdot}6667) = 41^\circ 55'$$

Now let us see what happens if the angle of incidence θ_i exceeds the critical angle θ_c. From equation (1)

$$\sin\theta_r = \frac{\sin\theta_i}{\left(\dfrac{n_2}{n_1}\right)} = \frac{\sin\theta_i}{\sin\theta_c} > 1 \qquad (\text{since } \theta_i > \theta_c) \qquad \text{...(3)}$$

This equations indicates that angle of refraction θ_r is imaginary; since sine of any real angle can never be greater than unity.

Thus

$$\cos\theta_r = \sqrt{(1-\sin^2\theta_r)} = \sqrt{\left[1-\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2\right]}$$

$$= i\sqrt{\left[\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1\right]} \qquad \text{...(4)}$$

i.e. $\cos\theta_r$ is purely imaginary $\left(\text{since } \dfrac{\sin\theta_i}{\sin\theta_r} > 1\right)$

Now to find the meaning of imaginary θ_r, let us consider the transmitted or refracted electromagnetic wave

$$E_2 = E_{02}\, e^{i\mathbf{k}_2 \cdot \mathbf{r} - i\omega t}$$

$$= \mathbf{E}_{02}\, e^{ik_2 \hat{\mathbf{n}}_2 \cdot \mathbf{r} - i\omega t} \qquad \text{...(5)}$$

($\hat{\mathbf{n}}_2$ being unit vector along $\mathbf{k}_2$)

As transmitted wave lies in xz plane, hence

$$\hat{\mathbf{n}}_2 \cdot \mathbf{r} = (\mathbf{i}\sin\theta_r + \mathbf{k}\cos\theta_r) \cdot (\mathbf{i}x + \mathbf{i}y + \mathbf{k}z)$$

$$= x\sin\theta_r + z\cos\theta_r$$

$$\mathbf{E}_2 = \mathbf{E}_{02}\, e^{ik_2(x\sin\theta_r + z\cos\theta_r) - i\omega t} \qquad \text{...(6)}$$

Substituting value of $\sin\theta_r$ and $\cos\theta_r$ from equation (3) and (4), we get

$$\mathbf{E}_2 = \mathbf{E}_{02}\exp\left[ik_2\left\{x\left(\frac{\sin\theta_i}{\sin\theta_c}\right) + iz\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}\right\}\right]\exp(-i\omega t)$$

$$= \mathbf{E}_{02}\exp\left[-k_2 z\left\{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1\right\}^{1/2}\right]\exp\left[ik_2\left(\frac{\sin\theta_i}{\sin\theta_c}\right)x - i\omega t\right] \qquad \text{...(7)}$$

This equation shows that for $\theta_i > \theta_c$, *the refracted wave is propagated only parallel to the surface and is attenuated exponentially beyond the interface.* Thus we conclude that for $\theta_i > \theta_c$ the wave actually penetrates the second medium where its amplitude is quickly damped according to factor $e^{-k_2 z\,[(\sin\theta/\sin\theta_c)^2 - 1]^{1/2} z}$. The existence of this damped wave within the second medium has been established in experiments employing electromagnetic waves of a few centimeters wavelength (microwaves).

Energy flow in the second medium

The time-averaged normal component of Poynting vector just inside the surface gives the energy flux carried by transmitted wave per unit area per second and is given by

$$\mathbf{S}_2 \cdot \mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\,[(\mathbf{E}_2 \times \mathbf{H}_2^*) \cdot \mathbf{n}]$$

But

$$\mathbf{H}_2 = \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2\omega_2} \qquad \left\{\text{where } k_2 = \frac{\omega_2}{v_2} = \omega_2\sqrt{\varepsilon/\mu_2} \text{ is real}\right\}$$

$$\therefore \qquad \mathbf{S}_2 \cdot \mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\left\{\left[\mathbf{E}_2 \times \frac{\mathbf{k}_2 \times \mathbf{E}_2^*}{\mu_2\,\omega_2}\right] \cdot \mathbf{n}\right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{\left[\frac{(\mathbf{E}_2 \cdot \mathbf{E}_2^*)\,\mathbf{k}_2\,(\mathbf{E}_2 \cdot \mathbf{k}_2)\,\mathbf{E}_2^*}{\mu_2\,\omega_2}\right] \cdot \mathbf{n}\right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{\left[\frac{(\mathbf{E}_2 \cdot \mathbf{E}_2^*)\,\mathbf{k}_2 \cdot \mathbf{n}}{\mu_2\,\omega_2}\right]\right\}$$

(since $\mathbf{k}_2$ and $\mathbf{E}_2$ are mutually perpendicular)

But $\qquad \mathbf{E}_2.\mathbf{E}_2^* = \mathbf{E}_{02}^2\, e^{-2k_2 z\,[(\sin\theta_i/\sin\theta_c)^2 - 1]^{1/2}}$

and $\qquad \mathbf{k}_2 \cdot \mathbf{n} = k_2 \cos\theta_r$

$$\therefore \qquad \mathbf{S}_2 \cdot \mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{k_2\,E_{12}^2}{\mu_2\omega_2}\, e^{-2k_2 z\,\{(\sin\theta_i/\sin\theta_c)^2 - 1\}^{1/2}} \cos\theta_r\right] = 0 \qquad \text{...(8)}$$

Since $\cos\theta_r$ is purely imaginary, the quantity within bracket is purely imaginary.

Thus we may say that ***the net energy flow through the surface into second medium is zero*** even though fields exist in the second medium. Hence total internal reflection occurs at $\theta_i \geq \theta_c$

Now, let us re-examine Fresnel's equations by inserting the values of $\sin\theta_r$ and $\cos\theta_r$ from equations (3) and (4) respectively. In the first case when **E** is perpendicular to the plane of incidence, we have(refer eqn. (8a) of section 9.4).

$$\left\{\frac{E'_{01}}{E_{01}}\right\}_\perp = \frac{\cos\theta_i - \dfrac{n_2}{n_1}\cos\theta_r}{\cos\theta_i + \dfrac{n_2}{n_1}\cos\theta_r} \qquad \text{...(9)}$$

$$= \frac{\alpha - i\,\beta}{\alpha + i\beta} \text{ where } \begin{cases} \alpha = \cos\theta_r\ i\beta = \dfrac{n_2}{n_1}\cos\theta_r \\ \text{(since } \cos\theta_r \text{ is purely imaginary)} \end{cases}$$

$$= \frac{(\alpha - i\beta)\,(\alpha - i\beta)}{(\alpha + i\beta)\,(\alpha - i\beta)} = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} - i\,\frac{2\alpha\beta}{\alpha^2 + \beta^2} \qquad \text{...(10)}$$

If we put

$$\frac{\alpha - i\beta}{\alpha + i\beta} = \rho e^{-i\delta_\perp} = (\rho\cos\delta_\perp) - i\,(\rho\sin\delta_\perp) \qquad \text{...(11)}$$

Comparing (10) and (11), we get

$$\rho\cos\delta_\perp = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \text{ and } \rho\sin\delta_\perp = \frac{2\alpha\beta}{\alpha^2 + \beta^2} \qquad \text{...(12)}$$

which yields $\rho = 1$ and $\tan\delta_\perp = \dfrac{2\alpha\beta}{\alpha^2 - \beta^2}$ ...(13)

Thus

$$\left\{\frac{E_{01}'}{E_{01}}\right\}_\perp = \frac{\alpha - i\beta}{\alpha + i\beta} = e^{i\delta_\perp} \text{ with } \tan{}^{*}\frac{\delta_\perp}{2} = \frac{\beta}{\alpha} \qquad \text{...(14)}$$

---

[*] If $\tan\delta = \dfrac{2\alpha\beta}{\alpha^2 - \beta^2}$ or $\dfrac{2}{\cot\delta/2 - \tan\delta/2} = \dfrac{2}{\alpha/\beta - \beta/\alpha}$

$\therefore \quad \tan\delta/2 = \beta/\alpha.$

Similarly in the second case when **E** is parallel to the plane of incidence (refer equation (16a) of section 9.4).

$$\left\{\frac{E_{01}'}{E_{01}}\right\}_{\parallel} = \frac{\frac{n_2}{n_1}\cos\theta_i - \cos\theta_r}{\frac{n_2}{n_1}\cos\theta_i + \cos\theta_r}$$

$$= \frac{\cos\theta_i - \frac{n_1}{n_2}\cos\theta_r}{\cos\theta_i + \frac{n_1}{n_2}\cos\theta_r} = \frac{\alpha - i\beta'}{\alpha + i\beta'}$$

where $\begin{cases} x = \cos\theta_i \\ i\beta' = \frac{n_1}{n_2}\cos\theta_r \end{cases}$

$$= e^{i\delta_{\parallel}} \text{ with } \tan\frac{\delta_{\parallel}}{2} = \frac{\beta'}{\alpha'} \qquad \text{...(15)}$$

Thus the components of electric vector parallel and perpendicular the plane of incidence on reflection experience phase retardations of $\delta_{\parallel}$ and $\delta_{\perp}$ respectively; but their amplitudes remain unaltered.

$$|(E_{01}')_{\parallel}| = (E_{01})_{\parallel}| \text{ and } |(E_{01}')_{\perp}| = (E_{01})_{\perp}| \qquad \text{...(16)}$$

Thus for each component the intensity of electromagnetic wave which is totally reflected is equal to the intensity of incident light. Consequently we say that *the incident wave is totally reflected.*

Relative phase difference between parallel and perpendicular components of reflected electromagnetic wave. The relative phase different between parallel and perpendicular components of the reflected wave is given by

$$\delta = \delta_{\parallel} - \delta_{\perp}$$

or

$$\frac{\delta}{2} = \frac{\delta_{\parallel}}{2} - \frac{\delta_{\perp}}{2}$$

$$\therefore \quad \tan\frac{\delta}{2} = \tan\left(\frac{\delta_{\parallel}}{2} - \frac{\delta_{\perp}}{2}\right) = \frac{\tan\left(\frac{\delta_{\parallel}}{2}\right) - \tan\left(\frac{\delta_{\perp}}{2}\right)}{1 + \tan\left(\frac{\delta_{\parallel}}{2}\right)\tan\left(\frac{\delta_{\perp}}{2}\right)}$$

$$= \frac{\frac{\beta'}{\alpha} - \frac{\beta}{\alpha}}{1 + \frac{\beta'}{\alpha}\cdot\frac{\beta}{\alpha}} = \frac{\alpha(\beta' - \beta)}{\alpha^2 + \beta'\beta}$$

But $\alpha = \cos\theta_i$, $\beta = \frac{n_2}{n_1}\cos\theta_r$ and $\beta' = \frac{n_1}{n_2}\cos\theta_r$

$$\therefore \quad \tan\frac{\delta}{2} = \frac{\cos\theta_i\left\{\frac{n_1}{n_2}\cos\theta_r - \frac{n_2}{n_1}\cos\theta_r\right\}}{\cos^2\theta_i + \frac{n_1}{n_2}\cos\theta_r\cdot\frac{n_2}{n_1}\cos\theta_r}$$

$$= \frac{\cos\theta_i \left\{ \frac{n_1}{n_2} - \frac{n_2}{n_1} \right\} \cos\theta_r}{\cos^2\theta_i + \cos^2\theta_r}$$

$$= \frac{\cos\theta_i \left\{ \frac{n_1}{n_2} - \frac{n_2}{n_1} \right\} \left[\left\{ \frac{n_1}{n_2} \sin\theta_i \right\}^2 - 1 \right]^{1/2}}{\cos^2\theta_i + \left\{ \frac{n_1}{n_2} \sin\theta_i \right\}^2 - 1}$$

[Using eqn. (4)]

$$= \frac{\cos\theta_i \frac{n_1}{n_2} \left\{ \frac{n_1^2}{n_2^2} - 1 \right\} \left[\left\{ \frac{n_1}{n_2} \sin\theta_i \right\}^2 - 1 \right]^{1/2}}{\frac{n_1^2}{n_2^2} \sin^2\theta_i - (1 - \cos^2\theta_i)}$$

$$= \frac{\cos\theta_i \frac{n_2}{n_1} \left\{ \frac{n_1^2}{n_2^2} - 1 \right\} \left[\left\{ \frac{n_1}{n_2} \sin^2\theta_i \right\}^2 - 1 \right]^{1/2}}{\left\{ \frac{n_1^2}{n_2^2} - 1 \right\} \sin^2\theta_i}$$

$$\therefore \quad \tan\frac{\delta}{2} = \frac{\cos\theta_i \sqrt{\left[\sin^2\theta_i - \left\{ \frac{n_2}{n_1} \right\}^2 \right]}}{\sin^2\theta_i} \qquad \text{...(17)}$$

Obviously the phase difference depends upon the angle of incidence θ_i greater than the critical angle. The plane polarised radiation after reflection becomes elliptically polarised. Any desired phase difference δ can be introduced between the two components of reflected light by the proper choice of θ. Therefore any desired type of elliptically polarised radiation can be conveniently produced by the method of total internal reflection. From equation (17) it is easy to see that $n_1/n_2 = \frac{3}{2}$, $\delta = \pi/4$ when $\theta_i = 53°15'$ or $\theta_i = 50°14'$ and so another total internal reflection at same angle the phase difference between reflected components becomes $\pi/2$. The reflected radiation is of course elliptically polarised since in general the components $E_\parallel$ and $E_\perp$ are of unequal amplitudes.

The phenomenon of total internal reflection is applicable in cases where it is required to transmit light without loss in intensity, It is also used to produce elliptically and circularly polarised light.

9.9. Reflection From a Conducting Plane (*i.e.* Metallic Reflection)

Let us consider the reflection and refraction at the interface separating a conducting and a non-conducting media. Let the non-conducting medium '1' be characterised by permittivity ε_1, permeability μ_1, refractive index n_1, while the conducting medium '2' by permittivity ε_2, permeability μ_2, refractive index n_2 and conductivity σ_2. Let a plane monochromatic electromagnetic wave be incident on the interface from medium 1 in medium 2. The reflection and refraction take place but quite differently from the case of interface separating non-conducting media.

The electric and magnetic field intensities in the two media may be expressed as :

In non-conducting medium 'I'

$$\left.\begin{aligned} \text{For incident wave } \mathbf{E}_1 &= \mathbf{E}_{01}\, e^{i\mathbf{k}_1 \cdot \mathbf{r} - i\omega_1 t} \\ \text{and} \qquad \mathbf{H}_1 &= \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\mu_1 \omega_1} \text{ where } |\mathbf{k}_1| = \frac{\omega_1}{v_1} = \sqrt{(\mu_1 \varepsilon_1)}\, \omega_1 \end{aligned}\right] \quad \text{...(1)}$$

$$\left.\begin{aligned} \text{For reflected wave } \mathbf{E}_1' &= \mathbf{E}_{01}'\, e^{i\mathbf{k}_1 \cdot \mathbf{r} - i\omega_1' t} \\ \text{and} \qquad \mathbf{H}_1' &= \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\mu_1 \omega_1'} \text{ where } |\mathbf{k}_1'| = \frac{\omega_1'}{v_1'} = \sqrt{(\mu_1 \varepsilon_1)}\, \omega_1' \end{aligned}\right] \quad \text{...(2)}$$

In conducting medium '2'

$$\left.\begin{aligned} \text{For refracted wave } \mathbf{E}_2 &= \mathbf{E}_{02}\, e^{i\mathbf{k} \cdot \mathbf{r} - i\omega_2 t} \\ \text{and} \qquad \mathbf{H}_2 &= \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2 \omega_2} \end{aligned}\right] \quad \text{...(3)}$$

where

$$|\mathbf{k}_2|^* = \alpha_2 + i\beta_2 \quad \text{...(4a)}$$

$$\left.\begin{aligned} \text{with } \alpha_2 &= \sqrt{(\mu_2 \varepsilon_2)}\, \omega_2 \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma_2}{\omega_2 \varepsilon_2}\right)^2\right\}} + 1}{2}\right]^{1/2} \\ \beta_2 &= \sqrt{(\mu_2 \varepsilon_2)}\, \omega_2 \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma_2}{\omega \varepsilon_2}\right)^2\right\}} - 1}{2}\right]^{1/2} \end{aligned}\right] \quad \text{...(4b)}$$

For good conductors $\sigma_2/\omega\,\varepsilon_2 \gg 1$, so

$$\alpha_2 \approx \beta_2 = \sqrt{\left(\frac{\mu_1 \omega_2 \sigma_2}{2}\right)}$$

Now using the boundary condition that the tangential component of electric field is continuous at the interface, we obtain

$$E_{1t} + E_{1t}' = E_{2t}$$

---

* In section 8.15, we have seen that in a conducting medium both electric and magnetic field vectors decay exponentially and they are no longer in phase, Moreover the propagation vector **k** is complex given by $k = \mu\varepsilon\omega^2 [1 + (i\sigma/\omega\varepsilon)]$ or equivalently $k = \alpha + i\beta$

$$\text{where } \alpha = \sqrt{(\mu\varepsilon)}\, \omega \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}} + 1}{2}\right]^{1/2}$$

$$\beta = \sqrt{(\mu\varepsilon)}\, \omega \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}} - 1}{2}\right]$$

In good conductors $\alpha \approx \beta = \sqrt{\left(\frac{\mu\omega\sigma}{2}\right)}$

i.e.
$$(E_{01})_t\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1 t} + (E_{01}')_t\, e^{i\mathbf{k}_1'\cdot\mathbf{r} - i\omega_1' t} = (E_{02})_t\, e^{i\mathbf{k}_2\cdot\mathbf{r} - i\omega_2 t} \qquad ...(5)$$

Since this equality is independent of time, we must have
$$\omega_1 = \omega_1' = \omega_2 = \omega, \text{ (say)} \qquad ...(6)$$

That ***is the incident, reflected and refracted waves all have the same frequency.***

Since equation (5) holds for all points of interface, hence at interface $z = 0$, we must have
$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_1'\cdot\mathbf{r})_{z=0} = (\mathbf{k}_2\cdot\mathbf{r})_{z=0} \qquad ...(7)$$

This equation shows that $\mathbf{k}_1$, $\mathbf{k}_1'$ and $\mathbf{k}_2$ all lie in the same plane. Equation (7) also implies (refer section 9.3)
$$\theta_i = \theta_r'$$

i.e. angle of incidence is equal to angle of reflection.

and
$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{k_2}{k_1} = \frac{k_2/\sqrt{\mu_0\varepsilon_0}}{k_2/\sqrt{\mu_0\varepsilon_0}} = \frac{k_2/\sqrt{\mu_0\varepsilon_0}}{\omega\sqrt{\mu_1\varepsilon_1}/\sqrt{\mu_0\varepsilon_0}} = \frac{k_2/\omega\sqrt{\mu_0\varepsilon_0}}{\sqrt{(\mu_1\varepsilon_1/\mu_0\varepsilon_0)}} = \frac{n_2}{n_1} \qquad ...(8)$$

which is *well known Snell's law of refraction.*

Hence n_1 and n_2 are refractive indices in medium 1 and medium 2 respectively, given by
$$n_1 = \sqrt{\left(\frac{\mu_1}{\mu_0}\frac{\varepsilon_1}{\varepsilon_0}\right)} = \sqrt{(K_{1m}K_{1e})}$$
$$n_2 = \frac{k_2}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{c}{\omega}k_2 \text{ where } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$
$$= \text{speed of light in vacuum}$$

Since k_2 is complex, therefore n_2 must be complex and may be expressed as
$$n_2 = (c/\omega)\,k_2 = (c/\omega)\,(\alpha_2 + i\beta) \qquad ...(9a)$$

where α_2 and β_2 are given by equation (4a) with $\omega_2 = \omega$.

Since n_1 is real, it is *customary* to write
$$n_2/n_1 = {}_1n_2 = n\,(1 + ik) \qquad ...(9b)$$

where k is known as *extinction coefficient.*

From equations (9a) and (9b) it is obvious that
$$n = \frac{c\alpha_2}{\omega n_1} \text{ and } nk = \frac{c\beta_2}{\omega n_1} \qquad ...(9c)$$

For extremely good conductors
$$\alpha_2 \approx \beta_2 = \sqrt{\left(\frac{\mu_2\omega\sigma_2}{2}\right)}, \qquad \text{so that } k \to 1. \qquad ...(10)$$

Now from Snell's law
$$\left.\begin{aligned} \frac{\sin\theta_i}{\sin\theta_r} &= \frac{n_2}{n_1} = {}_1n_2 \text{ i.e. } \sin\theta_r = \frac{\sin\theta_i}{{}_1n_2} \\ \text{and} \quad \cos\theta_r &= \sqrt{(1-\sin^2\theta_r)} = \sqrt{\left(1+\frac{\sin^2\theta_i}{({}_1n_2)^2}\right)} \end{aligned}\right] \qquad ...(11)$$

As the refractive index of a conducting medium is very large as compared to that of a non-conducting medium *i.e.* ${}_1n_2 = n_2/n_1$ is very large. This implies that $\cos\theta_r \approx 0$ or $\theta_r = 0$. Thus in a ***good conducting medium, the refracted ray propagates along the normal to the interface irrespective of angle of incidence.***

The refracted or transmitted wave may be expressed as

$$\mathbf{E}_2 = \mathbf{E}_{02}\, e^{i(\alpha_2 + i\beta_2)\hat{\mathbf{n}}\cdot\mathbf{r} - i\omega t} \quad \hat{\mathbf{n}}_2 \text{ being unit vector along } \mathbf{k}_2$$

$$= \mathbf{E}_{02}\, e^{-\beta_2 \hat{\mathbf{n}}_2\cdot\mathbf{r}}\, e^{i\alpha_2 \hat{\mathbf{n}}\cdot\mathbf{r} - i\omega t} \qquad \text{...(12)}$$

This equation indicates that the field vectors in refractd wave exponentially *i.e.* the refracted wave is highly damped wave. The penetration depth or skin depth is given by

$$\delta = \frac{1}{\beta_2} = \frac{1}{\sqrt{(\mu_2\varepsilon_2)}\,\omega\left[\dfrac{\left\{1+\left(\dfrac{\sigma_2}{\omega\varepsilon_2}\right)^2+1\right\}}{2}\right]^{1/2}}$$

$$= \frac{1}{\sqrt{(\mu_2\varepsilon_2)}\,\omega\,\sqrt{(\sigma_2/2\omega\varepsilon)}} = \frac{2}{\mu_2\sigma_2\omega} \qquad \text{...(13)}$$

for good conductors

Obviously the skin depth is independent of the angle of incidence. Now let us find the *Fresnel's equations* in the two usual cases.

Case (i) When E vector is perpendicular to the plane of incidence. Considering fig. 9.4, the continuity of tangential components of electric and magnetic field intensities **E** and **H** at the interface requires that (refer equation (1) and (2) of section 9.4)

$$E_{01} + E_{01}' = E_{02} \qquad \text{...(14)}$$

$$(H_{01} - H_{01}')\cos\theta_i = H_{02}\cos\theta_r \qquad \text{...(15)}$$

We have

$$\mathbf{H}_1 = \frac{\mathbf{k}_1\times\mathbf{E}_1}{\mu_1\omega_1} = \frac{k_1\,\hat{\mathbf{n}}_1\times\mathbf{E}_1}{\mu_1\varepsilon_1} = \sqrt{(\mu_1\varepsilon_1)}\,\frac{\hat{\mathbf{n}}\times\mathbf{E}_1}{\mu_1}$$

$$\left.\begin{array}{ll} \text{This given} & H_{01} = \sqrt{(\varepsilon_1/\mu_1)}\,E_{01} \\ \text{Similarly} & H_{01}' = \sqrt{(\varepsilon_1/\mu_1)}\,E_{01}' \\ \text{and} & H_{02} = \dfrac{k_2}{\mu_0\omega}\,E_{02} \end{array}\right\} \qquad \text{...(16)}$$

Using (16), equation (15) gives

$$\frac{\varepsilon_1}{\mu_1}(E_{01} - E_{01}')\cos\theta_i = \frac{k_1}{\mu_2\omega}E_{02}\cos\theta_r \qquad \text{...(17)}$$

Solving equation (14) and (17), we get

$$\left(\frac{E_{01}'}{E_{01}}\right)_{\perp} = \frac{\cos\theta_i - \dfrac{k_2}{\mu_2\omega}\sqrt{\left(\dfrac{\mu_1}{\varepsilon_1}\right)}\cos\theta_r}{\cos\theta_i + \dfrac{k_2}{\mu_2\omega}\sqrt{\dfrac{\mu_1}{\varepsilon_1}}\cos\theta_r} \qquad \text{...(18)}$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \dfrac{k_2}{\mu_2\omega}\sqrt{\left(\dfrac{\mu_1}{\varepsilon_1}\right)}\cos\theta_r} \qquad \text{...(19)}$$

These are the general *Fresnel's equations.*

Now let us consider that *conducting medium is good conductor* for which $\sigma_2/\omega\,\varepsilon_2 \ll 1$, so that $\cos\theta_r = 1$, then equations (18) and (19) take the form

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \frac{\cos\theta_i - \dfrac{k_2}{\mu_2\omega}\sqrt{\dfrac{\mu_1}{\varepsilon_1}}}{\cos\theta_i + \dfrac{k_2}{\mu_2\omega}\sqrt{\dfrac{\mu_1}{\varepsilon_1}}} \qquad ...(20)$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \dfrac{k_2}{\mu_2\omega}\sqrt{\dfrac{\mu_1}{\varepsilon_1}}} \qquad ...(21)$$

Now noting that

$$k_2 = \frac{\omega n_2}{c} \text{ and } n_1 = c\sqrt{\mu_1\varepsilon_1}, \text{ so that } \frac{k_1}{\mu_2\omega}\sqrt{\frac{\mu_1}{k_1}}\cdot\frac{n_2}{n_1}\cdot\frac{\mu_1}{\mu_2},$$

Equations (20) and (21) taken the form

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \frac{\cos\theta_i - \dfrac{n_2}{n_1}\dfrac{\mu_1}{\mu_2}}{\cos\theta_i + \dfrac{n_2}{n_1}\dfrac{\mu_1}{\mu_2}} \qquad ...(22)$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \dfrac{n_2}{n_1}\dfrac{\mu_1}{m_2}} \qquad ..(23)$$

Further for simplicity assuming that the media are non-magnetic *i.e.* $\mu_1 = \mu_2 = \mu_0$ and using (9b), we get

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{\cos\theta_i - n(1-i\kappa)}{\cos\theta_i + n(1-i\kappa)} = -\frac{(n-\cos\theta_i) + in\kappa}{(n+\cos\theta_i) - in\kappa} \qquad ...(24)$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + n(1-i\kappa)} = \frac{2\cos\theta_i}{(n+\cos\theta_i) - in\kappa} \qquad ...(25)$$

Now as $[(n-\cos\theta_i) + in\kappa] = ae^{i\delta_1}$

$$(n+\cos\theta_i) - in\kappa = be^{i\delta_2}$$

with

$$a = \sqrt{[(n-\cos\theta_i)^2 + n^2\kappa^2]},\ \delta_1 = \tan^{-1}\left\{\frac{n\kappa}{n-\cos\theta_i}\right\} \qquad ...(26)$$

$$b = \sqrt{[(n+\cos\theta_i)^2 + n^2\kappa^2]},\ \delta_2 = \tan^{-1}\left\{\frac{n\kappa}{n+\cos\theta_i}\right\} \qquad ...(27)$$

so equations (24) and (25) reduce to

$$\left(\frac{E'_{01}}{E'_{01}}\right)_{\perp} = \frac{ae^{i\delta_1}}{be^{-i\delta_2}} = \frac{a}{b}e^{i(\delta_1+\delta_2)}$$

$$= \sqrt{\left\{\frac{(n-\cos\theta_i)^2 + n^2\kappa^2}{(n+\cos\theta_i)^2 + n^2\kappa^2}\right\}}\, e^{i(\delta_1+\delta_2)} \qquad ...(28)$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{be^{-\delta_2}} = \frac{2\cos\theta_i}{\sqrt{[(n+\cos\theta_i)^2 + n^2\kappa^2]}}\, e^{i\delta_2} \qquad ...(29)$$

Equations (28) and (29) are final equations representing the reflected and transmitted amplitudes relative to incident amplitude. From these equations it is clear that both reflected and transmitted waves exist and they are not in phase with the incident wave. Further the largevalue of n implies that the amplitude of transmitted wave is very small while that of reflected wave is nearly equal to that of incident one.

The reflection coefficient $R_\perp$ is obtained by comparing the normal component of reflected poynting vector with that of incident Poynting vector *i.e.*

$$R_\perp = \frac{S'_1 \cdot \mathbf{n}}{S_1 \cdot \mathbf{n}} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_2}{\mu_1}}\,|E'_{01}|^2 \cos\theta'_r}{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}}\,|E_{01}|^2 \cos\theta_i} = \frac{|E'_{01}|_\perp^2}{|E_0|_\perp^2} \quad (\text{since } \theta'_r = \theta_i)$$

$$= \left|\frac{E'_{01}}{E_{01}}\right|_\perp^2 = \left\{\frac{E'_{01}}{E_{01}}\right\}_\perp \left\{\frac{E'^{*}_{01}}{E_{01}{}^{*}}\right\}_\perp$$

$$= \left[\frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2}\right]^{1/2} e^{i(\delta_1+\delta_2)}\left[\frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2}\right]e^{-(\delta_1+\delta_2)}$$

$$= \frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2} \qquad \text{...(30)}$$

Case (ii) When E is parallel to plane of incidence.

Considering fig. 9.5, the continuity of tangential components of electric and magnetic field intensities **E** and **H** at the interface requires [refer equations (10) and (11) of section (8.4)].

$$H_{01} + H'_{01} = H_{02} \qquad \text{...(31)}$$

$$(E_{01} - E'_{01})\cos\theta_i = E_{02}\cos\theta_r \qquad \text{...(32)}$$

Using (16), equation (31) gives

$$\sqrt{\frac{\varepsilon_1}{\mu_2}}\,(E_{01} + E'_{01}) = \frac{k_2}{\mu_2\,\omega}\,E_{02} \qquad \text{...(33)}$$

Solving (32) and (33), we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_\parallel = \frac{\sqrt{\frac{\mu_1}{\varepsilon_1}}\,\frac{k_2}{\mu_2\omega}\cos\theta_i - \cos\theta_r}{\sqrt{\frac{\mu_1}{\varepsilon_1}}\,\frac{k_2}{\mu_2\omega}\cos\theta_i + \cos\theta_r} \qquad \text{...(34)}$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_\parallel = \frac{2\cos\theta_i}{\sqrt{\frac{\mu_1}{\varepsilon_1}}\,\frac{k_2}{\mu_2\omega}\cos\theta_i + \cos\theta_r} \qquad \text{...(35)}$$

If the conducting medium is a good conductor, then $\cos\theta_r = 1$, and writing

$$\sqrt{\frac{\mu_1}{\varepsilon_1}}\cdot\frac{k_2}{\mu_2\omega} = \frac{\mu_1}{\mu_2}\cdot\frac{n_2}{n_1}$$

as before, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_i - 1}{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_i + 1} \text{ and } \left\{\frac{E_{02}}{E_{01}}\right\}_{\parallel} = \frac{2\cos\theta_i}{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_i + 1}$$

Again considering $\mu_1 = \mu_2 = \mu_0$ and writting $n_2/n_1 = {}_1n_2 = n\,(1 + i\kappa)$ above equation reduce to

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{n\,(1 + i\kappa)\cos\theta_i - 1}{n\,(1 - i\kappa)\cos\theta_i + 1} = \frac{\left\{n - \frac{1}{\cos\theta_i}\right\} + in\kappa}{\left\{n + \frac{1}{\cos\theta_i}\right\} + in\kappa} \qquad \text{...(36)}$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\cos\theta_i}{n\,(1 + i\kappa)\cos\theta_i + 1} = \frac{2}{\left\{n + \frac{1}{\cos\theta_i}\right\} + in\kappa} \qquad \text{...(37)}$$

Again writting

$$\left(n - \frac{1}{\cos\theta_i}\right) + in\kappa = a'e^{i\delta'_1}$$

and

$$\left(n + \frac{1}{\cos\theta_i}\right) + in\kappa = b'e^{i\delta'_2}$$

with

$$a' = \left[\left\{n - \frac{1}{\cos\theta_i}\right\} + n^2\kappa^2\right]^{1/2}; \delta'_1 = \tan^{-1}\frac{n\kappa}{\left(n - \frac{1}{\cos\theta_i}\right)} \qquad \text{...(38)}$$

$$b' = \left[\left(n + \frac{1}{\cos\theta_i}\right)^1 + n^2\kappa^2\right]^{1/2}; \delta'_2 = \tan^{-1}\frac{n\kappa}{\left[n + \frac{1}{\cos\theta_i}\right]} \qquad \text{...(39)}$$

Equations (36) and (37) take the form

$$\left(\frac{E'_{01}}{E_{01}}\right) = \frac{a'e^{i\delta'_1}}{b'e^{i\delta'_2}}\left[\frac{\left(n - \frac{1}{\cos\theta_i}\right)^2 + n^2\kappa^2}{\left(n + \frac{1}{\cos\theta_2}\right)^2 + n^2\kappa^2}\right]^{1/2} e^{i(\delta'_1 - \delta'_2)} \qquad \text{...(40)}$$

and

$$\left[\frac{E_{02}}{E_{01}}\right]_{\parallel} = \frac{2}{b'e^{i\delta'_2}} = \frac{2}{\left[\left(n + \frac{1}{\cos\theta_i}\right)^2 + n^2\kappa^2\right]^{1/2}} e^{-i\delta'_2}. \qquad \text{...(41)}$$

These equations indicate again that both reflected and transmitted waves exist and they are not in phase with the incident wave. Moreover the large value of n implies that amplitude of transmitted wave is extremely small,

$$R_{\parallel} = \left|\frac{E'_{02}}{E_{01}}\right|^2_{\parallel} = \frac{\left[n - \frac{1}{\cos\theta_i}\right]^2 + n^2\kappa^2}{\left[n + \frac{1}{\cos\theta_i}\right]^2 + n^2\kappa^2} \qquad \text{...(42)}$$

At normal incidence $\theta_i = 0$; $\cos\theta_i = 1$; therefore equations (30) and (42) take identical form

$$\left.\begin{aligned} R = R_{\perp} = R_{\parallel} &= \frac{(n-1)^2 + n^2\kappa^2}{(n+1)^2 + n^2\kappa^2} && \text{...(a)} \\ &= 1 - \frac{4n}{(n+1)^2 + n^2\kappa^2} && \text{...(b)} \end{aligned}\right\} \quad \text{...(43)}$$

For extremely good conductors $\kappa \to 1$ and

$$n \to \frac{c\alpha_2}{\omega n_1} \to \frac{c}{\omega n_1}\sqrt{\frac{\mu_2\omega\sigma_2}{2}} \to \frac{c}{\omega c\sqrt{\mu_1\varepsilon_1}}\sqrt{\frac{\mu\omega\sigma_2}{2}} \to \sqrt{\frac{\sigma_2}{2\omega\varepsilon_1}} \qquad (\text{since } \mu_1 = \mu_2)$$

Also

$$n = \sqrt{\frac{\sigma_2}{2\omega\varepsilon_1}} \gg 1.$$

Making these substitutions, equation (43b) gives

$$R = 1 - \frac{2}{n} = 1 - 2\sqrt{\frac{2\omega\varepsilon_1}{\sigma_2}} \qquad \text{...(44)}$$

This equation shows that at low frequencies for substance of high conductivity, the reflection coefficient R is close to unity and therefore all the energy will be reflected. This explains why metals are opaque to light. The little energy that flows into the metal is rapidly dissipated by the heat loss associated with the induced currents. For example taking $\sigma = 5{\cdot}6 \times 10^7$ mho/m, the value for copper and assuming $\mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, we find that for frequency 10^{10} sec^{-1} (3 cm wavelength), $R = 0{\cdot}9997$; for lower frequencies R is even closer to unity *i.e.* perfect reflection takes place. It is only for radiation of very short wavelength that the deviation of R from unity becomes significant for extremly good conductors like copper, silver and aluminium. The skin depth for these materials is very small *i.e.* these materials absorb waves strongly. Thus all waves which are strongly absorbed are also strongly reflected by metals *i.e. all good conductors are good absorbers and good reflectors.*

Experimental verification of the law metallic reflection. At optical *frequencies* and normal incidence at metals, the reflection coefficient, $R = 1 - \frac{2}{n} = 1 - 2\sqrt{\frac{2\omega\varepsilon_1}{\sigma_2}}$. The value of this reflection coefficient is usually determined by measuring the *emissivity* e of the surface, then by Kirchoff's law

$$e = 1 - R = \frac{2}{n} = 2\sqrt{\frac{2\omega\varepsilon_1}{\sigma_2}} \qquad \text{...(45)}$$

This implies $e\sqrt{\sigma_2} = 2\sqrt{2\omega\varepsilon_1} =$ *constant at a given frequency i.e.* for all *metals* $e\sqrt{\sigma_2}$ should be constant at a given frequency of incident radiation. Extensive measurement by Hagen and Rubens confirmed this prediction for a wide range of metals within frequency range $10^{23} \sim 10^{14}$ Hertz. The average value of $e\sqrt{\sigma}$ obtained from observation on 12 metals at $\lambda = 2{\cdot}55 \times 10^{-5}$ m. was $6{\cdot}96 \times 10^6$, where as the value calculated from conductivity and frequency was $6{\cdot}68 \times 10^6$. The close agreement between experimental and theoretical result verifies the law of metallic reflection. Hagen and Rubens also showed that the relation (44) is not valid below $\lambda = 0{\cdot}5 \times 10^{-5}$ m. Performing experiment below this specific wavelength, they showed that the reflecting power of most metals increases rapidly with wavelength and approaches the value given by (44).

9.10. Group Velocity.

If a region consists of two or more wave trains, then the *physical velocity of propagation of waves is called the* **group velocity.** The velocity of individual wave is known as phase velocity and is given by $v_p = \omega/k$ where $k = |\mathbf{k}| =$ magnitude of propagation vector and ω is angular frequency. The phase velocity is not the velocity of propagation of any physical quantity; this simply represents the velocity of a point of constant phase on the wave. Signals are transmitted with the group velocity and not with the phase velocity. The group velocity $\mathbf{v}_G$ is given by the derivative

$$\mathbf{v}_G = \frac{\partial \omega}{\partial \mathbf{k}} \quad \text{...(1)}$$

the ratio ω/k is called the phase velocity.

Let us derive relation (1). Let the two wave trains propagating along z-axis have the same amplitude but slightly different frequencies ω_1 and ω_2 and phase velocities v_1 and v_2, so that they can be expressed by the equations

$$E_1 = E_0 \cos \omega_1 \left(t - \frac{z}{v_1}\right) \quad \text{...(2)}$$

$$E_2 = E_0 \cos \omega_2 \left(t - \frac{z}{v_2}\right) \quad \text{...(3)}$$

The resultant amplitude of wave may be obtained by the principle of superposition as

$$\begin{aligned} E = E_1 + E_2 &= E_0 \cos \omega_1 \left(t - \frac{z}{v_1}\right) + E_0 \cos \omega_2 \left(t - \frac{z}{v_2}\right) \\ &= 2\,E_0 \cos \left\{\frac{\omega_1}{2}\left(t - \frac{z}{v_1}\right) + \frac{\omega_2}{2}\left(t - \frac{z}{v_2}\right)\right\} \\ &\quad \cos \left\{\frac{\omega_1}{2}\left(t - \frac{z}{v_1}\right) - \frac{\omega_2}{2}\left(t - \frac{z}{v_2}\right)\right\} \end{aligned} \quad \text{...(4)}$$

Let us write $\omega_1 = \omega$; $\omega_2 = \omega + d\omega$

Then $\omega_1 - \omega_2 = -\,d\omega$ and $\omega_1 + \omega_2 = 2\omega$ as $d\omega \ll \omega$

Similarly $\dfrac{\omega_1}{v_1} + \dfrac{\omega_2}{v_2} = 2\dfrac{\omega}{v}$ where $v = \dfrac{v + v_2}{2}$

Then equation (4) becomes

$$E = 2E_0 \cos \omega \left(t - \frac{z}{v}\right) \cos \left\{-\frac{d\omega}{v_2}t + \frac{1}{2}\left(\frac{\omega_2}{v_2} - \frac{\omega_1}{v_1}\right)z\right\}$$

Now

$$\frac{\omega_2}{v_2} - \frac{\omega_1}{v_1} = d\left(\frac{\omega}{v}\right)$$

so that

$$E = 2E_0 \cos \frac{d\omega}{2}\left[t - \frac{d\,(\omega/v)}{d\omega}z\right] \cos \omega \left(1 - \frac{z}{v}\right) \quad \text{...(5)}$$

The equation represents a wave, travelling with phase velocity v whose amplitude is

$$2E_0 \cos \frac{d\omega}{2}\left(t - \frac{z}{v_G}\right) \quad \text{...(6)}$$

where

$$v_G = \frac{d\omega}{d\,(\omega/v)} = \frac{d\omega}{dk} \quad \text{...(7)}$$

where $k = \omega/v$.

Equation (6) represents that amplitude itself is being propagated, along Z axis with velocity v_G and since amplitude velocity is signal velocity, this is *group velocity* and is given by equation (7) or (1) in vector form.

9.11. Propagation of Electromagnetic Waves Between Parallel Conducting Planes.

We know that in a good conductor the field penetrates only to a distance comparable with the skin depth, which reduces to zero in the limits of perfect conductivity. Thus a perfect conductor behaves as a perfect reflector and hence allows no field to penetrate within it. Consequently the electric and magnetic fields are zero within the conducting plane and our boundary conditions reduce to the following two statements.

(i) *The tangential component of* **E** *at the surface of the conducting plane is zero*, since it must be zero within the plane and we know that the tangential component is continuous.

(ii) *The normal componet of* **B** *at the surface of the conducting plane is zero*, since it must be zero with the plane and the normal component of **B** is continuous.

The other two conditions on the normal component of **D** and the tangential component of **H** lead to information about the surface charge and surface current respectively. Since the surface is conducting and can carry charge and current ; therefore the normal component of **D** outside the conducting plane measures the surface charge density and the tangential component of **H** measures the surface current density, which appear on the surface. Because *of no penetration of electromagnetic waves, we need to consider only incident waves striking the conducting plane and the corresponding reflected waves.*

Now let us consider the propagation of electromagnetic waves in the region between two parallel perfectly conducting planes separated by a distance L. Let the planes be in the $X-Z$ plane situated at $y=0$ and $y=L$ as shown in fig. 9.10. If an electromagnetic wave is incident on any plane, it will be reflected from the wall whenever it strikes it and hence by a process of multiple reflections the wave will propagate in a direction parallel to planes. Since in fig 9.10 the x and z directions are physically indistinguishable no generality is lost if we consider only waves with wave vectors in y–z plane in particular, those making an angle θ with y-axis. Such waves will strike on perfectly conducting surface at $y=L$ and will be reflected as waves whose propagation vector k_1' makes an angle θ with the minus y-axis. When these waves are reflected again by the surface at $y=0$, they become waves of initial type again and have propagation vector **k**. Thus the propagation between two conducting planes can be described in terms of exponential factors

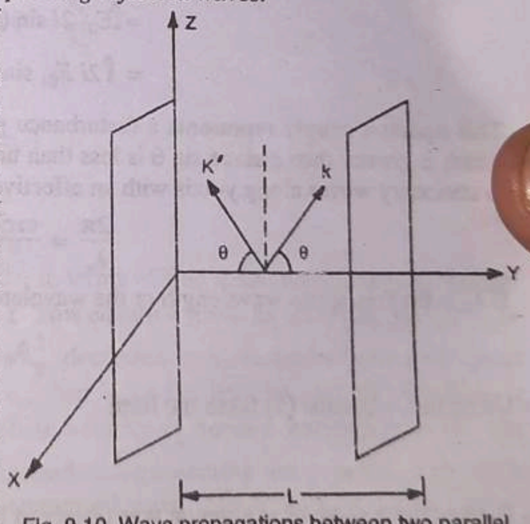

Fig. 9.10. Wave propagations between two parallel conducting planes.

$$\left.\begin{aligned}&\text{for initial wave } e^{i\mathbf{k}\cdot\mathbf{r}} \rightarrow e^{ik(y\cos\theta + z\sin\theta) - i\omega t}\\ &\text{For } \textit{reflected wave (once)}\ e^{i\mathbf{k}'\cdot\mathbf{r}} \rightarrow e^{ik(-y\cos\theta + z\sin\theta) - i\omega t}\end{aligned}\right\} \quad \text{...(1)}$$

where $|\mathbf{k}| = |\mathbf{k}'| = \omega/c$, since both waves are in same medium.

Since in electromagnetic waves **k**, **H** and**E** are mutually perpendicular, therefore for such waves there are only two possible polarisation which may be described by saying that for one **E** is parallel to X-axis or at right angles to z for both the direct and reflected waves (*i.e.* $E_z=0$) and for the other **H** is parallel to X-axis or at right angles to z for both the direct and reflected waves (*i.e.* $H_z=0$). These are known as the *transverse electric* (abbreviated *TE*). and *transverse- magnetic* (abbreviated *TM*) modes of propagation. Here we shall consider only *TE* waves.

The electric fields for incident and reflected waves in *TE* case may be expressed as

$$\left.\begin{aligned} \mathbf{E}_1 &= \mathbf{i}E_{01}\, e^{ik(y\cos\theta + z\sin\theta) - i\omega t} && \text{(incident wave)} \\ \mathbf{E}_1' &= \mathbf{i}E_{01}'\, e^{ik(-y\cos\theta + z\sin\theta) - i\omega t} && \text{(reflected wave)} \end{aligned}\right\} \quad ...(1)$$

where $\mathbf{i}$ is unit vector along X-axis.

By the principle of superposition the electric field in the region between two conducting planes in *TE* case will be given by

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_1' \\ &= \mathbf{i}\,[E_{01}\, e^{ik(y\cos\theta + z\sin\theta) - i\omega t}] + [E_{01}'\, e^{ik(-y\cos\theta + z\sin\theta) - i\omega t}] \end{aligned} \quad ...(2)$$

According to boundary condition that the tangential component of electric field must vanish at the surface of conducting plane, we must have

$$0 = E_{01}\, e^{ikz\sin\theta\, i\omega t} + E_{01}'\, e^{ikz\sin\theta - i\omega t}$$

This equation is satisfied for all z and all t if

$$E_{01} = E_{01}' \quad ...(3)$$

This condition simply indicates that the reflection at the conducting plane involves a phase change of π and no change in amplitude.

Then E will be given by

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{i}} E_{01}\,(e^{iky\cos\theta} - e^{-iky\cos\theta})\, e^{i(kz\sin\theta - i\omega t)} \\ &= \hat{\mathbf{i}} E_{01}\, 2i\sin(k\cos\theta \bullet y) e^{i(kz\sin\theta - \omega t)} \\ &= \hat{\mathbf{i}}\, 2i\, E_{01}\sin\left(\frac{\omega\cos\theta}{c} y\right) e^{i[\{(\omega\sin\theta)/c\}z - \omega t]} \end{aligned}$$

This equation simply represents a disturbance propagating along z-axis with an apparent veloity* $c/\sin\theta$ which is greater than c since $\sin\theta$ is less than unity and varying sinusoidally along y-axis, or corresponding to stationary waves along y-axis with an effective wavelength λ_c (along y-axis) given by

$$\frac{2\pi}{\lambda_c} = \frac{\omega\cos\theta}{c} \text{ or } \lambda_c = \frac{2\pi c}{\omega\cos\theta} \quad ...(5)$$

If λ_0 is the free space wavelength or the wavelength in the absence of reflection, then

$$\lambda_0 = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \quad ...(6)$$

Using this, equation (5) takes the form

$$\lambda_c = \frac{\lambda_0}{\cos\theta} \quad ...(7)$$

The standing wave phenomenon was observed by Lippmann not in free space but in a sensitive emulsion of a photographic plate. The emulsion was blackened along surfaces half a wave-length apart in the standing wave pattern the so called *Lippmann fringes*, The phenomenon can also be demonstrated with the longer waves of the microwave region.

So far we have considered the boundary condition at $y = 0$: the same boundary condition at $x = L$ requires that (refer eqn. 4)

$$\sin\left(\frac{\omega\cos\theta}{c} L\right) = 0 \quad \text{for all } z \text{ and all } t.$$

This gives

---

*The phase velocity $(c/\sin\theta) > c$, the speed of light. This apparently contradicts relativity *i.e.* no signal can be propagated with a velocity greater than the velocity of light. The resolution of this apparent difficulty is that energy is propagated with a smaller velocity than c, namely, the so called group velocity. Signals are transmitted with group velocity and not with the phase velocity.

$$\frac{\omega \cos \theta L}{c} = n\pi$$

or
$$L = n\frac{\pi c}{\omega \cos \theta} = \frac{n\lambda_c}{2} \qquad \text{using (5)} \qquad \text{...(8a)}$$

i.e.
$$\lambda_c = \frac{2L}{n} \qquad \text{...(8b)}$$

This equation indicates that the dependence of disturbance on y can not have any arbitrary wavelength ; but only an infinite set of discrete wavelengths. For a given n or a given mode, the effective wavelength λ_c is determined by equation (8b). This is independent of the free space wavelength λ_c or the corresponding frequency ω.

Now it is convenient to introduce phase velocity v_c and the *effective wavelength* λ_c along z-axis.

The apparent phase velocity v_g along the z-direction is given by

$$v_g = \frac{c}{\sin\theta} \text{ (refer equation (4))} \qquad \text{...(9)}$$

$$\lambda_g = \frac{v_g}{\nu} = \frac{v_g}{c/\lambda_0} = \frac{v_g\,\lambda_0}{c} = \frac{\left(\frac{c}{\sin\theta}\right)\lambda_0}{c} = \frac{\lambda_0}{\sin\theta} \qquad \text{...(10)}$$

For many purposes it is convenient to eliminate θ. From equations (7) and (10) we have

$$\cos\theta = \frac{\lambda_0}{\lambda_c} \text{ and } \sin\theta = \frac{\lambda_0}{\lambda_g}.$$

Squaring and adding, we get

$$\left(\frac{\lambda_0}{\lambda_c}\right)^2 + \left(\frac{\lambda_0}{\lambda_g}\right)^2 = 1$$

or
$$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} \qquad \text{...(11)}$$

This equation determines the wavelength λ_g along the z-axis, in terms of free space wavelength λ_0 and the wavelength λ_c along y-direction or the direction normal to z. This equation gives an intersting fact. If λ_c is fixed and λ_0 is increased or the frequency is decreased, $1/\lambda_q^2$ decreases or λ_g increases with consequent increase of phase velocity v_g, until finally when λ_0 equals λ_c, then λ_g, and v_g both become infinite. If λ_0 is further increased beyond λ_c, then the wavelength λ_g and phase velocity v_g become imaginary, so that the exponential in (5) containing z viz. $e^{i\omega \sin\theta z/c} = {}^{\prime}e^{i\omega z/v_g}$ instead of representing wave propagated along z-axis, represents a decreasing exponential and hence the attenuated wave. *The free space wavelength at which the disturbance changes over from being propagated to being attenuated is called the* **cut off wavelength** and the corresponding frequency is called the **cut off frequency.** The reason for the subscript c is now clear : it means '*cut off*. The cut off wavelength is the longest wavelength of a wave in free space that can be propagated for a given mode 'n'.

The problem of propagation of electromagnetic waves between two parallel planes is very simple and yet instructive case of *wave-guide propagation. A wave guide is simply a hollow pipe of infinite length* and has recently come into much prominence in connection with the applications of microwaves or electromagnetic waves whose lengths are of the order of magnitude of laboratory dimensions. We have seen that the waves travelling in the direction along the pipe can be propagated and the free space wavelength of these propagating waves is less than or equal to λ_0 ; otherwise the waves will be attenuated.

Group Velocity. We have seen that the phase velocity $v_0 = c/\sin\theta$ exceeds velocity of light. This apparently contracts the theory of relativity where we have postulated that no signal can be propagated with a velocity greater than the velocity of light. The resolution of the apparent difficulty is that the energy is propagated along the guide with a smaller velocity than the velocity of light, namely with the so called *group velocity. Signals are transmitted with the group velocity and not with the phase velocity. The phase velocity is not the velocity of propagation of any physical quantity : this simply represents the velocity of a point of constant phase on the wave.*

To determine the group velocity *i.e.* the velocity of energy propagation we shall calculate the time average energy density and time averaged-Poynting vector. Then dividing the Poynting vector (power flow) by the energy density, the velocity of energy propagation (group velocity) may be obtained.

In terms of λ_g and λ_c, the electric field (4) may be written as

$$\mathbf{E} = \hat{\mathbf{i}} E' \sin\left(\frac{2\pi\lambda}{\lambda_c}\right) e^{i[(2\pi z/\lambda_g) - \omega t]} \qquad ...(12)$$

(where $E' = 2iE_{01}$)

The magnetic field induction in the guide is readily obtained from

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ and assuming that } \mathbf{B}(\mathbf{r}, t) = B(\mathbf{r})\, e^{-i\omega t}$$

we quickly find

$$B(\mathbf{r}, t) = \hat{\mathbf{j}} E' \frac{2\pi}{\omega\lambda_g} \sin\frac{2\pi y}{\lambda_c} e^{i}\{(2\pi z/\lambda_{g)} - \omega t\} + \hat{\mathbf{k}} iE' \frac{2\pi}{\omega\lambda_g} \cos\frac{2\pi y}{\lambda_c} e^{i}\{(2\pi z/\lambda_g) - \omega t\} \qquad ...(13)$$

The time average Poynting vector is

$$\begin{aligned} \langle S_z \rangle &= \tfrac{1}{2} R_e (\mathbf{E}^* \times \mathbf{H}) = \tfrac{1}{2} R_e (E_x{}^* H_y) \\ &= \tfrac{1}{2} R_e \left[E'^* \sin\left(\frac{2\pi y}{\lambda_c}\right) \frac{1}{\mu_0} E' \left(\frac{2\pi}{\omega\lambda_g}\right) \sin\left(\frac{2\pi y}{\lambda_c}\right) \right] \\ &= 2\, E'^* E' \left(\frac{2\pi}{\mu_0 \omega \lambda_g}\right) \sin^2\left(\frac{2\pi y}{\lambda_c}\right) \end{aligned}$$

Integrating this expression from $y = 0$ to $y = L$ across the guide yields the total average power travelling along the guide

$$\langle S_x \rangle = \int_0^L \langle S_z \rangle\, dy = \tfrac{1}{4} E'^* E' \left(\frac{2\pi}{\mu_0 \omega \lambda_g}\right) L \qquad ...(14)$$

$$\left(\text{since } \int_0^L \sin^2 \frac{2\pi y}{\lambda_c}\, dy = \frac{L}{2}\right)$$

The time averaged energy density is

$$\langle u \rangle = \tfrac{1}{4} \text{Re}[\mathbf{E}^* \cdot \mathbf{D} + \mathbf{B}^* \cdot \mathbf{H}]$$

$$= \tfrac{1}{4} \text{Re} \left[\varepsilon_0 E'^* E' \sin^2\left(\frac{2\pi y}{\lambda_c}\right) + \frac{1}{\mu_0} E'^* E' \left(\frac{2\pi}{\omega\lambda_g}\right)^2 \sin^2\left(\frac{2\pi y}{\lambda_c}\right) + \frac{1}{\mu_0} E'^* E' \left(\frac{2\pi}{\omega\lambda_g}\right)^2 \cos^2\left(\frac{2\pi y}{\lambda_c}\right) \right] \qquad ...(15)$$

Integrating in y-direction from $y = 0$ to $y = L$ across the guide, effectively replaces $\sin^2\left(\frac{2\pi y}{\lambda_c}\right)$ and $\cos^2\left(\frac{2\pi y}{\lambda_c}\right)$ by $\frac{L}{2}$. Thus

$$\begin{aligned}\langle S\rangle &= \int_0^L \langle u\rangle\, dy = \tfrac{1}{4}E'^{*}E'\frac{L}{2}\left[\varepsilon_0+\frac{4\pi}{\mu_0\omega^2}\left(\frac{1}{\lambda_g^2}+\frac{1}{\lambda_c^2}\right)\right]\\ &= \tfrac{1}{4}E'^{*}E'\frac{L}{2}\left[\varepsilon_0+\frac{4\pi^2}{\mu_0\omega^2\lambda_0^2}\right]\\ &= \tfrac{1}{4}E'^{*}E'\frac{L}{2}\left[\varepsilon_0+\frac{1}{\mu_0c^2}\right] = \tfrac{1}{4}E'^{*}E'\frac{L}{2}[\varepsilon_0+\varepsilon_0]\\ &= \tfrac{1}{4}E'^{*}E'\varepsilon_0 L \qquad \text{...(16)}\end{aligned}$$

The velocity of *energy propagation* or *group velocity*

$$\begin{aligned}v_G = \frac{\langle S\rangle}{\langle U\rangle} &= \frac{\tfrac{1}{4}E'^{*}E'\left(\dfrac{2\pi}{\mu_0\,\omega\lambda_g}\right)L}{\tfrac{1}{4}E'^{*}E\varepsilon_0 L} = \frac{2\pi}{\mu_0\varepsilon_0\omega\lambda_g}\\ &= \frac{2\pi c^2}{\omega\lambda_g} = c\frac{\lambda_0}{\lambda_g} = c\sin\theta \qquad \text{...(17)}\end{aligned}$$

Obviously group velocity v_G is smaller than the velocity of light.

The group velocity v_G may be obtained by usual relation $v_G = \frac{\partial\omega}{\partial \mathbf{k}}$, the ratio $\frac{\omega}{\mathbf{k}}$ is called the phase velocity which in this case becomes

$$v_c = \frac{\partial\omega}{\partial k_g}. \qquad \text{...(18)}$$

Equation (11) may be expressed as

$$k^2 = k_c^2 + k_g^2 \qquad (\text{as } k = 2\pi/\lambda).$$

In free space $k = \omega/c$, therefore

$$\omega^2/c^2 = k_c^2 + k_g^2$$

or

$$\{\omega = c\sqrt{(k_c^2+k_g^2)}\}. \qquad \text{....(19)}$$

Differentiating this equation with respect to k_g, we get

$$\begin{aligned}v_G = \frac{\partial\omega}{\partial k_g} &= c\,.\,\tfrac{1}{2}(k_c^2+k_g^2)^{-1/2}\times 2K_g = \frac{ck_g}{\omega/c} \text{ using(19)}\\ &= c^2.\left(\frac{k_g}{\omega}\right) = \frac{c^2}{c/\sin\theta} = c\sin\theta. \qquad \text{...(20)}\end{aligned}$$

e have $v_g = c/\sin\theta$ and $v_G = c\sin\theta$.

Thus it is readily apparent.

$$v_g v_G = c^2. \qquad \text{...(21)}$$

9.12. Wave Guides

A wave guide is a hollow pipe of infinite extent. We shall now consider the propagation of electromagnetic waves along a hollow conducting pipe (wave guide) of arbitrary (simply connected) cross section uniform along its length. We shall assume that the walls of the wave guide are perfectly conducting and take the z-axis along the wave guide. For simplicity we further assume that the interior of wave guide is vacuum. As shown in chapter 7, each component of **E** and **B** must satisfy wave equation which in vacuum has the form

$$\left.\begin{aligned}\nabla^2\mathbf{E}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}&=0 \text{ or } \nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \quad ...(a)\\ \nabla^2\mathbf{B}-\mu_0\varepsilon_0\frac{\partial^1\mathbf{B}}{\partial t^2}&=0 \text{ or } \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0 \quad ...(b)\end{aligned}\right\} \quad ...(1)$$

In addition to these wave equations Maxwell's equations and boundary conditions must be satisfied. As the wave is propagated along Z-axis, we assume solutions of the form

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t)&=\mathbf{E}(x,y)\,e^{ik_gz-i\omega t} \quad ...(a)\\ \mathbf{B}(\mathbf{r},t)&=\mathbf{B}(x,y)\,e^{ik_gz-i\omega t} \quad ...(b)\end{aligned}\right\} \quad ...(2)$$

Appropriate linear combinations can be formed to give travelling or standing waves in the Z-direction. The wave number k_g is known from preceding section but however we assume that it is unknown parameter which may be real or complex. With this assumed Z dependence of fields, the wave equation reduces to two dimensional form

$$\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}-k_g^2\,\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}=0$$

$$\left.\begin{aligned}\text{or}\qquad \frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]\mathbf{E}&=0 \quad ...(a)\\ \text{and}\qquad \frac{\partial^2\mathbf{B}}{\partial x^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]\mathbf{B}&=0 \quad ...(b)\end{aligned}\right\} \quad ...(3)$$

Now if $\nabla_\perp^2$ is transverse part of Laplacing operator ∇^2, then

$$\nabla_\perp^2=\nabla^2-\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad ...(4)$$

In these notations equation (3) takes the form

$$\left[\nabla_\perp^2+\left(\frac{\omega^2}{c^2}-k_g^2\right)\right]\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix}=0 \quad ...(5)$$

Now Maxwell's curl equations for free space are

$$\left.\begin{aligned}\nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t} \quad (a)\\ \text{and}\qquad \nabla\times\mathbf{B}&=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \text{ or } \nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad ...(b)\end{aligned}\right\} \quad ...(6)$$

which in terms of components can be written as

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=-\frac{\partial B_x}{\partial t}\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=-\frac{\partial B_y}{\partial t}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=-\frac{\partial B_z}{\partial t}\end{aligned}\right] \quad ...(7a)$$

and

$$\left.\begin{aligned} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} &= \frac{1}{c^2}\frac{\partial E_x}{\partial t} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= \frac{1}{c^2}\frac{\partial E_y}{\partial t} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \frac{1}{c^2}\frac{\partial E_z}{\partial t} \end{aligned}\right] \qquad \text{...(7b)}$$

From the form of exponential $e^{ik_g z - i\omega t}$ it is apparent that

$$\frac{\partial}{\partial z} \to ik_g \text{ and } \frac{\partial}{\partial t} \to -i\omega$$

Therefore equations (7) lead to

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - ik_g E_y &= i\omega B_x && \text{....}(a) \\ ik_g E_x - \frac{\partial E_z}{\partial y} &= i\omega B_z && \text{....}(b) \\ \frac{\partial \mathbf{E_y}}{\partial x} - \frac{\partial \mathbf{E_x}}{\partial y} &= i\omega B_z && \text{...}(c) \end{aligned}\right] \qquad \text{...(8)}$$

and

$$\left.\begin{aligned} \frac{\partial B_z}{\partial y} - ik_g B_y &= -\frac{i\omega}{c^2}\mathbf{E_x} && \text{....}(a) \\ ik_g B_x - \frac{\partial B_z}{\partial x} &= -\frac{i\omega}{c^2}\mathbf{E_y} && \text{....}(b) \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -\frac{i\omega}{c^2}\mathbf{E_z} && \text{...}(c) \end{aligned}\right] \qquad \text{...(9)}$$

Substituting values of B_y from (8b) in (9a), we get

$$\frac{\partial B_z}{\partial y} - ik_g\left(\frac{ik_g\mathbf{E_x} - \dfrac{\partial \mathbf{E_z}}{\partial x}}{i\omega}\right) = -\frac{i\omega}{c^2}\mathbf{E_x}$$

i.e.

$$\frac{\partial B_z}{\partial y} + \frac{k_g}{\omega}\frac{\partial \mathbf{E_z}}{\partial x} = \left(\frac{ik_g^2}{\omega} - \frac{i\omega}{c^2}\right)\mathbf{E_x}$$

i.e.

$$\mathbf{E_x} = \frac{i}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)}\left[k_g\frac{\partial \mathbf{E_z}}{\partial x} - \omega\frac{\partial B_y}{\partial y}\right] \qquad \text{..(10)}$$

Now eliminating B_x from (8a) and (9b), we get

$$\mathbf{E_y} = \frac{i}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]}\left[k_g\frac{\partial \mathbf{E_z}}{\partial q} - \omega\frac{\partial B_y}{\partial x}\right] \qquad \text{...(11)}$$

Similarly eliminating $\mathbf{E}_y$ and $\mathbf{E}_x$ in turn from (8) and (9), we get

$$B_x = \frac{i}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\left(k_g \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial \mathbf{E}_z}{\partial y}\right) \qquad ....(12)$$

$$B_y = \frac{i}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\left(k_g \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial \mathbf{E}_z}{\partial x}\right) \qquad ...(13)$$

Equation (10), (11), (12) and (13) show that it is sufficient to determine $\mathbf{E}_z$ and B_z as the appropriate solutions of the two dimensional wave equation (5). The other components can then be calculated from (10), (11), (12) and (13). In general cylindrical guide with perfectly conducting walls, such as that shown in fig. 9.11, is under consideration, then the appropriate boundary conditions are that the tangential component of **E** and the normal component of **B** should vanish on the surface of the conductor. The boundary condition for two dimensional wave equation for $\mathbf{E}_z$ viz.

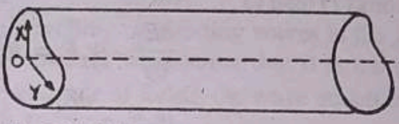

Fig. 9.11 Hollow cylindrical wave guide of arbitrary cross sectional shape

$$\nabla_\perp^2 \mathbf{E}_z + \left(\frac{\omega^2}{c^2} - k_g^2\right)\mathbf{E}_z = 0 \qquad ...(14)$$

that the tangential component of **E** should vanish on the surface of the conductor requires

$$\mathbf{B}_z \,|_S = 0 \qquad ...(15)$$

The normal component of **B** at the surface requires

$$|\mathbf{B}\cdot\mathbf{n}|_S = 0$$

where **n** is unit vector normal the surface

i.e.

$$\left| B, (x, y)\, e^{ik_g z - i\omega t} \cdot \mathbf{n} \right|_S = 0$$

this condition implies

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \qquad ...(16)$$

where $\partial/\partial x$ is the normal derivative at the point on the surface. The two dimensional wave equations (5) for $\mathbf{E}_z$ and B_z together with the boundary conditions on $\mathbf{E}_z$ and B_z at the surface of the cylinder allow only certain values of axial wave number k_g for a given frequency ω. Because of different boundary conditions on $\mathbf{E}_z$ and B_z they can not be generally satisfied simultaneously. Consequently the fields may be divided into two distinct classes.

Transverse Magnetic (TM) Mode. In this case $B_z = 0$ always *i.e.* magnetic fields is always perpendicular to the direction of propagation; hence the name transverse magnetic. These type of wave are also sometimes named as *electric type waves* or **E** waves.

Substituting $B_z = 0$, equations (10) to (13) take to form

$$\mathbf{E}_x = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\frac{\partial \mathbf{E}_z}{\partial x} = \frac{ik_g}{(k_o^2 - k_g^2)}\frac{\partial \mathbf{E}_z}{\partial x} \qquad ...(17)$$

$$\mathbf{E_y} = \frac{ik_g}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{E_z}}{\partial y} = \frac{ik_g}{(k_o^2 - k_g^2)} \frac{\partial \mathbf{E_z}}{\partial y} \qquad ...(18)$$

$$\mathbf{B_x} = -\frac{i\omega}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)c^2} \frac{\partial \mathbf{E_z}}{\partial y} = -\frac{i\omega}{(k_o^2 - k_g^2)\,c^2} \frac{\partial \mathbf{E_z}}{\partial y} \qquad ...(19)$$

$$\mathbf{B_y} = \frac{i\omega}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)c^2} \frac{\partial \mathbf{E_z}}{\partial y} = \frac{i\omega}{(k_o^2 - k_g^2)\,c^2} \frac{\partial \mathbf{E_z}}{\partial x} \qquad ...(20)$$

where $k_0 = \dfrac{\omega}{c}$ = free space wave number

Thus in *TM* mode, all the transverse components of **E** and **B** can be expressed in terms of longitudinal component of the electric field. This component may be obtained by solving the two dimensional wave equation (5) for $\mathbf{E_z}$ *viz.*

$$\nabla_\perp^2 \mathbf{E_z} + \left(\frac{\omega^2}{c^2} - k_g^2\right)\mathbf{E_z} = 0 \qquad ...(21)$$

or

$$\nabla_\perp^2 \mathbf{E_z} + (k_0^2 - k_g^2)\,\mathbf{E_z} = 0$$

The boundary condition for this equation is $\mathbf{E_z}|_S = 0$ ...(i)

(ii) Transverse electric (TE) Mode. In this case $\mathbf{E_z} = 0$ always *i.e.* electric field is always perpendicular to the direction of propagation of the wave, hence the name *transverse electric mode.* These types of waves are also called magnetic type waves or *H*-waves.

Substituting $\mathbf{E_z} = 0$, equations (10) to (13) take the form

$$\mathbf{E_x} = \frac{i\omega}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{B_z}}{\partial y} = \frac{i\omega}{k_0^2 - k_g^2} \frac{\partial \mathbf{B_z}}{\partial y} \qquad ...(22)$$

$$\mathbf{E_y} = -\frac{i\omega}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{B_z}}{\partial x} = -\frac{i\omega}{k_0^2 - k_g^2} \frac{\partial \mathbf{B_z}}{\partial y} \qquad ...(23)$$

$$B_z = \frac{ik_g}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial B_z}{\partial x} = -\frac{ik_g}{k_0^2 - k_g^2} \frac{\partial B_z}{\partial x} \qquad ...(24)$$

$$B_y = \frac{ik_g}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial B_z}{\partial y} = -\frac{ik_g}{k_0^2 - k_g^2} \frac{\partial B_z}{\partial y} \qquad ...(25)$$

Thus in *TE* mode all transverse components of **E** and **B** can be expressed in terms of longitudinal component of magnetic field. This component may be obtained by solving two dimensional wave equation (5) for B_z *viz.*

$$\nabla_\perp^2 B_z + \left(\frac{\omega^2}{c^2} - k_g^2\right) B_z = 0$$

or

$$\nabla_\perp^2 B_z + (k_0^2 - k_g^2)\, B_z = 0 \qquad \text{...(26)}$$

with the boundary condition

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \qquad \text{...(i)}$$

According to equations (22)–(25), this condition ensures that normal component of **B** is zero.

Thus the problem of determining the electromagnetic field in a wave-guide reduces to that of finding solutions of two dimensional wave equation

$$[\nabla_\perp^2 + (k_o^2 - k_g^2)]\, \psi = 0 \text{ or } [\nabla_\perp^2 + k_c^2]\, \psi = 0 \qquad \text{...(27)}$$

subject to boundary conditions $\psi|_S = 0$ and $\partial\psi/\partial n\,|_S = 0$.

Here $$k_c^2 = k_0^2 - k_g^2 \text{ or } k_0^2 = k_c^2 + k_g^2 \qquad \text{...(28a)}$$

Equation (28 a) is equivalent to $$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} \qquad \text{...(28 b)}$$

which is same as equation (11) of preceding section 9.11 Thus the conclusions drawn in the preceding section are quite general and are not limited to the case of propagation between parallel conducting planes.

For a given cross section the solution ψ exists only for a certain definite eigen values of the parameter k_c^2. It is easy to see that the constant k_c^2 must be non-negative, since roughly speaking ψ must be oscillatory in order to satisfy required boundary condition on opposite sides of the cylinder. These will from a spectrum of eigen values $(k_c^2)_\lambda$ and corresponding solutions ψ_λ ($\lambda = 1, 2, 3$) form an orthogonal set. These different solutions are called the *modes of the guide*. For a given frequency ω the wave number k_g is determined for each value of λ

$$(k_g^2)_\lambda = k_0^2 - (k_c^2)_\lambda = \omega^2/c^2 - (k_c^2)_\lambda \qquad \text{...(29)}$$

Since $k_g^2 = k_0^2 - k_c^2$, therefore we note that for $k_0 > k_c$ or $\omega > \omega_c$, the wave number k_g is real ; hence waves of such modes can propagate in the guide But if $k_c < k_c$ or $\omega < \omega_c$, k_g is imaginary which is turn implies the attenuation of **E** and **B** given by (2) : hence such modes can not propagate and are called *cut off modes*. The frequency given by

$$(\omega_c)\lambda = [c]\,(k_c)$$

is called cut off frequency. Then the *wave number* can be written ...(30)

as

$$(k_g)\lambda = \frac{1}{c}\sqrt{[\omega^2 - (\omega_c^2)\,\lambda]} \qquad \text{...(31)}$$

From this it follows that a guide acts a sort of high pass filter in the sense that only frequencies greater that cut off frequency can be propagated in the guide. Moreover at any given frequency only a finite mumber of modes can propagate. It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.

The velocity of propagation *of the wave (i.e. group velocity) along the wave guide* is given by the derivative

$$v_z = \frac{\partial\omega}{\partial k_g} = \frac{\partial}{\partial k_g}\, c\,(k_g^2 + k_c^2)^{1/2}$$

$$= \frac{ck_g}{\sqrt{(k_0^2 + k_c^2)}} = \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \qquad \text{...(32)}$$

For a given k_c : this varies from o to c when k_g varies from o to ∞.

The **phase velocity** v_g in the guide is given by

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(\omega^2 - \omega_c^2)/c}} = \frac{c}{\sqrt{\left[1 - \left(\frac{\omega_c^2}{\omega^2}\right)\right]}}$$

Obviously $v_g > c$ and becomes infinite exactly at cut off frequency. The energy in the guide is propagated with the group velocity.

TEM **Waves.** In addition to *TE* and *TM* modes, there is a degenerate mode, called the transverse electromagnetic (*TEM*) mode, in which both E_z and B_z vanish. If we substitute $E_z = B_z = 0$ in equations (10)–(13), we note that there is no non-zero component of **E** or **B**. This implies that ***TEM* wave can not be propagated along the wave-guide.**

9.13. Rectangular Wave Guide

The most commonly used wave guide is that of rectangular cross-section having inner dimension a and b as shown in fig. 9.21.

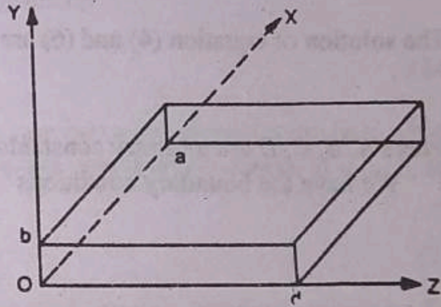

Fig. 9.12

The solution of two dimensional wave equation

$$(\nabla_\perp^2 \neq k_c^2)\,\psi = 0 \qquad \text{...(1)}$$

can be carried out in rectangular coordinates as follows :

TE **Mode.** For *TE* mode $E_z = 0$: hence equation (1) is to be written for B_z ; which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z = 0 \qquad \text{...(2)}$$

The boundary conditions are

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0$$

i.e.

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

We shall solve equation (2) by the method of separation of variables. Therefore writing

$$B_z(x, y) = X(x)\,Y(y) = XY \qquad \text{...(3)}$$

where X is a function of x only and Y is a function of y only.

Susbtituting equation (3) in (2) and dividing by XY, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

or
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

In above equation L.H.S. is a function of x only, while R.H.S. is a function of y only. Hence this equation will be satisfied if both sides are equal to a constant say p^2 *i.e.*

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = +p^2$$

or
$$\frac{\partial^2 X}{\partial x^2} + (k_c^2 - p^2)X = 0$$

or
$$\frac{\partial^2 X}{\partial x^2} + q^2 X = 0 \qquad \text{...(4)}$$

where
$$q^2 = k_c^2 - p^2 \qquad \text{...(5)}$$

and
$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = p^2$$

or
$$\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0 \qquad \text{...(6)}$$

The solution of equation (4) and (6) are

$$X(x) = A\cos qx + B\sin qx \qquad \text{...(7)}$$
$$Y(y) = C\cos py + D\sin py \qquad \text{...(8)}$$

where A, B, C, D are arbitrary constants.

We have the boundary conditions

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$
$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

These conditions are equivalent to

$$\frac{\partial X}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and
$$\frac{\partial Y}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b.$$

Differentiating equations (7) and (8), we get

$$\frac{\partial X}{\partial x} = -Aq\sin qx + Bq\cos qx$$
$$\frac{\partial Y}{\partial y} = -Cp\sin qy + Dp\cos py$$

Applying boundary condition $\left.\frac{\partial Z}{\partial x}\right|_{x=a} = 0$, we get

$Bq = 0$ $\therefore$ This gives $B = 0$

Now applying boundary condition $\left.\frac{\partial X}{\partial x}\right|_{x=a} = 0$, we get

$$-Aq\sin qa = 0$$

We must take $A \neq 0$ since otherwise $X = 0$ and $B_z = 0$. Hence

$$\sin qa = 0 \text{ or } qa = m\pi, \text{ that is,}$$

$$q = \frac{m\pi}{a} \quad (m \text{ integer}) \qquad \text{...(11)}$$

In precisely the same manner we conclude that $D = 0$ and p must be restricted to values $p = n\pi/b$ where n is an integer. In this way be obtain the solutions

$$X(x) = A\cos\left(\frac{m\pi}{a}\right)x\,;\quad Y(y) = C\cos\left(\frac{n\pi}{b}\right)y \qquad \text{...(12a)}$$

where

$$m = 1, 2, 3, \ldots\ldots \quad n = 1, 2, 3, \ldots.$$

and

$$k_c^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The solution for $B_z(x, y)$ is consequently

$$B_z(x, y) = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad \text{...(13a)}$$

when

$$(k_c^2)_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \qquad \text{...(13b)}$$

Here the indices mn specify the mode. The *cut off frequency* ω_{mn} is given by

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad \text{...(14)}$$

The modes corresponding to m and n are represented as TE_{mn}. The case $m = n = 0$ gives a static field which does not represent a wave propagation ; hence the mode TE_{00} represents non-trivial solution. If $a > b$, the lowest *cut off frequency* results for $m = 1$ and $n = 0$,

i.e.

$$\omega_{10} = \frac{\pi c}{a} \quad \text{or} \quad (k_c)_{10} = \frac{\pi}{a} \qquad \text{...(15)}$$

The mode (TE_{10}) represents the *dominant TE mode* and is the one used in most practical situations. The values E_x, E_y, B_x and B_y for TE mode may be obtained from equations (22)–(25) of preceding section by substituting the solution for B_z, which is

$$\begin{aligned} B_z(r, t) &= B_z(x, y)\, e^{ik_g z - i\omega t} \\ &= B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, e^{ik_g z - i\omega t} \end{aligned} \qquad \text{...(16)}$$

Thus we have

$$\left.\begin{aligned} E_x &= \frac{in\pi\omega}{k_c^2 b} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ E_y &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_x &= -\frac{im\pi k_g}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_y &= -\frac{in\pi k_g}{k_c^2 a} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{ik_g z - i\omega t} \end{aligned}\right\} \quad \text{...(17)}$$

For *TE* mode, these equations yield

$$\left(\text{put } m = 1,\ n = 0, k_c^2 = \frac{\pi^2}{a^2}\right)$$

$$\left.\begin{aligned} E_x = B_y = E_z = 0\,;\ B_z &= B_0 \cos\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ \text{and} \qquad B_x &= -\frac{ik_g a}{a} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ E_y &= \frac{i\omega a}{\pi c} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t}\,. \end{aligned}\right\} \quad \text{...(18)}$$

The presence of a factor i in B_x (and E_y) means that there is a spatial (or temporal) phase difference of $\pi/2$ between B_x (and E_y) and B_z in the propagation.

***TM* Mode :** For *TM* mode $B_z = 0$; hence equation (1) is to be written for E_z which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \quad \text{...(19)}$$

The boundary conditions are $E_z = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$.

Solving equation (19) as for *TE* case, we note that the solution of equation (19) is of the form

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad \text{...(20)}$$

where k_c^2 and hence ω_{mn} are still given by equations (12b) and (14). This implies that *TE and TM modes of a rectangular guide have the same set of cut off frequencies.* However in this case $m = 1$ and $n = 0$ represents non-trival solution since this gives $E_z = 0$ and hence all components of **E** and **B** will be zero.

It is obvious that in this case the lowest mode has $m = n = 1$ and may be represented by TM_{11}. The **cut off frequency of lowest mode** is given by

$$\omega_{11} = \pi c\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = \frac{\pi c}{a}\left[1 + \frac{a^2}{b^2}\right]^{1/2} \quad \text{...(21)}$$

Since $a < b$, therefore the *cut off frequency of lowest TM mode is greater than that of the lowest TE mode by the factor* $\left[1 + \left(\frac{a^2}{b^2}\right)\right]^{1/2}$ The fields E_x, E_y, B_x, B_y for *TM* mode may be obtained from equations (17)–(22) of preceding section if we substitute

$$E_z(\mathbf{r}, t) = E_0(x, y)\, e^{ik_g z - i\omega t}$$

$$E_z(\mathbf{r}, t) = E_0 \sin\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{-ik_g z - i\omega t} \qquad \text{...(22)}$$

or

Thus we have

$$\left.\begin{aligned} E_x &= \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b.}\right] e^{ik_g z - i\omega t} \\ E_y &= \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_x &= -\frac{i\omega n\pi}{k_c^2 bc} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_y &= \frac{i\omega n\pi}{k_c^2 ac^2} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \end{aligned}\right] \qquad \text{...(22)}$$

Ex. 2. *What must be the width of a rectangular guide such that the energy of electromagnetic radiation, whose free space wavelength is 3·0 cm, travels down the guide at 95% of the speed of light ?* **(Meerut 1970)**

Solution. We know that the energy of electromagnetic radiation is propagated down the guide with *group velocity* given by

$$v_z = \frac{ek_g}{k_0} = \frac{c\sqrt{(k_0^2 - k_c^2)}}{k_0} = c\left[1 - \frac{k_c^2}{k_0^2}\right]^{1/2} \qquad \text{...(1)}$$

For rectangular wave guide

$$(k_c^2)_{mn} = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{a^2}\right]$$

For *TE mode* and $a > b$; the maximum cut off frequency is given by $m = 1$ and $m = 0$,

$$(k_c^2)_{10} = \pi^2\left[\frac{1\cdot}{a^2} + \frac{0}{a^2}\right] = \frac{\pi^2}{a^2} \qquad \text{...(2)}$$

If λ_0 is *free space wavelength,* then

$$k_c = 2\pi/\lambda_0 \qquad \text{...(3)}$$

Using (2) and (3), equation (1) gives

$$v_z = c\left[1 - \frac{(\pi^2/a^2)}{(2\pi/\lambda c)^2}\right]^{1/2} = c\left[1 - \frac{\lambda_0^2}{4\pi^2}\right]^{1/2}$$

Squaring, we get

$$v_z^2 = c^2\left[1 - \frac{\lambda_0^2}{4a^2}\right]$$

or

$$\frac{v_z^2}{c^2} = 1 - \frac{\lambda_0^2}{4a^2}$$

This gives

$$a = \frac{\lambda_0}{2\left[1 - \frac{v_z^2}{c^2}\right]^{1/2}}$$

Here $v_z = \frac{95}{100}\, c = 0{\cdot}95\, c$ and $\lambda_0 = 3\text{ cm} = 0{\cdot}03\text{ m}$

$$\therefore \qquad a = \frac{0{\cdot}03}{2\left[1-\left(\frac{0{\cdot}95}{c}\right)^2\right]^{1/2}} = -\frac{0{\cdot}03}{2\,[1-(0{\cdot}95)^2]^{1/2}}$$

$$= \frac{0{\cdot}03}{2\times 0{\cdot}12} = 0{\cdot}048\text{ m} = \mathbf{4{\cdot}8\text{ cm.}}$$

9.14. Circular Wave Guide

The next most important example of wave guides after the rectangular guide is that in which the bounding surfaces are circular cylinders. This includes both the circular guide and the coaxial line, which has two concentric cylinder, with the wave propagated in the angular space between them. In either case the two dimensional wave equation *viz.*

$$(\nabla_{\perp}^2 + k_c^2)\,\psi = 0 \qquad \text{...(1)}$$

is to be solved in cylinderical coordinates, but here we shall restrict ourselves for cases where cross-section is circular. Let a be the radius of circular cross-section

We have

$$\nabla_{\perp}^2 = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$$

Therefore two dimensional wave equation (1) in this case becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\psi + k_c^2\,\psi = 0 \qquad \text{...(2)}$$

If a is the radius of circular cross-section, then the boundary conditions are

$$\psi\,\Big|_{r=a} = 0 \text{ and } \frac{\partial\psi}{\partial n}\Big|_{r=a} = 0 \qquad \text{for all values of } \theta \qquad \text{...(3)}$$

Now let us write Maxwell's equations in cylinderical coordinates (r, θ, z). Maxwell's curl equations in free space are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{...(4)}$$

In cylindrical coordinates equations take the form

$$\left(\frac{1}{r}\frac{\partial E_r}{\partial\theta} - \frac{\partial E_\theta}{\partial z}\right)\hat{n}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,E_\theta) - \frac{\partial E_r}{\partial\theta}\right]\hat{n}$$

$$= -\left[\frac{\partial B_r}{\partial t}\hat{n}_r + \frac{\partial B_\theta}{\partial t}\hat{n}_\theta + \frac{\partial B_z}{\partial t}\hat{n}_z\right]$$

and

$$\left[\frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial r}\right]\hat{n}_r + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,B_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{n}_z$$

$$= \frac{1}{c^2}\left[\frac{\partial E_r}{\partial t}\hat{n}_r + \frac{\partial E_\theta}{\partial t}\hat{n}_\theta + \frac{\partial E_z}{\partial t}\hat{n}_z\right]$$

Comparing coefficients of $\hat{n}_r$, $\hat{n}_\theta$ and $\hat{n}_z$ on either sides, we get

guide is that in which the
and the coaxial line, which
veen them. In either case the

...(1)

r cases where cross-section

...(2)

...(3)

ll's curl equations in free

...(4)

$\left.\frac{E_r}{\theta}\right] \hat{n}$

$\left.\right] \hat{n}_z$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial E_z}{\partial r} - \frac{\partial E_\theta}{\partial r} &= -\frac{\partial B_r}{\partial t} \\ \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} &= -\frac{\partial B_\theta}{\partial t} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial \theta}\right] &= -\frac{\partial B_z}{\partial t} \end{aligned}\right] \text{and}$$

and

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} &= \frac{1}{c^2}\frac{\partial E_r}{\partial t} \\ \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} &= \frac{1}{c^2}\frac{\partial E_\theta}{\partial t} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial \theta}\right] &= \frac{1}{c^2}\frac{\partial E_z}{\partial t} \end{aligned}\right] \quad \text{...(5)}$$

But

$$\mathbf{E}(r, \theta, z, t) = \mathbf{E}(r, \theta)\, e^{ik_g z - i\omega t}$$

$$\mathbf{B}(r, \theta, z, t) = \mathbf{B}(r, \theta)\, e^{ik_g z - i\omega t}$$

From the form of exponential $e^{ik_g z - i\omega t}$, it is apparent that

$$\partial/\partial z \to ik_g \quad \text{and} \quad \partial/\partial t \to -i\omega t$$

Substituting these values in (5), we get

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial E_z}{\partial \theta} - ik_g E_\theta &= i\omega B_r & \text{...(i)} \\ ik_g E_r - \frac{\partial E_r}{\partial r} &= i\omega B_\theta & \text{...(ii)} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial \theta}\right] &= i\omega B_z & \text{...(iii)} \\ \frac{1}{r}\frac{\partial B_z}{\partial \theta} - ik_g B_\theta &= \frac{i\omega}{c^2} E_r & \text{...(vi)} \\ ik_g B_r - \frac{\partial E_z}{\partial r} &= -\frac{i\omega}{c^2} E_\theta & \text{...(v)} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial \theta}\right] &= -\frac{i\omega}{c^2} E_z & \text{...(vi)} \end{aligned}\right] \quad \text{...(6)}$$

Eliminating B_θ from (ii) and (iv), we get

$$E_r = \frac{l\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right]}\left[\frac{k_g}{\omega}\frac{\partial E_z}{\partial r} + \frac{1}{r}\frac{\partial B_z}{\partial \theta}\right] \quad \text{...(7)}$$

Eliminating B_r from (i) and (v), we get

$$E_\theta = -\frac{i\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right]}\left[\frac{\partial B_z}{\partial r} - \frac{1}{r}\frac{k_g}{\omega}\frac{\partial E_z}{\partial \theta}\right] \quad \text{...(8)}$$

Eliminating E_θ from (i) and (v), we get

$$B_r = +\frac{i}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]}\left[k_g\frac{\partial B_z}{\partial r} - \frac{\omega}{c^2 r}\frac{\partial E_z}{\partial \theta}\right] \qquad \text{...(9)}$$

and finally eliminating E_r from (ii) and (iv), we get

$$B_\theta = \frac{i}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]}\left[\frac{\omega}{c^2}\frac{\partial E_z}{\partial r} + \frac{k_g}{r}\frac{\partial B_z}{\partial \theta}\right] \qquad \text{...(10)}$$

It is obvious that all the transverse components of field viz. E_r , E_θ , B_r , B_θ may be obtained from B_z and E_z . Thus we have to solve two dimensional wave equation (2) for E_z and B_z . Let us now consider two usual cases :

Case (i). TM Mode. For *TM* mode $B_z = 0$; hence equations (7), (8), (9) and (10) take the form

$$\left.\begin{aligned}
E_r &= \frac{ik_g}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]}\frac{\partial E_z}{\partial r} && \text{...(a)}\\
E_\theta &= \frac{ik_g}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]}\frac{\partial E_z}{\partial \theta} && \text{...(b)}\\
B_r &= -\frac{i\omega}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]c^2 r}\frac{\partial E_z}{\partial \theta} && \text{...(c)}\\
B_\theta &= \frac{i\omega}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]c^2}\cdot\frac{\partial E_z}{\partial r} && \text{...(d)}
\end{aligned}\right] \qquad \text{...(11)}$$

Obviously in *TM* mode all transverse components are expressible in terms of E_z ; hence we have to solve two dimensional wave equation for E_z ; which is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]E_z + k_g^2 E_z = 0 \qquad \text{...(12a)}$$

with boundary condition $E_z = 0$ at $r = a$ for all values of θ . ....(12b)

Let us solve above equation by the method separation of variables. Therefore writing

$$E_z(r, \theta) = R(r)\,\Theta(\theta) \qquad \text{...(13)}$$

where $R(r)$ is a function of r only, while $\Theta(\theta)$ is a function of θ only. Substituting $E_z = \Theta R$ in (12a) and dividing throughout by $R\,\Theta$, we get

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} + k_c^2 = 0 \qquad \text{...(14)}$$

Muliplying throughout by r^2, we get

$$\frac{r}{R}\frac{\partial}{\partial r}\left(\frac{\partial R}{\partial r}\right) + k_c^2 r^2 = -\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} \qquad \text{..(15)}$$

In above equation L.H.S. is a function of r only ; while R.H.S. is a function of θ only. This is only possible if each side is equal to the same constant say n^2 *i..e*

$$-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = n^2 \qquad \text{...(16a)}$$

and

$$\frac{r}{R}\frac{r}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k_c^2 r^2 = n^2 \qquad \text{...(17)}$$

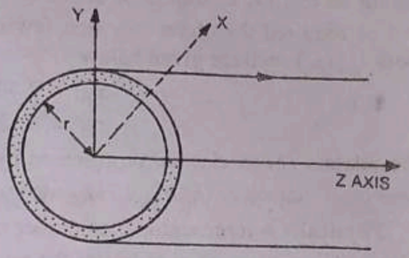

Fig. 9.13. Circular wave guide.

Equation (16a) may be expressed as

$$\frac{\partial^2\Theta}{\partial\theta^2} + n^2\Theta = 0$$

the solution of equations (16b) is

$$\Theta = A_n \cos(n\theta) + B_n \sin(n\theta) \qquad \text{...(18)}$$

Further the relative amplitudes of A_n and B_n determine the orientation of the field in the guide and for a circular guide and for any particular values of n, the θ-axes can always be oriented to make either A_n or B_n equal to zero. Let us choose B_n equal to zero, so that

$$\Theta = A_n \cos n\theta$$

Now equation (17a) may be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \left(k_c^2 - \frac{n^2}{r^2}\right)R = 0 \qquad \text{...(1}$$

or

$$\frac{1}{r}\frac{\partial}{\partial(k_c r)}\left[r\frac{\partial R}{\partial(k_c r)}\right] + \left[1 - \frac{n^2}{(k_c r)^2}\right] = 0 \qquad \text{...(20)}$$

Equation (20) is a Bessel's equation in terms of $(k_c r)$ which has two independent solutions namely Bessel's function $J_n(k_c r)$ and Neumann's function $N_n(k_c r)$.

The Bessel's and Neumann's function $[J_n(x)$ and $N_n(x)]$ have the following properties :

(i) At $x = 0$, the Bessel's function $J(x)$ is proportional to x^n while Neumann's function $N_n(x)$ becomes infinite.

(ii) For large values of x, $J_n(x)$ and $N_n(x)$ approach the values

$$\left.\begin{aligned} J_n(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{(2n+1)}{4}\pi\right) \\ N_n(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{2n+1}{4}\pi\right) \end{aligned}\right] \qquad \text{...(21)}$$

From this it is clear that in a circular guide, where the field must be finite at $r = 0$, we must use only Bessel's function $J_n(k_c r)$. Thus the accepted solution of equation (10) is

$$R = J_n(k_c c) \qquad \text{...(22)}$$

Hence

$$E_z(k_c r) = R(r)\,.\,\Theta(\theta)$$

$$= J_n(k_c r)\,A_n \cos n\theta = A_n J_n(k_{cr}) \cos n\theta \qquad \text{...(23a)}$$

$\therefore$

$$E_z(r, \theta, z, t) = A_n J_n(k_c r) \cos n\theta\, e^{ik_g z - i\omega t} \qquad \text{...(23b)}$$

The boundary condition for *TM* mode is

$$E_z\Big|_{r=a} = 0. \text{ This implies } J_n(k_c a) = 0 \qquad \text{...(24)}$$

Equation (24) has infinite number of roots but since for the propagation to be possible $k_g = \sqrt{(k_0^2 - k_c^2)}$ should be real *i.e.* k_c should be small $k_c^2 \ll k_0^2$ or $k_c^2 \ll \omega^2/c^2$ or otherwise extremely high frequencies will be required therefore only first few roots of equation (24) will be of practical interest. The first few roots $J_n(x_{nm}) = 0$ are given below

$$\left.\begin{aligned} x_{01} &= 2{\cdot}405\,;\; x_{11} = 3{\cdot}832\,;\; x_{21} = 5{\cdot}135 \\ x_{01} &= 5{\cdot}520\,;\; x_{12} = 7{\cdot}016\,;\; x_{22} = 8{\cdot}417 \end{aligned}\right\} \qquad \text{...(25)}$$

The various *TM* modes are represented by TM_{01}, TM_{02}, etc. *i.e.* in general TM_{nm}. It is obvious that since there is no root x_{00} or $(k_c\,a)_{00}$. *TM_{00} wave can not exist.*

Physically n represented the number of cycles of variation of E_z found as θ varies around the complete cylinder. The subscript m indicates the number of zeros of electric field along a radial path from the centre to the inside surface of the guide wall. Thus if we let x_{nm} be the mth value of x for which $J_n(x) = 0$; then the cut off value of k_c is given by

$$k_c a = x_{nm} \quad \text{or} \quad k_c = \frac{x_{nm}}{a} \qquad \text{...(26)}$$

Hence *cut off wavelength* is given by

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi a}{x_{nm}} \qquad \text{...(27)}$$

the phase velocity is

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(k_0^2 - k_c^2)}} \qquad \text{...(28a)}$$

$$= \frac{\omega}{k_0 \sqrt{\left[1 - \left(\dfrac{k_c^2}{k_0^2}\right)\right]}}$$

$$= \frac{c}{\sqrt{\left[1 - \dfrac{k_c^2}{k_0^2}\right]}}$$

$$= \frac{c}{\sqrt{\left[1 - \dfrac{\omega_c^2}{\omega_0^2}\right]}} \qquad \text{...(28b)}$$

It is obvious that phase velocity is greater than the speed of light, and *group velocity* or *velocity of propagation of wave is*

$$v_z = \frac{\partial \omega}{\partial k_g} = \frac{\partial}{\partial k_g}(ck_o) = \frac{\partial}{\partial k_g}\left[c\,(k_g^2 + k_c^2)^{1/2}\right]$$

$$= \frac{ck_g}{\sqrt{(k_g^2 + k_c^2)}}$$

$$= \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \qquad \text{...(29)}$$

For a given k_c, this values varies from o to c when k_c varies from o to ∞.

Substituting solution for E_z from (23b) and using

$$k_c^2 = k_0^2 - k_g^2 = (\omega^2/c^2) - k_g^2$$

the *transverse field components* may be expressed from equation (11) *viz.*

$$\left.\begin{aligned} E_r &= \frac{ik_g}{k_c^2} A_n J_n'(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} && \ldots\text{(a)} \\ E_\theta &= -\frac{ik_g n}{k_c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} && \ldots\text{(b)} \\ B_r &= +\frac{i\omega n}{k_c^2 c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} && \ldots\text{(c)} \\ B_\theta &= \frac{i\omega}{k_c^2 c^2} A_n J_n'(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} && \ldots\text{(d)} \end{aligned}\right] \quad \ldots(30)$$

where J_n' represents derivative of J_n with respect to r.

Case (ii). *TE* Mode. For *TE* mode $E_2 = 0$; hence equations (7), (8), (9) and (10) take the form

$$\left.\begin{aligned} E_r &= \frac{i\omega}{[(\omega^2/c^2) - k_g^2]_r} \frac{\partial B_z}{\partial \theta} && \ldots\text{(a)} \\ E_\theta &= \frac{i\omega}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial r} && \ldots\text{(b)} \\ B_r &= \frac{ik_g}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial r} && \ldots\text{(c)} \\ B_\theta &= \frac{ik_g}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial \theta} && \ldots\text{(d)} \end{aligned}\right] \quad \ldots(31)$$

Obviously in *TE* mode all transverse components E_r, E_θ, B_r, B_θ are expressible in terms of B_z, hence we have to obtain the solution for two dimensional wave equation (1) for B_z. The solution is obtained exactly as in *TM* mode we get

$$B_z(r, \theta) = A_n J_n(k_c r) \cos n\theta \quad \ldots(32)$$

The boundary condition for *TE* mode is $\partial B_z / \partial r = 0$ at $r = a$ for all values of θ.

This implies
$$\left.\frac{\partial J_n(k_c r)}{\partial r}\right|_{r=a} = 0 \text{ or } J'_n(k_c a) = 0 \quad \ldots(33)$$

The derivative $J_n'(k_c r)$ with respect to r of the Bessel function may be obtained from

$$J'_n(k_c r) = \frac{\partial J(k_c r)}{\partial r} = \frac{n}{k_c r} J_n(k_c r) - J_{n+1}(k_c r) \quad \ldots(34)$$

The first few roots of $J'_n(x_{nm}) = 0$ are

$$\left.\begin{aligned} x_{01} &= 3{\cdot}832, \ x_{11} = 1{\cdot}842, \ x_{21} = 3{\cdot}05 \\ x_{02} &= 7{\cdot}016, \ x_{12} = 5{\cdot}330, \ x_{22} = 6{\cdot}71 \end{aligned}\right\} \quad \ldots(35)$$

If we let x_{nm} by the m^{th} value of x for which $J_n'(x) = 0$, then cut off value of k_c is given by

$$k_c a = x_{nm} \text{ or } k_c = \frac{x_{nm}}{a}. \quad \ldots(36a)$$

Hence *cut off wavelength*

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi a}{x_{nm}} \qquad \text{...(36b)}$$

where x_{nm} are given by equation (35). Thus we can calculate the cut off wavelength of various modes. The equations for phase velocity and group velocity are the same as those obtained for *TM* mode except for the use of equation (35) for x_{nm} in place of equation (25).

The field components for *TE* mode may by expressed as

$$\left.\begin{aligned} E_z &= 0 \text{ and } B_z = A_n J_n (k_c r) \cos n\theta \, e^{ik_g z - i\omega t} \\ E_r &= -\frac{i\omega n}{k_c^2 r} A_n J_n (k_c r) \sin n\theta \, e^{ik_g z - i\omega t} \\ E_\theta &= -\frac{i\omega}{k_c^2} A_n J'_n (k_c r) \cos n\theta \, e^{ik_g z - i\omega t} \\ B_r &= \frac{ik_g}{k_c^2} A_n J'_n (k_c r) \cos n\theta \, e^{k_g z - i\omega t} \\ B_\theta &= -\frac{ik_g n}{k_c^2 r} A_n J_n (k_c r) \sin n\theta \, e^{ik_g z - i\omega t} \end{aligned}\right\} \qquad \text{...(37)}$$

9.15. Resonant Cavities

If the cylindrical wave guide of finite length has its end faces then it forms what is known as *resonant cavity* (or cavity resonator). Such cylindrical cavity resonator may be rectangular or circular cavity resonator depending upon whether the cross-section of the cavity resonator is rectangular of circular. We shall restrict ourselves to case where cross-section is rectangular. We shall assume that :

(i) the end faces are plane and perpendicular to the axis of cavity.

(ii) the walls are perfectly conducting.

(iii) the interior of cavity is vacuum.

An electromagnetic wave propagating along the axis of a cavity will be reflected back and forth from the end faces, thus giving rise to *standing wave pattern.* If r is the amplitude of reflection coefficient, then the amplitude of transverse electric field is r times of the amplitude of incident wave ; consequently the amplitude of resulting disturbance, obtained by superposition of incident and reflected waves, will vary from something proportional to $(1 + r)$; where the two waves add, to something proportional to $(1 - r)$; where they oppose each other. The ratio of maximum to minimum E or $\dfrac{1+r}{1-r}$ is called the *standing wave ratio* in voltage. We see that it can go to unity $r = 0$ when there is no reflection to $r = 1$ when there is perfect reflection. For metallic reflection which we shall assume $r = 1$, so that the *standing wave ratio is infinite.* The points along the guide where the resultant transverse electric field is maximum are called *standing wave maxima* and those where it is minimum are called *standing wave minima.* For in infinite standing wave ratio the standing wave minima become the nodes and in any case they are separated by a half wave length.

It is observed that certain standing wave field distributions be set up in the cavity which correspond to different cavity modes. The possible frequencies are known as resonant frequencies of the cavity. Consider a rectangular cavity consisting of the region bounded by the six planes $x = 0, x = a, y = 0, y = b, z = 0, z = d$.

Case (i). *TE* Mode. In this mode $E_x = 0$; let us put

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_z \qquad \text{...(1)}$$

where

$$\mathbf{E}_\perp = (\mathbf{i}\,\mathbf{E}_x + \mathbf{j}\,E_y)$$

represents the transverse or perpendicular component of electric field $\mathbf{E}_z$ and $\mathbf{E}_r$ is component of electric field along which wave is propagating *i.e.* along *z*-axis.

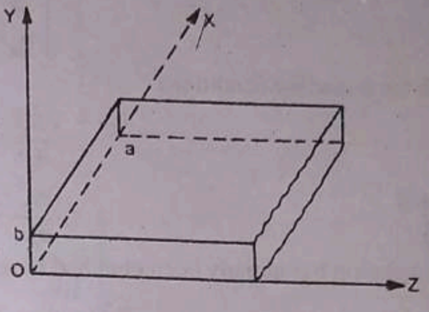

Fig. 9.14. Rectangular resonant cavity.

The transverse component $\mathbf{E}_\perp^+$ of electric field propagating along positive *Z*-direction may be expressed as

$$\mathbf{E}_\perp^+(\mathbf{r}, t) = \mathbf{E}_\perp^+(x, y,)\, e^{ik_g z - i\omega t} = \mathbf{E}_\perp^+(x, y)\, e^{ik_g z - i\omega t} \quad \text{...(2)}$$

The transverse component of electric field of reflected wave propagating along negative *Z*-axis is expressed as

$$\mathbf{E}_\perp^-(\mathbf{r}, t) = \mathbf{E}_\perp^-(x, y)\, e^{ik_g z - i\omega t} \quad \text{...(3)}$$

The resultant *transverse electric field*, which results due to superposition of two *TE* waves, one incident propagating along positive *Z*-axis and the other reflected propagating along negative *Z*-axis, is given by

$$\begin{aligned} \mathbf{E}_\perp(\mathbf{r} . t) &= \mathbf{E}_\perp^+(\mathbf{r}, t) + \mathbf{E}_\perp^-(\mathbf{r}, t) \\ &= [E_\perp^+(x, y)\, e^{ik_g z} + E_\perp^-(x, y)\, e^{-ik_g z}]\, e^{-i\omega t} \end{aligned} \quad \text{...(4)}$$

The boundary condition is that the tangential component of $\mathbf{E}(\mathbf{r}, t)$ must vanish at $z = 0$ and $z = d$. The boundary condition $\mathbf{E}_\perp(\mathbf{r}, t) = 0$ at $z = 0$ for all values of x, y and t implies

$$\mathbf{E}_\perp^+(x, y) + \mathbf{E}_\perp^-(x, y) = 0$$

i.e.

$$\mathbf{E}_\perp^{(-)}(x, y) = -\,\mathbf{E}_\perp^+(x, y) \quad \text{...(5)}$$

this equation (5) gives

$$\begin{aligned} \mathbf{E}_\perp(\mathbf{r}, t) &= \mathbf{E}_\perp^+(x, y)\,(e^{ik_g z} - e^{-ik_g z})\, e^{-i\omega t} \\ &= 2i\,\mathbf{E}_\perp^+(x, y) \left(\frac{e^{ik_g z} - e^{-ik_g z}}{2i} \right) e^{-i\omega t} \\ &= 2i\,\mathbf{E}_\perp^+(x, y) \sin k_g z\, e^{-i\omega t} \end{aligned} \quad \text{...(6)}$$

The boundary condition $\mathbf{E}_\perp(\mathbf{r}, t) = 0$ at $z = d$ implies that

$$\sin(k_g d) = 0 \text{ or } k_g d = p\pi,\; p = 0, 1, 2, 3, \ldots$$

This gives

$$k_g = \frac{\pi p}{d} \quad \text{...(7)}$$

Substituting this value of k_g in equation (6), we get

$$E_\perp(\mathbf{r}, t) = 2i\,\mathbf{E}_\perp^+(x, y) \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t} \quad \text{...(8)}$$

This equation may be expressed as

$$iE_z(\mathbf{r}, t) + jE_y(\mathbf{r}, t) = 2i\,\{\mathbf{i}E_x^+(x, y) + \mathbf{j}\mathbf{E}_y^+(x, y)\}\, \sin \frac{p\pi z}{d}\, e^{-i\omega t}$$

This gives

$$\left.\begin{aligned} E_x(\mathbf{r}, t) &= 2i\,E_x^+(x, y) \sin \frac{p\pi z}{d}\, e^{-i\omega t} \quad &\text{...(a)} \\ E_y(\mathbf{r}, t) &= 2i\,\mathbf{E}_y^+(x, y)\} \sin \frac{p\pi z}{d}\, e^{-i\omega t} \quad &\text{...(b)} \end{aligned}\right\} \quad \text{...(9)}$$

In *TE* mode, the value of B^+ may be obtained by solving the equation

$$(\nabla_\perp^2 + k_c^2)\, B_z^+ = 0$$

or

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mathbf{k}_c^2\right] B_z^+ = 0$$

with the boundary conditions

$$\left.\begin{aligned} \frac{\partial B_z^+}{\partial x} &= 0 \text{ at } x = 0 \text{ and } x = a \\ \frac{\partial B_z^+}{\partial y} &= 0 \text{ at } y = 0 \text{ and } y = b \end{aligned}\right\}$$

This solution has already been obtained in section 8.13 in the case of rectangular wave guide and is given by

$$B_z^+(\mathbf{r}, t) = B_0 \cos\frac{m\pi x}{h} \cos\frac{n\pi y}{b}\, e^{-ik_z z - i\omega t} \qquad \text{...(10a)}$$

i.e.

$$B_z^+(x, y) = B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad \text{...(10b)}$$

with

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \text{...(11)}$$

This in turn provides

$$\left.\begin{aligned} E_x^+(x, y) &= -\frac{in\pi\omega}{k_c^2 b} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi x}{b} & \text{...(a)} \\ E_y^+(x, y) &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} & \text{...(b)} \end{aligned}\right\} \qquad \text{...(12)}$$

Substituting these values in (9), we get

$$\mathbf{E}_x(\mathbf{r}, t) = \frac{2\omega}{k_c^2}\left\{\frac{n\pi}{b}\right\} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\left\{\frac{p\pi z}{d}\right\} e^{-i\omega t} \qquad \text{...(13a)}$$

$$\mathbf{E}_y(\mathbf{r}, t) = \frac{2\omega}{k_c^2}\left\{\frac{m\pi}{d}\right\} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi z}{b} \sin\left\{\frac{p\pi y}{d}\right\} e^{-i\omega t} \qquad \text{...(13b)}$$

In *TE* mode we already have

$$E_z(\mathbf{r}, t) = 0. \qquad \text{...(13c)}$$

The accompanying magnetic field components for *TE* mode may be obtained by using Maxwell's curl equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

which in terms of components can be written as

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} & \text{...(a)} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} & \text{...(b)} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} & \text{...(c)} \end{aligned}\right\} \qquad \text{...(14)}$$

From the form of exponential $e^{-i\omega t}$, it is apparent that

$$\frac{\partial}{\partial t} \Rightarrow -i\omega$$

therefore equations (14) may be expressed as rectangular wave guide and is given by

$ik_z z - i\omega t$

...(10a)

...(10b)

...(11)

...(a)

...(b) ...(12)

$\left.\frac{p\pi z}{d}\right\} e^{-i\omega t}$ ...(13a)

$\left.\frac{p\pi y}{d}\right\} e^{-i\omega t}$ ...(13b)

...(13c)

obtained by using Maxwell's curl

...(14)

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= i\omega B_x && \text{...(a)} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= i\omega B_y && \text{...(b)} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y} &= i\omega B_z && \text{...(c)} \end{aligned}\right\} \qquad \text{...(15)}$$

In *TE* mode $E_z = 0$, therefore above equtions yield

$$\left.\begin{aligned} -\frac{\partial E_y}{\partial z} &= i\omega B_x && \text{...(a)} \\ \frac{\partial E_x}{\partial z} &= i\omega B_y && \text{...(b)} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= i\omega B_z && \text{...(c)} \end{aligned}\right\} \qquad \text{...(16)}$$

From equation (16a), we have

$$\begin{aligned} B_x &= -\frac{1}{i\omega}\frac{\partial E_y}{\partial z} \\ &= -\frac{1}{i\omega}\frac{\partial}{\partial z}\left\{-\frac{2m\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi y}{d}\, e^{-i\omega t}\right\} \\ &= -\frac{2i}{k_c^2}\left[\frac{m\,\pi}{b}\right]\left[\frac{p\pi}{d}\right] B_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\, e^{-i\omega t} \end{aligned} \qquad \text{...(17a)}$$

From equation (16 b) we have

$$\begin{aligned} B_y &= \frac{1}{i\omega}\frac{\partial E_x}{\partial z} \\ &= \frac{1}{i\omega}\frac{\partial}{\partial z}\left\{\frac{2\pi n\omega}{k_c^2 b} B_0 \cos\frac{m\pi y}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\, e^{-i\omega t}\right\} \\ &= -\frac{2i}{k_c^2}\left[\frac{n\pi}{b}\right]\left[\frac{b\pi}{d}\right] B_0 \cos\frac{m\pi y}{a}\sin\frac{n\pi y}{h}\cos\frac{p\pi z}{d}\, e^{-i\omega t} \end{aligned} \qquad \text{...(17d)}$$

Similarly from equation (16c), we have

$$\begin{aligned} B_z &= \frac{1}{i\omega}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right] \\ &= \frac{1}{i\omega}\left[\frac{\partial}{\partial x}\left\{-\frac{2m\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\, e^{-i\omega t}\right\}\right] \\ &\qquad - \frac{\partial}{\partial y}\left\{\frac{2n\pi\omega}{k_c^2 b} B_0 \cos\frac{m\pi y}{b}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\, e^{-i\omega t}\right\}\Bigg]\Bigg] \\ &= \frac{1}{i\omega}\left[-\frac{2m\pi\omega}{k_c^2 a}\cdot\frac{m\pi}{a}\cdot B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\, e^{-i\omega t}\right. \\ &\qquad \left. - \frac{2n\pi\omega}{k_c^2 b}\cdot\frac{n\pi}{b} B_0 \cos\frac{n\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\, e^{-i\omega t}\right] \\ &= \frac{2i}{k_c^2}\left[\left\{\frac{m\pi}{a}\right\}^2 + \left\{\frac{n\pi}{b}\right\}^2\right] B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\, e^{-i\omega t} \end{aligned}$$

$$= 2i\ B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \frac{p\pi z}{d}\ e^{-i\omega t} \qquad \text{...(17c)}$$

Equation (13) and (17) express components of field in resonant cavity for *TE* mode.

The *resonant frequency* of the cavity may be obtained by the use of *guide equations*

$$k_g^2 = k_0^2 - k_c^2 . \qquad \text{...(18a)}$$

The equation implies that for $k_g < k_c$; k_g is imaginary. This leads to attenuation rather than propagation. Equation (18a) may be expressed as

$$k_0 = k_c^2 + k_g^2 . \qquad \text{...(18b)}$$

Substituting values of k_g and k_c from equations (7) and (11), and using $k_0 = \omega/c$, we get

$$\frac{\omega^2}{c^2} = \left[\frac{\pi m}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 + \left[\frac{\pi p}{d}\right]^2$$

or

$$\omega^2 = c^2\pi^2 \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right] \qquad \text{...(19)}$$

i.e. resonant angular frequency

$$\omega_r = \pi c \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right]^{1/2} \qquad \text{...(20a)}$$

and *resonant linear frequency*

$$f_r = \frac{\omega_r}{2\pi} = \frac{c}{2}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right]^{1/2} \qquad \text{...(20b)}$$

This equation represents that the resonant frequencies form a discrete set depending on the choice of m, n and p. Because propagation in cavity resonators may take place in more than one direction and in various modes, cavity resonators in general have a large number of possible modes of resonance. A cavity resonator for a specific application may be designed and excited in such a manner however, that only one mode of resonance is obtained over a limited frequency range.

From equations (13) and (17) it is obvious that if any of the two integers m, n and p are zero, all the field components would become zero. This implies that TE_{000} , TE_{001} , TE_{010} ,TE_{100} , modes do not exist in the cavity. The physically possible lowest modes are of the type TE_{101} , TE_{011} , (*i.e.* where only one integer is zero). The smallest allowed frequency of these modes is called the *fundamental frequency* or *first harmonic*. The higher allowed values which are integral multiples of the fundamental are called *over ones or higher harnomics*

Case (ii) TM Mode. In *TM* mode $B_z = 0$: and the value of $\mathbf{E}_z^+$ may be obtained by solving the equation

$$(\nabla_\perp^2 + k_c^2)\, E_z^+ = 0 \quad \text{or} \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right] E_z^+ = 0 \qquad \text{...(21)}$$

with the boundary condition $E_z^+ = 0$ at $x = 0, x = a$; $y = 0$ and $y = b$.

This solution has already been obtained in section 9.13 for rectangular wave guide and is given by

$$E_z^+ (\mathbf{r}, t) = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}\ e^{ik_g z - i\omega t} \qquad \text{...(22a)}$$

or

$$E_z^+ (x, y) = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{...(22b)}$$

with the condition

$$k_0^2 = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 \qquad \text{...(23)}$$

The transverse components E_x^+ and E_y^+ may be obtained from equations [refer equations (17) and (18) of section 9.12]

$$\left.\begin{aligned} E_x^+ &= \frac{ik_g}{k_c^2}\frac{\partial E_z^+}{\partial x} && \text{...(a)} \\ E_y^+ &= \frac{ik_g}{k_c^2}\frac{\partial E_z^+}{\partial y} && \text{...(b)} \\ \text{where}\quad k_c^2 &= k_g^2 - k_g^2 && \text{...(c)} \end{aligned}\right\} \qquad \text{...(24)}$$

Substituting value of E_z^+ from (22a) in above equations, we get

$$\left.\begin{aligned} E_x^+(\mathbf{r},t) &= \frac{ik_g}{k_c^2}\left[\frac{m\pi}{a}\right] E_0 \cos\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,e^{ik_gz-i\omega t} && \text{...(a)} \\ E_y^+(\mathbf{r},t) &= \frac{ik_g}{k_c^2}\left[\frac{n\pi}{b}\right] E_0 \sin\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b}\,e^{ik_gz-i\omega t} && \text{...(b)} \end{aligned}\right\} \qquad \text{...(25)}$$

If $\mathbf{E}_\perp^+$ is the transverse component of electric field propagating along positive z-direction, then

$$\mathbf{E}_\perp^+(\mathbf{r},t) = i\,\mathbf{F}_z(\mathbf{r},t) + \mathbf{j}\mathbf{E}_y^+(\mathbf{r},t)$$

and it may be expressed as

$$\mathbf{E}_\perp^+(\mathbf{r},t) = \mathbf{E}_\perp^+(x,y)\;e^{ik_gz-i\omega t}$$

If $\mathbf{E}_\perp^-$ is the transverse component of electric field in *TM* wave propagating along negative z-axis, then

$$\mathbf{E}_\perp^-(\mathbf{r},t) = \mathbf{F}_\perp^-(x,y)\;\epsilon^{-ik_gz-i\omega t}$$

∴ Resultant *transverse electric field* is given by

$$\begin{aligned} \mathbf{E}_\perp(\mathbf{r},t) &= \mathbf{E}_\perp^+(\mathbf{r},t) + \mathbf{E}_\perp^-(\mathbf{r},t) \\ &= \mathbf{E}_\perp^+(x,y)\,e^{ik_gz-i\omega t} + \mathbf{E}_\perp^-(x,y)\,e^{-ik_gz-i\omega t} \end{aligned} \qquad \text{...(26)}$$

The boundary condition that the tangential component of electric field vanishes at $z=0$ implies that $\mathbf{E}_\perp(\mathbf{r},t) = 0$ at $z=0$ or $\mathbf{E}_\perp^+(x,y) = -\,\mathbf{E}_\perp^-(x,y)$ for all values of z and t.

Using this substitution, equation (26) becomes

$$\mathbf{E}_\perp(\mathbf{r},t) = 2i\,\mathbf{E}_\perp^+(x,y)\sin k_gz\,e^{-i\omega t} \qquad \text{...(27)}$$

The boundary condition that the tangential component of electric field vanishes at $z=d$ implies that

$$\sin k_gd = 0 \quad\text{or}\quad k_gd = p\pi \quad\text{or}\quad k_g = \frac{p\pi}{d} \qquad \text{...(28)}$$

where p is an integer. With this subtitution equation (27) takes the form

$$\mathbf{E}_\perp(\mathbf{r},t) = 2i\,\mathbf{E}_\perp^+(x,y)\,\sin\frac{p\pi z}{d}\,e^{-i\omega t} \qquad \text{...(29)}$$

In terms of components equation (29) may be expressed as

$$\left.\begin{aligned} E_x(\mathbf{r},t) &= 2i\,E_x^+(x,y)\,\sin\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(a)} \\ E_y(\mathbf{r},t) &= 2i\,E_y^+(x,y)\,\sin\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(b)} \end{aligned}\right\} \qquad \text{...(29)}$$

Substituting values from (25) we get

$$\left.\begin{aligned} E_x(\mathbf{r},t) &= -\frac{2k_z}{k_c^2}\left(\frac{n\pi}{a}\right)E_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(a)\\ E_y(\mathbf{r},t) &= -\frac{2k_z}{k_c^2}\left(\frac{n\pi}{a}\right)E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(b)\end{aligned}\right\} \quad \ldots(30)$$

We already have

$$B_z(\mathbf{r},t) = 0. \quad \ldots(31)$$

The components B_x, B_y and B_z may be obtained by using Maxwell curl equation

$$\nabla \times \mathbf{B} = -\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

which in terms of components can be written as

$$\left.\begin{aligned} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial x} &= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} \quad \ldots(a)\\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= -\frac{1}{c^2}\frac{\partial E_y}{\partial t} \quad \ldots(b)\\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -\frac{1}{c^2}\frac{\partial E_z}{\partial t} \quad \ldots(c)\end{aligned}\right\} \quad \ldots(32)$$

From the form of exponential $e^{-i\omega t}$, it is apparent that $\partial/\partial t \rightarrow -i\omega$ and keeping in mind that $B_z = 0$ for *TM* mode; the equations (32) takes the form

$$\left.\begin{aligned} \frac{\partial B_y}{\partial z} &= -\frac{i\omega}{c^2}E_x \quad \ldots(a)\\ \frac{\partial B_y}{\partial z} &= +\frac{i\omega}{c^2}E_y \quad \ldots(b)\\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \frac{i\omega}{c^2}E_z \quad \ldots(c)\end{aligned}\right\} \quad \ldots(33)$$

Substituting value of E_y from (30b), in (33b), we get

$$\frac{\partial B_x}{\partial z} = +\frac{i\omega}{c^2}\frac{2k_g}{k_c^2}\left\{\frac{n\pi}{b}\right\}E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t}$$

Integrating with respect to z and using equation (28) viz. $k_g = p\pi/d$, we get

$$B_x = +\frac{2i\omega E_0}{k_c^2c^2}\left\{\frac{n\pi}{b}\right\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(34a)$$

Similarly substituting value of E_x from (30a) in (33a), integrating with respect to z and using equation (28), we get

$$B_y = -\frac{2i\omega E_0}{k_c^2c^2}\left\{\frac{m\pi}{a}\right\}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(34b)$$

Now using (34a) and (34b), equations (33c) gives

$$E_y = \frac{c^2}{i\omega}\left[\frac{\partial B_y}{i\omega} - \frac{\partial B_x}{\partial y}\right]$$

$$= \frac{c^2}{i\omega}\left[\frac{\partial}{\partial x}\left\{-\frac{2i\omega E_0}{k_c^2 c^2}\left\{\frac{m\pi}{a}\right\}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t}\right\}\right.$$
$$\left. - \frac{\partial}{\partial y}\left|\frac{2i\omega E_0}{k_c^2 c^2}\left\{\frac{n\pi}{b}\right\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t}\right\}\right]$$

$$E_z = \frac{2E_0}{k_c^2}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t}$$

Using equation (23), we have

$$E_z = 2\,E_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} \quad \text{...(34c)}$$

Writing $k_g = p\pi/d$ (equation 28), the *field components for TM mode in resonant cavity* may be expressed as

$$\left.\begin{aligned}
B_x(\mathbf{r}, t) &= \frac{2i\omega}{k_c^2 c^2}\left(\frac{n\pi}{b}\right)E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(a)}\\
B_y(\mathbf{r}, t) &= -\frac{2i\omega}{k_c^2 c^2}\left(\frac{m\pi}{a}\right)E_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(b)}\\
B_z(\mathbf{r}, t) &= 0 && \text{...(c)}\\
E_x(\mathbf{r}, t) &= -\frac{2}{k_c^2}\left(\frac{mx}{a}\right)\left(\frac{p\pi}{d}\right)E_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(d)}\\
E_y(\mathbf{r}, t) &= -\frac{2}{k_c^2}\left(\frac{nx}{b}\right)\left(\frac{p\pi}{d}\right)E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(e)}\\
E_z(\mathbf{r}, t) &= 2\,E_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(f)}
\end{aligned}\right\}$$

The resonant frequency is again calculated by guide equation

$$k_0^2 = k_c^2 + k_g^2$$

i.e.

$$\frac{\omega^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \quad \text{...(35)}$$

or *resonant angular frequency*

$$(\omega_r)_{mnp} = \pi c\left\{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right\}^{1/2} \quad \text{...(36)}$$

which is same as in case of *TE* mode. From field equations (35) it is apparent that the following mode TM_{mnp} do not exist

$$TM_{000},\ TM_{001},\ TM_{010},\ TM_{101},\ TM_{011}$$

The physically possible lowest mode is of type TM_{110}

If two or more cavity modes have the same resonant frequency, then the system is said to be degenerate and the resonant frequency is called degenerate frequency. The order of the degeneracy is equal to the number of modes beloging to the degenerate frequency.

The cavity resonators display the properties typical of *resonant circuits* in that they can store energy in oscillating electric and magnetic fields. Furthermore practical cavity resonators dissipate a fraction of the stored energy in each cycle of oscillation. The fraction of energy dissipated in a cavity resonator is about $\frac{1}{20}$ th of the fraction dissipated per cycle in an *L—C* circuit : that is why the cavity resonators are usually superior to conventional LC circuits by a factor of about twenty. The additional advantage of cavity resonators is that they have resonant frequencies with range upward from a few hundred mega cycles-just the region where it is impossible to construct a $L—C$ circuit.

QUESTIONS AND PROBLEMS

1. (a) Obtain the boundary conditions satisfied by the electromagnetic field vectors **E**, **D**, **B** and **H** on the plane interface between two media. **(Meerut 1994, 76)**

   (b) A plane wave travelling in a homogeneous isotropic dielectric is incident obliquely at the plane surface of another dielectric, Determine the reflection and transmission coefficient if the incident wave is polarised with its electric vector.

   (i) perpendicular to the plane of incidence (ii) in the plane of incidence. **(Meerut 1971)**

2. A plane electromagnetic wave is incident on a plane boundary between two non-conducting media. Specify the boundary conditions to be satisfied by the wave and hence derive the Fresnel formula for the reflected and transmitted intensities. Discuss important consequences of these formulae.

3. (a) Derive Fresnel's equaitons for reflection and refraction of electro-magnetic waves at a plane boundary separating two non-conducting isotropic linear media. **(Meerut 1998, Rohilkhand 1997, 96, 91; 87)**

   (b) Discuss metallic reflection and refraction. **(Rohilkhand 1987, 84)**

   (c) Find out an expression for the reflection power of a metallic surface. **(Rohilkhand 1997, 82; Agra 1977, 72)**

4. Derive Fresnel's equations for reflection of light from a dielectric surface. How are the relations verified experimentally ? **(Agra 1970)**

5. What are Fresnel's equations for reflection of plane polarized light from transparent media ? Indicate how they are derived and how they can be experimentally verified. **(Agra 1936)**

6. Derive Fresnel's equations for reflection and refraction of electromagnetic wave at a plane boundary separating two media when the incident wave is polarised with is $\vec{E}$ vector parallel to the plane of incidence. Hence show that there is an angle of incidence for which there is no reflected wave. Name this angle. **(Garhwal 1993)**

7. Plane electromagnetic waves are incident obliquely at the plane interface between two dielectric media with electric vector lying in the plane of incidence. Obtain the amplitude of electric vectors in the reflected and transmitted waves as fractions of that in the incident wave. What is the reflection coefficient ? Show that there is an angle of incidence for which there is no reflected wave. Find this angle ? What is this angle called ? **(Rohilkhand 1988, 86, 80)**

8. Discuss the reflection and refraction of electromagentic waves at the interface of two non-conducting media and derive the Fresnel equations. **(Meerut 1992)**

9. Discuss the reflection and refraction of electromagnetic waves at a plane interface between dielectrics. Under what condition will the incident wave be totally internally reflected ? **(Meerut 1997)**

10. Show on the basis of electromagnetic theory, that all good codnuctors are good absorbers and good reflectors. **(Garhwal 1993)**

11. Obtain the necessary relation between the incident and reflected electromagnetic field amplitudes for reflection at a plane metal surface and show how to determine the complex refractive index optically in terms of suitable parameters **(Meerut 1987)**

12. (a) $A_{\parallel}$ and $A_{\perp}$ are parallel and perpendicular components of plane electromagnetic wave incident at an angle θ, obtain expression for the corresponding components in the reflected beam if the second medium is denser (refractive index n).

    (b) Show that if first medium is denser in (a), there is a difference of phase between the reflected parallel and perpendicular components given by

$$\tan\frac{\delta}{2} = \frac{\cot\theta\sqrt{[\sin^2\theta - (1/n^2)]}}{\sin^2\theta}$$

    (c) The degree of polarisation is defined as

$$P = \frac{I_{\perp} - I_{\parallel}}{I_{\perp} + I_{\parallel}}$$

where $I_{\perp}$ and $I_{\parallel}$ are perpendicular and parallel components of intensity. Calculate the value of ρ for ordinary light reflected from glass (index 1·5) at an angle of incidence of 45°. **(Meerut 1970)**

13. Deduce Brewster's law on the basis of electromagnetic theory and explain how it can be verified experimently. What should be the altitude (angle above horizon) of the sun so that sunlight reflected from a still lake is completely plane polarised. (UPCS 1997)

[Hint. Brewster's law $n = \tan\theta_p$; $\theta_p = \tan^{-1}(n) = \tan^{-1}(4/3 = 53^\circ)$ and altitude $\theta = 90^\circ - \theta_p = 37^\circ$].

14. Derive Fresnel's equations for reflection and refraction of electromagnetic waves at a plane boundary separating two dielectric media. Discuss the phenomenon of total internal reflection. (Rohilkhand 1997, 96)

15. Discuss the phenomenon of total internal reflection on the basis of electromagnetic theory of light. (Meerut 1980)

16. Establish the Maxwell's equations for the electromagnetic field. Discuss the phenomenon of total internal reflection on the basis of these equations. (Agra 1997, 89)

17. What happens when a plane wave is incident on a plane surface with boundaries having different electric and magnetic properties ? Discuss the condition of the incident angle when the ray suffers maximum degree of polarisation after reflection. (Agra 1974)

18. (a) Establish Maxwell's equations for the electromagnetic field.

(b) Discuss the phenomenon of reflection from a conducting medium on the basis of these equations.

(Rohilkhand 1986; Agra 1995)

19. Electromagnetic waves are incident normally on a metal surface of refractive inding n and extenction coefficient k. Show that the ratio of the reflected intensity to incident intensity is, $R = \dfrac{n^2 + k^2 - 2n + 1}{n^2 + k^2 + 2n + 1}$

[Hint. put $\dfrac{n_2}{n_1} = n_2 = n + ik$ in section 9·9]

(Rohilkhand 1993)

20. What are pecularities of metallic reflection ? Give its theory in brief and describe method for its verification.

(Agra 1938)

21. What is a wave guide ? Describe the propagation of electromagnetic wave along a hollow wave guide of uniform cross-section. Explain how the cut off modes arise. Show that the phase velocity of waves becomes infinite exactly at the cutoff. (Meerut 1989, 87)

22. What is a wave guide ? For transverse electric waves perfectly propagating in a rectangular wave guide with perfectly conducting walls; find

(i) the cut off wavelength.

(ii) the magnetic field induction.

(iii) The velocity with which energy is transmitted along the guide. (Meerut 1989, 74)

23. What is a resonant cavity ? Discuss its superiority over reasonant circuit. For *TE* wave in the cavity with perfectly conducting walls, derive expression for (i) cut off wavelength, (ii) resonant frequency, (iii) electric and magnetic fields.

24. What is an electromagnetic cavity resonator ? Describe the excitation of T.E.M. waves in the class of cavity resonators produced by placing plane end faces on a length of a cylindrical wave guide. (Meerut 1983)

25. What are cavity resonators ? How do they resonate ? How they resemble with electronic resonant circuits ? Discuss in detail the theory of a rectangular cavity resonator, showing what lowest modes are possible in TE mode. (Meerut 1991, 81)

26. How is the polarisation of an electromagnetic wave affected when it crosses the plane interface between two dielectrices ? Explain the significance of Brewster's angle. What is the total internal reflection and why do we get it ? (Kanpur 1983; Rohilkhand 1985, 81)

27. Write short notes on

(i) Total internal reflection (Meerut 98)

(ii) Brewster's law (Meerut 1997)

(iii) Fresnel equations. (Rohilkhand 1999)

12

Electrodynamics of a Moving Charge and Radiating Systems

12.1. Introduction

A stationary charge produces electrostatic field only. A uniformly moving charge produces electromagnetic field of short range, so they can not radiate electromagnetic energy. We shall see that *the electromagnetic energy can be radiated only if a charged particle is accelerated.* The fields produced by moving charges can be computed either by the concept of retarded potentials or the theory of relativity. In the former case the field point parameters are related to the source point. Since the cause occurs at the source point and the effect follows at the field point; the procedure to relate the cause and effect is known as *retardation.* In the relativistic method two inertial frames of reference are chosen, with field point in one, the source point in the other and the charge is assumed to be relativistically invariant.

In this chapter we shall find the electromagnetic fields by the concept of retarded potentials and shall calculate quantities of interest such as total radiation emitted by an accelerated charge and the angular distribution of radiation to discuss the behaviour of radiated energy. Finally we shall discuss the fields and radiations produced by an electric dipole with simple applications to short linear antenna and antenna arrays.

12.2. Solution of Inhomogeneous Wave Equation by Fourier Analysis.

The scalar and vector wave equations in *free space* are

$$\left.\begin{aligned}\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi &= -\frac{\rho}{\varepsilon_0} \quad \text{or} \quad \Box^2\phi = -\frac{\rho}{\varepsilon_0} \quad \text{...(a)}\\ \left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} &= -\mu_0\mathbf{J} \quad \text{or} \quad \Box^2\mathbf{A} = -\mu_0\mathbf{J} \quad \text{...(b)}\end{aligned}\right\}. \qquad \text{...(1)}$$

These two equations may be replaced by a single equation of the form

$$\Box^2\psi(\mathbf{r}, t) = -g(\mathbf{r}, t) \qquad \text{...(2)}$$

where $\psi(\mathbf{r}, t)$ is the source function at the field point $(\mathbf{r}, t)$ and covers both ϕ and $\mathbf{A}$: while $g(\mathbf{r}, t)$ is the source function and covers both ρ and $\mathbf{J}$.

According to Fourier's theorem a complex single harmonic periodic function may be expressed as a sum of simple harmonic periodic functions of all possible frequencies. Accordingly the source function $g(\mathbf{r}, t)$ may be analysed by the Fourier integral

$$g(\mathbf{r}, t) = \int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega \qquad ...(3)$$

which has the *Fourier inversion*

$$g_\omega(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\mathbf{r}, t)\, e^{i\omega t}\, dt \qquad ...(4)$$

Similarly the general complex potential function $\psi(\mathbf{r}, t)$ may be analysed as

$$\psi(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega \qquad ...(5)$$

and the corresponding Fourier inversion is

$$\psi_\omega(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(\mathbf{r}, t)\, e^{i\omega t}\, dt \qquad ...(6)$$

Substituting values of $g(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$ for (3) and (5) in equation (2), we get

$$\Box^2 \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega = \int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega = -\int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

or

$$\nabla^2 \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega + \frac{\omega^2}{c^2}\int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega = -\int_{-\infty}^{+\infty} g_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

or

$$\int_{-in}^{+\infty} \left\{\nabla^2 \psi_\omega(\mathbf{r}) + \frac{\omega^2}{c^2}\psi_\omega(\mathbf{r}) + g_\omega(\mathbf{r})\right\} e^{-i\omega t}\, d\omega = 0,$$

This equation is true for any arbitrary instant and any arbitrary ω; hence this is satisfied only if

$$\nabla^2 \psi_\omega(\mathbf{r}) + \frac{\omega^2}{c^2}\psi_\omega(\mathbf{r}) + g_\omega(\mathbf{r}) = 0$$

i.e.

$$\nabla^2 \psi_\omega + \frac{\omega^2}{c^2}\psi_\omega = -g_\omega \qquad ...(7)$$

This equation is similar to Poisson's equation $\nabla^2\psi = -\rho/\varepsilon_0$, except for the second term on L.H.S. The solution of equation (7) may be assumed as the superposition of unit point solutions corresponding to a source at the point $\mathbf{r}'$ given by $g(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$:, where $\delta(\mathbf{r} - \mathbf{r}')$ is the *Dirac-function.* Then if G is a unit source potential called the *Green's function,* then equation (7) in terms of G takes the form

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \qquad ...(8)$$

where G is a function of both $\mathbf{r}$ and $\mathbf{r}'$. According to property of Dirac δ-function $\delta(\mathbf{r} - \mathbf{r}') = 0$ for $\mathbf{r} \neq \mathbf{r}'$. This means when the field and source points are different, the R.H.S. of equation (8) is zero and the relation is then homogeneous.

The partial solution cooresponding to frequency ω of the total source is given by the superposition :

$$\psi_\omega(\mathbf{r}) = \int g_\omega(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, dv'. \qquad ...(9)$$

To find the value of G, we note that the solution of equation (8) is spherically symmetric in R, the distance between points $\mathbf{r}$ and $\mathbf{r}'$, except at $R = 0$ where $\mathbf{r} = \mathbf{r}'$. For spherically symmetric situation

$$\nabla^2 G = \frac{1}{R}\frac{d^2}{dR^2}(RG).$$

Hence at all points other than $R = 0$, equation (8) is identical with the equation

$$\frac{1}{R}\frac{d^2(RG)}{dR^2} + \frac{\omega^2}{c^2} G = 0. \qquad ...(10)$$

Writing $k = \dfrac{\omega}{c}$ as usual, equation (10) takes the form

$$\frac{1}{R}\frac{d^2(RG)}{dR^2} + k^2 G = 0$$

i.e.

$$\frac{d^2(RG)}{dR^2} + k^2 RG = 0. \qquad ...(11)$$

On integration this equation immediately gives

$$RG = A\, e^{\pm i\omega R}. \qquad \text{where } A \text{ is constant}$$

or

$$G = \frac{A}{R} e^{\pm i\omega R} \qquad ...(12)$$

It is obvious that in the neighbourhood of singular point $R = 0$, G behaves as A/R.

The Gauss divergence theorem applicable to any vector $\mathbf{V}$ is

$$\int \nabla \cdot \mathbf{V}\, dv = \int \mathbf{V} \cdot d\mathbf{S}.$$

Letting $\mathbf{V} = \nabla\left[\dfrac{1}{R}\right]$, we get

$$\int \nabla \cdot \nabla\left[\frac{1}{R}\right] dv' = \int \nabla\left[\frac{1}{R}\right] \cdot d\mathbf{S}'$$

or

$$\int \nabla^2\left[\frac{1}{R}\right] dv' = -\int \frac{\mathbf{R} \cdot d\mathbf{S}'}{R^3} = -\int d\Omega. \qquad ...(13)$$

where Ω is the solid angle subtended at point $\mathbf{r}$ by the surface of integration $\mathbf{S}'$ over the variables $\mathbf{r}'$. Since $\mathbf{S}'$ includes $\mathbf{r}$, we have simply $\int d\Omega = 4\pi$. Hence equation (13) gives

$$\int \nabla^2\left[\frac{1}{R}\right] dv' = -4\pi.$$

The implies

$$\int \nabla^2\left(\frac{A}{R}\right) dv' = -4\pi A \qquad \text{(since } A \text{ is constant)}$$

i.e.

$$\int \nabla^2 G\, dv' = 4\pi A \qquad ...(14)$$

in the neighbourhood of singular point $R = 0$.

Now considering volume integral of equation (8) over the neighbourhood of the singular point $R = 0$ where G behaves as A/R, we get

$$\int \nabla^2 G(\mathbf{r}, \mathbf{r}')\, dv' + \frac{\omega^2}{c^2} \int G(\mathbf{r}, \mathbf{r}')\, d' = -\int \delta(\mathbf{r} - \mathbf{r}')\, dv'. \qquad ...(15)$$

Noting that second term vanishes in the limit $R \to 0$ while considering the volume integral in neighbourhood of singular point $R = 0$ and using (14) along with the property of Dirac δ-function, *viz.* $\int \delta(\mathbf{r} - \mathbf{r}')\, dv' = 1$ for $R = 0$, equation (15) gives

$$-4\pi A = -1$$

or

$$A = \frac{1}{4\pi} \qquad \text{...(16)}$$

Substituting this value of A in equation (12), we get

$$G = \frac{1}{4\pi R} e^{\pm ikR} \qquad \text{...(17)}$$

as the solution of equation (8).

Substituting value of G from (17) in equation (9), we get

$$\psi_\omega(\mathbf{r}) = \frac{1}{4\pi} \int \frac{g_\omega(\mathbf{r}')}{R(\mathbf{r}, \mathbf{r}')} e^{\pm ikR(\mathbf{r}, \mathbf{r}')}\, dv'. \qquad \text{...(18)}$$

We note that the effect of second term in equation (7) is simply to introduce a complex exponential factor in the familiar Coulomb potential expression *viz.*

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{R(\mathbf{r}, \mathbf{r}')}\, dv' = \frac{1}{4\pi} \int \frac{g(\mathbf{r}')}{R(\mathbf{r}, \mathbf{r}')}\, dv'. \qquad \text{...(19)}$$

Now substituting value of $\psi_\omega(\mathbf{r})$ from (18) in equation (5), we get

$$\psi(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \psi_\omega(\mathbf{r})\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{4\pi} \iint \frac{g_\omega(\mathbf{r}')\, e^{-i(\omega t \pm kR)}}{R(\mathbf{r}, \mathbf{r}')}\, d\omega\, dv'. \qquad \text{...(20a)}$$

Since $k = \omega/c$, equation (20a) may be expressed as

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \iint \frac{g_\omega(\mathbf{r}')\, e^{i\omega(t \pm R/c)}}{R(\mathbf{r}, \mathbf{r}')}\, d\omega\, dv' \qquad \text{...(20b)}$$

Not let us introduce a new time t' given by

$$t'(\mathbf{r}, \mathbf{r}') = t \pm \frac{R}{c}. \qquad \text{...(21)}$$

which corresponds to shifting of origin of time by an amount equal to the time taken by the light signal to travel a distance R from the source point r' to the field point $\mathbf{r}$. Then we may represent

$$\int_{-\infty}^{+\infty} g(\mathbf{r})\, e^{-i\omega(t \pm R/c)}\, d\omega = g\left(\mathbf{r}', t \pm \frac{R}{c}\right) \qquad \text{...(22)}$$

So that equation (20b) takes the form

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{g(\mathbf{r}', t \pm R/c)}{R(\mathbf{r}, \mathbf{r}')}\, dv' \qquad \text{...(23)}$$

Mathematically both plus and minus signs in above equation are valid; but only the minus sign appears to have physical significance. The minus sign indicates that the signal arriving at $(\mathbf{r}, t)$ should have started from $\mathbf{r}'$ at instant $(t - R/c)$. In other words the minus sign corrsspoonds to the cause preceding the effect. Equation (23) with minus sign in known as the *retarded potential solution of inhomogeneous wave equation*. The plus sign in (23) indicates that the signal arriving at $(\mathbf{r}, t)$ should have started from $\mathbf{r}'$ at instant $(t + R/c)$; it implies that the effect precedes the cause in time. That is why we shall ignore the solution with plus sign, since it clearly violates experience; the field does not appear to precede the charges and currents which cause it. The solution with plus sign in known as *abvanced solution.* Here we shall restrict ourselves to the minus sign *i.e.* to the use of retarded solutions.

The retarded solution for scalar and vector potentials may now be written as

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0}\int \frac{[\rho(\mathbf{r}')]\,dv'}{R(\mathbf{r}, \mathbf{r}')} \quad &\text{...(a)} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}(\mathbf{r}')]\,dv'}{R(\mathbf{r}, \mathbf{r}')} \quad &\text{...(b)} \end{aligned}\right\} \qquad \text{...(24)}$$

where $[\rho(\mathbf{r}')]$ and $[\mathbf{J}(\mathbf{r}')]$ within square brackets represent, however, not the values of charge and current densities at the time of observation t, but at the earlier time $t' = t - R/c$. In other words the retarded potentials ϕ and $\mathbf{A}$ are calculated from charge and current densities at a time which is *set back* by R/c relative to the time of observation.

Having known the electromagnetic potentials $\mathbf{A}$ and ϕ, the electromagnetic field vectors, in general may be calculated by the relations;

$$\mathbf{B} = \text{curl}\ \mathbf{A} \text{ and } \mathbf{E} = -\text{grad}\ \phi - \frac{\partial \mathbf{A}}{\partial t}. \qquad \text{...(25)}$$

12.3. Lienard-Wiechert Potentials

The Lienard-Wiechert potentials are scalar and vector potentials produced by a moving point charge. To obtain these potentials one of the most instructive procedures is to consider a moving volume carrying with it a fixed charge distribution, for example a uniformly charged spherical volume, moving through space along a prescribed trajectory. The field due to a point charge may then be obtained by taking properly the limit of the field due to such a charge distribution.

The scalar and vector potentials due to moving charge and current distributions at point $\mathbf{r}$ and time t are given by the *retarded potentials*

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) dV'}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(\mathbf{r}', t - \frac{R}{c}\right)}{\mathbf{R}(\mathbf{r}, \mathbf{r}')} dV' \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} dV' \\ &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{R}{c}\right)}{R(\mathbf{r}, \mathbf{r}')} dV' \end{aligned}$$

where
$$\mathbf{R}(\mathbf{r}, \mathbf{r}') = (\mathbf{r} - \mathbf{r}'). \qquad \text{...(1)}$$

The difficulty in integration arises because time $t' = \left(t - \frac{R}{c}\right)$ is not fixed and hence the volume of integration. In other words the volume in which ρ and $\mathbf{J}$ are different from zero, is not specified. To remove the difficulty let us introduce one dimensional Dirac delta function having properties

$$\left.\begin{aligned}&\text{(i)}\quad \delta(x) = 0 \quad \text{for} \quad x \neq 0 \text{ and } \delta(0) \neq 0\\ &\text{(ii)}\quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1\\ &\text{(iii)}\quad f(x)\,\delta(x-a) = f(a)\,\delta(x-a)\\ &\text{(iv)}\quad \int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\, dx = f(a)\end{aligned}\right\} \qquad \text{...(2)}$$

where $f(x)$, is regular, continuous and differentiable function of x.

With this δ function the scalar and vector potentials given by (1) may be written in the form

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\int_{-\infty}^{+\infty} \frac{\rho(\mathbf{r}', t')\,\delta\left[t' - \left(t - \frac{R}{c}\right)\right]}{R(\mathbf{r}, \mathbf{r}')}\, dV'\, dt'$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{V'}\int_{-\infty}^{+\infty} \frac{\mathbf{J}(\mathbf{r}', t')}{R(\mathbf{r}, \mathbf{r}')}\,\delta\left[\left(t' - \left(t - \frac{R}{c}\right)\right)\right] dV'\, dt' \qquad \text{...(3)}$$

Now $\rho(\mathbf{r}', t')\, dV' = dq$ is the charge in the volume element dV' and similarly $\mathbf{J}(\mathbf{r}, t') = \rho(\mathbf{r}', t')\, \mathbf{v}(t')$ is the *current density produced by an element of charge density* $\rho(\mathbf{r}', t')$ *moving with velocity* $\mathbf{v}(t')$. This gives

$$\mathbf{J}(r', t')\, dV' = \rho(r', t')\, \mathbf{v}(t')\, dV' = \mathbf{v}(t')\, dq. \qquad \text{...(4)}$$

Thus, we get

$$\left.\begin{aligned}\phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0}\int_{V}\int_{-\infty}^{+\infty} \frac{dq\,\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int_{V'}\int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, dq\,\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\end{aligned}\right] \qquad \text{...(5)}$$

If the *charge distribution limits to point charge q*, we can immediately write equation (5) as

$$\left.\begin{aligned}\phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\,\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\end{aligned}\right] \qquad \text{...(6)}$$

Thse integrals can be evaluated if we choose instead of t' a new fixed time t_1 give by

$$t_1 = t' - t + \frac{R}{c} \qquad \text{...(7a)}$$

We get

$$\frac{dt_1}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'} \qquad \text{...(7b)}$$

We have $R^2 = \mathbf{R}\cdot\mathbf{R}$

or

$$2R\frac{dR}{dt'} = \mathbf{R}\cdot\frac{d\mathbf{R}}{dt'} + \frac{d\mathbf{R}}{dt'}\cdot\mathbf{R}$$

i.e.

$$\frac{dR}{dt'} = \frac{\mathbf{R}}{R}\cdot\frac{d\mathbf{R}}{dt'} \qquad \text{...(8)}$$

From fig. 12.1 we have

$$\mathbf{R} = (\mathbf{R} + d\mathbf{R}) + \mathbf{v}\, dt'$$

i.e.

$$\frac{d\mathbf{R}}{dt'} = -\mathbf{v}.$$

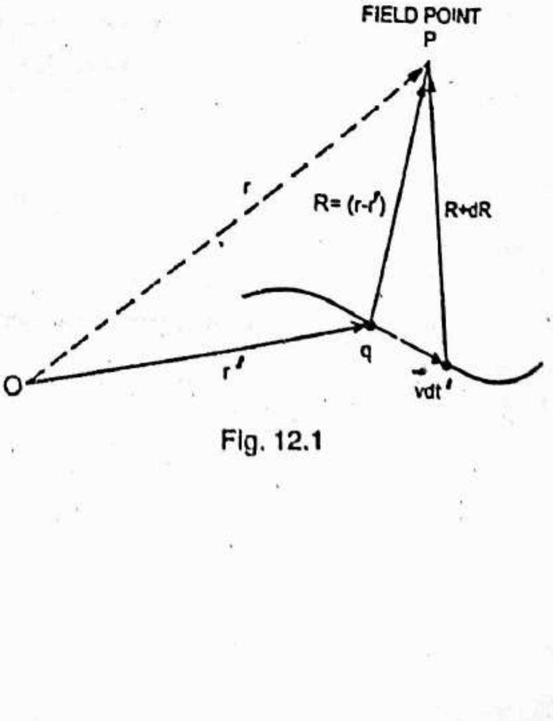

Fig. 12.1

Using this; equation (8) gives

$$\frac{dR}{dt'} = -\mathbf{v} \cdot \frac{\mathbf{R}}{R} \qquad \text{...(9)}$$

Substituting this value of dR/dt in equation (7b), we get

$$\frac{dt_1}{dt'} = 1 - \frac{\mathbf{v}}{c} \cdot \frac{\mathbf{R}}{R} = \frac{R - (\mathbf{v} \cdot \mathbf{R}/c)}{R}$$

$$dt' = \frac{R\, dt_1}{R - \left\{\frac{\mathbf{v}}{c}\right\} \cdot \mathbf{R}} \qquad \text{...(10)}$$

Making substitutions from (7a) and (10) in (6), we get

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{\delta(t_1)\, dt_1}{R - \left\{\frac{\mathbf{v} \cdot \mathbf{R}}{\mathbf{c}}\right\}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t_1)\, dt_1}{R - \left\{\frac{\mathbf{v} \cdot \mathbf{R}}{\mathbf{c}}\right\}} \end{aligned}\right] \qquad \text{...(11)}$$

Now using property of δ function, we have

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}\right]_{\text{at } t_1 = 0} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi} \left[\frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}\right]_{\text{at } t_1 = 0} \end{aligned}\right] \qquad \text{...(12)}$$

But $t_1 = 0$ implies that $t' - t + R/c = 0$ or $t' = t - R/c$, the *retarded time.*

Thus equations (12) may be expressed as

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}\right]_{\text{retarded}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi} \left[\frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}\right]_{\text{retarded}} \end{aligned}\right] \qquad \text{...(13)}$$

These potentials are *velocity dependent* and were first of all given by *A. Lienard and E Wiechert* for a moving point charge and after their names they are known as Lienard-Weichert potentials. These potentials may be expressed in a more compact form if we substitute

$$k = 1 - \frac{\mathbf{v} \cdot \mathbf{n}}{c} = 1 - \vec{\beta} \cdot \mathbf{n} \qquad \text{...(14)}$$

where $\vec{\beta} = \frac{\mathbf{v}}{c}$ and $\mathbf{n} = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{\mathbf{R}}{R}$, is a unit vector directed from source point q to the field point P. Then we may write equations (13) as,

$$\left.\begin{aligned} \phi(\mathbf{r}, t) &= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{kR}\right]_{\text{retarded}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi} \left[\frac{\vec{\beta} c}{kR}\right]_{\text{retarded}} \end{aligned}\right\} \text{where } \vec{\beta} = \mathbf{v}/c \qquad \text{...(15)}$$

The square brackets with subscript retarded mean that the quantity in brackets is to be evaluated at the retarded time $t' = t - R(t')/c$. For non-relativitic motion $\beta \rightarrow 0$; so that $k \rightarrow 1$, then the potentials (15) reduce to the well known *non-relativistic results* given by

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{q}{[R]_{\text{retarded}}} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r', t')\, dV'}{[R]_{\text{retarded}}}$$

$$= \frac{1}{4\pi\varepsilon_0} \int \left[\frac{\rho}{R}\right]_{\text{retarded}} dV' \qquad \text{...(16a)}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \left[\frac{\mathbf{v}}{R}\right]_{\text{retarded}} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')\, dV'}{[R]_{\text{retarded}}}$$

$$= \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}}{R}\right]_{\text{retarded}} dV' \qquad \text{...(16b)}$$

12.4. The Electromagnetic Fields from Lienard-Wiechert Potentials of a Moving Point Charge.

So far we have found the electromagnetic potentials A and ϕ for a charged particle undergoing arbitrary motion. Now we want to find the electromagnetic fields **E** and **B** given by

$$\mathbf{B} = \text{curl } \mathbf{A} \qquad \text{...(1)}$$

and

$$\mathbf{E} = -\text{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{...(2)}$$

Let us start with *Lienard-Wiechert potential* expressions (6) of section 12.3, given by

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')} \qquad \text{...(3)}$$

$$\mathbf{A}(\mathbf{r}, t) - \frac{\mu_0 q}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')} \qquad \text{...(4)}$$

Substituting these values in equation (2), we get

$$\mathbf{E}(\mathbf{r}, t) = -\left[\frac{q}{4\pi\varepsilon_0} \nabla \int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')} + \frac{\mu_0 q}{4\pi} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\right] \qquad \text{...(5)}$$

Now we note that the only dependence on the spatial coordinates r of the observation (field) point is through R. Hence the gradient operation is equivalent to

$$\text{grad} = \nabla \rightarrow \mathbf{n} \frac{\partial}{\partial R} \qquad \text{...(6)}$$

where **n** is a unit vector along $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$.

Therefore equation (5) takes the form

$$\mathbf{E}(\mathbf{r}, t) = -\left[\frac{q}{4\pi\varepsilon_0} \mathbf{n} \int_{-\infty}^{+\infty} \frac{\delta(t' - t + R/c)\, dt}{R(\mathbf{r}, \mathbf{r}')} + \frac{\mu_0 q}{4\pi} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\right]$$

$$= \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \left[\frac{\mathbf{n}}{R^2} \delta(t' - t + R/c) - \frac{\mathbf{n}}{Rc} \delta'(t' - t + R/c)\right] dt'$$

$$+ \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{v}(t')\, \delta'(t' - t + R/c)\, dt'}{R(\mathbf{r}, \mathbf{r}')}\Bigg]$$

The prime on the delta function means differentiation with respect to its argument.

--Now using $\mu_0\varepsilon_0 = 1/c^2$ and hence substituting $\mu_0 = 1/c^2\varepsilon_0$ and $\vec{\beta} = \mathbf{v}/c$, we get

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{R^2}\,\delta(t' - t + R/c) + \left(\frac{\overrightarrow{\beta} - \mathbf{n}}{cR}\right)\delta'(t' - t + R/c)\right]dt' \qquad \text{...(7)}$$

Similarly substituting value of **A** from (4) in equation (1) we get .

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0 q}{4\pi}\,\nabla \times \int_{-\infty}^{+\infty}\frac{\mathbf{v}(t')\,\delta(t' - t + R/c)\,dt'}{R(\mathbf{r}, \mathbf{r}')}\\ &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial R} \times \frac{\{\mathbf{v}(t')\,\delta(t' - t + R/c)\}}{R(\mathbf{r}, \mathbf{r}')}\right]dt'\\ &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\mathbf{n} \times \mathbf{v}\left\{\frac{\delta(t' - t + R/c)}{R^2} + \frac{\delta'(t' - t + R/c)}{Rc}\right\}\Bigg]dt'\\ &= \frac{\mu_0 qc}{4\pi}\int_{-\infty}^{+\infty}\left[\mathbf{n} \times \overrightarrow{\beta}\left\{\frac{\delta(t' - t + R/c)}{R^2} + \frac{\delta'(t' - t + R/c)}{R/c}\right\}\right]dt' \qquad \text{...(8)}\end{aligned}$$

Since $t_1 = t' + R/c$

i.e.
$$\frac{dt_1}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'} = 1 = \frac{\mathbf{v}\cdot\mathbf{R}}{c\mathbf{R}} \qquad \text{...(9)}$$

$$\because\left[\text{since } \frac{dR}{dt} = -\frac{\mathbf{v}\cdot\mathbf{R}}{cR} \text{ using eqn. (9) of section 12.3}\right]$$

$$= 1 - \overrightarrow{\beta}\cdot\mathbf{n} = \kappa \qquad (\text{where } \overrightarrow{\beta} = \mathbf{v}/c \text{ and } \kappa = 1 - \overrightarrow{\beta}\cdot\mathbf{n})$$

so that
$$dt' = \frac{dt_1}{\kappa} \qquad \text{...(10)}$$

Substituting (9) and (10) in equations (7) and (8), we get

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{\kappa R^2}\,\delta(t_1 - t) + \frac{(\overrightarrow{\beta} - \mathbf{n})}{\kappa cR}\,\delta'(t_1 - t)\right]dt_1 \qquad \text{...(11)}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\int_{-\infty}^{+\infty}\left[(\mathbf{n} \times \overrightarrow{\beta})\left\{\frac{-\delta(t_1 - t)}{\kappa R^2} + \frac{1}{\kappa cR}\,\delta'(t_1 - t)\right\}\right]dt_1 \qquad \text{...(12)}$$

Integrating by parts on the derivative of delta functinon, we readily get

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa R^2} - \frac{d}{dt_1}\left(\frac{\overrightarrow{\beta} - \mathbf{n}}{\kappa cR}\right)\right] \qquad \text{...(13)}$$

and
$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\left[\frac{\mathbf{n} \times \overrightarrow{\beta}}{\kappa R^2} - \frac{d}{dt_1}\left(\frac{\mathbf{n} \times \overrightarrow{\beta}}{\kappa cR}\right)\right]. \qquad \text{...(14)}$$

From (10), $dt_1 = \kappa\, dt'$, therefore equations (13) and (14) take the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa}\frac{1}{dt'}\left(\frac{\mathbf{n} - \overrightarrow{\beta}}{\kappa R}\right)\right] \qquad \text{...(15)}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\left[\frac{\overrightarrow{\beta} \times \mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\overrightarrow{\beta} \times \mathbf{n}}{\kappa R}\right)\right]. \qquad \text{...(16)}$$

Referring fig. 12.1, we have

$$\frac{dR}{dt'} = -\mathbf{v}\cdot\mathbf{n} \text{ and } \frac{d\mathbf{R}}{dt'} = -\mathbf{v}. \qquad \text{...(17)}$$

We have

$$\mathbf{R} = \mathbf{n}R$$

$\because$
$$\frac{d\mathbf{R}}{dt'} = \mathbf{n}\,\frac{dR}{dt'} + R\,\frac{d\mathbf{n}}{dt'}$$

$\therefore$ Rate of change of unit vector **n** with time is given by

$$\frac{d\mathbf{n}}{dt'} = -\,\frac{\mathbf{v}}{R} + \frac{\mathbf{n}\,(\mathbf{v}\cdot\mathbf{n})}{R}$$

$$= -\,\frac{\vec{\beta}c}{R} + \frac{\mathbf{n}\,(\vec{\beta}c\cdot\mathbf{n})}{R} = \frac{c}{R}\,\{\mathbf{n}\,(\vec{\beta}\cdot\mathbf{n}) - (\mathbf{n}\cdot\mathbf{n})\,\vec{\beta}\}$$

Therefore
$$\frac{1}{c}\,\frac{d\mathbf{n}}{dt'} = \frac{\mathbf{n}\times(\mathbf{n}\times\vec{\beta})}{R}. \qquad \text{...(18)}$$

Making this substitution, wherever it appears explicity in (15) and (16), we get

$$\mathbf{E}\,(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa^2R^2} + \frac{\mathbf{n}}{c\kappa}\,\frac{d}{dt'}\left(\frac{1}{\kappa R}\right) - \frac{\vec{\beta}}{\kappa^2R^2} - \frac{1}{c\kappa}\,\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right] \qquad \text{...(19)}$$

and
$$\mathbf{B}\,(\mathbf{r}, t) = \frac{\mu_0 cq}{4\pi}\left[\left\{\frac{\vec{\beta}}{\kappa^2R^2} + \frac{1}{c\kappa}\,\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right\}\times\mathbf{n}\right] \qquad \text{...(20)}$$

From equations (19) and (20), we observe that the magnetic induction is related simply to the electric field by the relation

$$\mathbf{B} = \frac{\mathbf{n}\times\mathbf{E}}{c} \qquad \text{...(21)}$$

where the equation is understood to be in terms of the retarded quantities in square brakets.

The remaining derivatives needed in equations (19) and (20) are

$$\frac{d\vec{\beta}}{dt'} = \dot{\vec{\beta}} \qquad \text{...(22)}$$

and
$$\frac{1}{c}\,\frac{d}{dt'}\,(\kappa R) = \frac{R}{c}\,\frac{dk}{dt'} + \frac{\kappa}{c}\,\frac{dR}{dt'}$$

$$= \frac{R}{c}\,\frac{d}{dt'}\,(1-\mathbf{n}\cdot\vec{\beta}) + (1-\mathbf{n}\cdot\vec{\beta})(-\mathbf{n}\cdot\vec{\beta})$$

$$\left(\text{since } \frac{1}{c}\,\frac{dR}{dt'} = -\mathbf{n}\cdot\vec{\beta}\right)$$

$$= \frac{R}{c}\left\{\frac{d\mathbf{n}}{dt'}\cdot\vec{\beta} - \mathbf{n}\cdot\frac{d\vec{\beta}}{dt'}\right\} - \mathbf{n}\cdot\vec{\beta} + (\mathbf{n}\cdot\vec{\beta})^2$$

$$= -\,\{\mathbf{n}\times(\mathbf{n}\times\vec{\beta})\}\cdot\vec{\beta} - \frac{(\mathbf{n}\cdot\dot{\vec{\beta}})\,R}{c} - \mathbf{n}\cdot\vec{\beta} + (\mathbf{n}\cdot\vec{\beta})^2$$

$$= -\,\{(\mathbf{n}\cdot\vec{\beta})\,\mathbf{n} - (\mathbf{n}\cdot\mathbf{n})\,\vec{\beta}\}\cdot\vec{\beta} - \frac{(\mathbf{n}\cdot\dot{\vec{\beta}})\,R}{c} - \mathbf{n}\cdot\vec{\beta} + (\mathbf{n}\cdot\vec{\beta})^2$$

$$= \beta^2 - \mathbf{n}\cdot\vec{\beta} - \frac{R}{c}\,(\mathbf{n}\cdot\dot{\vec{\beta}}) \qquad \text{...(23)}$$

Then equation (14) takes the form

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}}{\kappa^2R^2}+\frac{\mathbf{n}}{c\kappa}\left(-\frac{1}{\kappa^2R^2}\right)\frac{d}{dt'}(\kappa R)-\frac{\vec{\beta}}{c\kappa^2R}-\frac{\dot{\vec{\beta}}}{c\kappa^2R}-\frac{\vec{\beta}}{c\kappa}\left(-\frac{1}{\kappa^2R^2}\right)\frac{d}{dt'}(\kappa R)\right]$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^2R^2}-\frac{(\mathbf{n}-\vec{\beta})}{c\,\kappa^3R^2}\frac{d}{dt'}(\kappa R)-\frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^2R^2}-\frac{(\mathbf{n}-\vec{\beta})}{\kappa^3R^2}\left\{\beta^2-\mathbf{n}\cdot\vec{\beta}-\frac{R}{c}(\mathbf{n}\cdot\dot{\vec{\beta}})\right\}-\frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^3R^2}\left\{\kappa-\beta^2+\mathbf{n}\cdot\vec{\beta}+\frac{R}{c}(\mathbf{n}\cdot\dot{\vec{\beta}})\right\}-\frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})}{\kappa^3R^2}\left\{(1-\mathbf{n}\cdot\vec{\beta})-\beta^2+\mathbf{n}\cdot\vec{\beta}+\frac{R}{c}\,\mathbf{n}\cdot\dot{\vec{\beta}}\right\}-\frac{\dot{\vec{\beta}}}{c\kappa^2R}\right]$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2}+\frac{\mathbf{n}-\vec{\beta}}{\kappa^3Rc}\{(\mathbf{n}\cdot\dot{\vec{\beta}})-\kappa\,\dot{\vec{\beta}}\}\right]$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2}+\frac{(\mathbf{n}-\vec{\beta})}{\kappa^3Rc}\{(\mathbf{n}\cdot\dot{\vec{\beta}})-(1-\mathbf{n}\cdot\vec{\beta})\,\dot{\vec{\beta}}\}\right] \qquad \text{...(24)}$$

using
$$\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}})=(\mathbf{n}\cdot\dot{\vec{\beta}})(\mathbf{n}-\vec{\beta})-\{\mathbf{n}\cdot(\mathbf{n}-\vec{\beta})\}\,\dot{\vec{\beta}}$$
$$=(\mathbf{n}\cdot\dot{\vec{\beta}})\,(\mathbf{n}-\vec{\beta})-(1-\mathbf{n}\cdot\vec{\beta})\,\dot{\vec{\beta}}$$

We may write equation (24) as

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2}+\frac{1}{\kappa^3Rc}\{\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\right] \qquad \text{...(25)}$$

Now using $n=\dfrac{\mathbf{R}}{R}$ and $\kappa=1-\mathbf{n}\cdot\vec{\beta}=1-\dfrac{\mathbf{R}\cdot\vec{\beta}}{R}$ *i.e.* $\kappa R=R-\mathbf{R}\cdot\vec{\beta}$

we may write equation (25) as

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}-(R\,\vec{\beta})(1-\beta^2)}{R-\mathbf{R}\cdot\vec{\beta}}+\frac{\mathbf{R}\times(\mathbf{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c\,(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \qquad \text{...(26)}$$

and the *magnetic field induction is* given by

$$=\frac{q}{4\pi\varepsilon_0Rc}\left[\frac{\mathbf{n}\times(\mathbf{R}-R\,\vec{\beta})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3}+\frac{\mathbf{n}\times\{\mathbf{R}\times(\mathbf{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c\,(R-\mathbf{R}\cdot\vec{\beta})^2}\right]$$

Using $\mu_0\varepsilon_0 = 1/c^2$ and $\mathbf{R} \times \mathbf{R} = 0$, we may write above equation

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0\varepsilon_0 q}{4\pi}\left[\frac{(\vec{\beta} \times \mathbf{n})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{n}\times\{\mathbf{R}\times\{\mathbf{R}-R\vec{\beta}\}\times\dot{\vec{\beta}}\}\}}{cR\,(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \quad ...(27)$$

Equation (26) and (27) represent required *electromagnetic* fields due to a moving point charge.

From equations (26) and (27) it is clear that electromagnetic fields **E** and **B** contain two terms in square brackets : the first term depends on velocity $\mathbf{v} = \vec{\beta} c$ and is independent of acceleration while the second term depends on acceleration $\mathbf{a} = c\dot{\vec{\beta}}$. Thus electromagnetic fields **E** and **B** divide themselves naturally into two types of fields namely *velocity fields* which are independent of acceleration and *acceleration fields* which depend linearly on $\dot{\vec{\beta}}$. The velocity fields are essentially *static field* falling off as R^{-2} ; whereas the acceleration fields are typical radiation fields, both **E** and **B** being transverse to radius vector and varying as R^{-1}.

12.5. The Electromagnetic Fields of a Uniformly Moving Point Charge.

The electric and magnetic fields of a point charge in motion are given by equation (26) and (27) of section (12.4) *i.e.*

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R}-R\vec{\beta})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\{R\times(\mathbf{R}-\mathbf{R}\vec{\beta})\times\dot{\vec{\beta}}\}}{c\,(\mathbf{R}-\mathbf{R}\cdot\dot{\vec{\beta}})^3}\right]_{retarded} \quad ...(1)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{n}\times\mathbf{E}}{c} = \frac{\mu_0 c q}{4\pi}\left[\frac{(\vec{\beta}\times\mathbf{n})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{n}\times\{\mathbf{R}\times(\mathbf{R}-R\vec{\beta})\times\dot{\vec{\beta}}\}}{cR\,(R-\mathbf{R}\cdot\dot{\vec{\beta}})^3}\right]_{retarded} \quad ...(2)$$

For a *uniformly moving point charge*, $\mathbf{v} = c\beta =$ constant.

This implies $\dot{\beta} = 0$.

Therefore for a *uniformly moving point charge*, equation (1) takes the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R}-R\vec{\beta})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \quad ...(3)$$

Now as the particle is moving with constant velocity, we have

$$\left.\begin{aligned}[\vec{\beta}]_{retarded} &= \vec{\beta} \\ \text{and hence } [1-\beta^2]_{retarded} &= 1-\beta^2\end{aligned}\right] \quad ...(4)$$

Hence equation (3) may be expressed as

$$\mathbf{E}(\mathbf{r}, t) = \frac{q(1-\beta^2)}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R}-R\vec{\beta})}{(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded} \quad ...(5)$$

The magnetic field may be computed by the relation

$$\mathbf{B}(\mathbf{r}, t) = \left[\frac{\mathbf{n} \times \mathbf{E}(\mathbf{r}, t)}{c}\right]_{retarded} \qquad ...(6)$$

Equations (5) and (6) represent electromagnetic fields in terms of retarded position of the point charge. Now for a uniformly moving point charge, it is desirable to represent these fields in terms of the present position of the charge. For the purpose we consider the geometrical configuration of fig. 12.2. relating the present and retarded positions of the charge. The positions A and A' occupied by the charge represent respectively the retarded and present positions of the charge with reference to field point (or point of observation) P.

The retarded distance of field point $P(\mathbf{r})$ from the source point $A(\mathbf{r}')$ is shown by $[R(t')] = R'$ and the present distance of field point P from the source point A is shown by $R(t) = R$. The electromagnetic signal starting from the source point A' reaches the field point P in time R'/c. In the meanwhile the charge itself moving with velocity $\mathbf{v}$ reaches the point A, the present position therefore the distance $A'A = vR'/c = \beta R'$; θ' is the angle between $\mathbf{v}$ and $\mathbf{R}'$ and θ is the angle between $\mathbf{v}$ and $\mathbf{R}$, AN is the perpendicular drawn from the present position on $A'P$ and PM is the perpendicular drawn from the field point P on the direction of velocity $\mathbf{v}$ of the moving point charge.

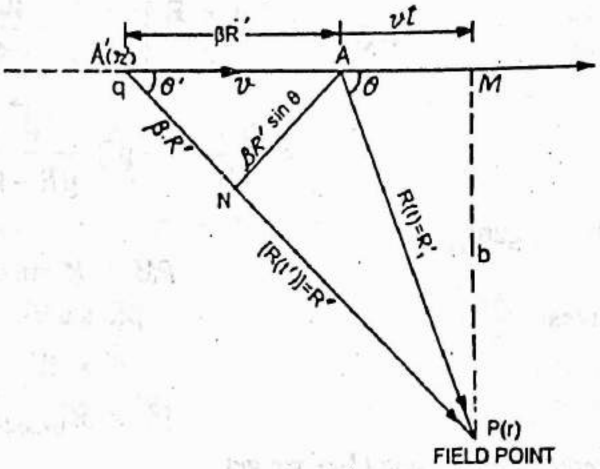

Fig. 12.2. Present and retarded positions of a point charge in uniform motion.

Now referring to fig. 12.2, we have

$$[\mathbf{R} - R\vec{\beta}]_{retarded} = \mathbf{R}' - R'\,\vec{\beta} = \mathbf{R} \qquad \text{(from triangle } AA'P\text{)} \;...(7)$$

and

$$b = R' \sin\theta' = R \sin\theta.$$

This implies $\beta R' \sin\theta' = \beta R \sin\theta$

i.e. $$\beta^2 R'^2 \sin^2\theta' = \beta^2 R^2 \sin^2\theta$$

i.e. $$\beta^2 R'^2 (1 - \cos^2\theta') = \beta^2 R^2 (1 - \cos^2\theta)$$

i.e. $$\beta^2 R'^2 - (\beta R' \cos\theta')^2 = \beta^2 R^2 - (\beta R \cos\theta)^2$$

i.e. $$\beta^2 R'^2 - (\vec{\beta} \cdot \mathbf{R}')^2 = \beta^2 R^2 - (\vec{\beta} \cdot \mathbf{R})^2 \qquad ...(8)$$

Squaring equation (7), we get

$$R'^2 - 2R'(\vec{\beta} \cdot \mathbf{R}') + R'^2 \beta^2 = R^2 \qquad ...(9)$$

Substracting equation (8) from (9), we get

$$R'^2 - 2R'(\vec{\beta} \cdot \mathbf{R}') + (\vec{\beta} \cdot \mathbf{R}')^2 = R^2 - \beta^2 R^2 + (\vec{\beta} \cdot \mathbf{R})^2 \qquad ...(10)$$

Now

$$\begin{aligned}
[\{R - \mathbf{R} \cdot \vec{\beta}\}^2]_{retarded} &= (R' - \mathbf{R}' \cdot \vec{\beta})^2 = R'^2 - 2R'(\mathbf{R}' \cdot \vec{\beta}) + (\mathbf{R}' \cdot \vec{\beta})^2 \\
&= R'^2 - 2R'(\vec{\beta} \cdot \mathbf{R}') + (\vec{\beta} \cdot \mathbf{R}')^2 \\
&= R^2 - \beta^2 R^2 + (\vec{\beta} \cdot \mathbf{R})^2 \qquad \text{using (10)} \\
&= R^2 - \beta^2 R^2 + \beta^2 R^2 \cos^2\theta \\
&= R^2 - \beta^2 R^2 (1 - \cos^2\theta)
\end{aligned}$$

$$[\{R-\mathbf{R}\cdot\vec{\beta}\}^2]_{retarded} = R^2(1-\beta^2\sin^2\theta) \qquad \text{...(11)}$$

Substituting values from (7) and (11) in equation (5), we get

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{\mathbf{R}(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}} \qquad \text{...(12)}$$

Now from (2) using $\vec{\beta} = 0$, we have

$$\mathbf{B}(\mathbf{r},t) = \left[\frac{\mathbf{n}\times\mathbf{E}}{c}\right]_{retarded} = \frac{\mu_0 cq}{4\pi}\left[\frac{(\vec{\beta}\times\mathbf{R})(1-\beta^2)}{(R-\mathbf{R}\cdot\vec{\beta})^3}\right]_{retarded}$$

$$= \frac{\mu_0 cq}{4\pi}(1-\beta^2)\,\frac{[\vec{\beta}\times\mathbf{R}]_{retarded}}{[(R-\mathbf{R}\cdot\vec{\beta})_{retarded}]^3} \qquad \text{...(13)}$$

Now from figure

$$PM = R'\sin\theta' = R\sin\theta.$$

This gives

$$\beta R'\sin\theta' = \beta R\sin\theta$$

$\therefore$

$$\vec{\beta}\times\mathbf{R}' = \vec{\beta}\times\mathbf{R}$$

i.e.

$$[\vec{\beta}\times\mathbf{R}]_{retarded} = \vec{\beta}\times\mathbf{R} \qquad \text{...(14)}$$

Using equation (11) and (14), we get

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 cq}{4\pi}\,\frac{(\vec{\beta}\times\mathbf{R})(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}} \qquad \text{...(15)}$$

Equations (12) and (15) represent desired values of electromagnetic field vectors **E** and **B** for a uniformly moving charge.

Discussion. From equations (12) and (15), we come to the following conclusions.

(i) Static Case. For smaller velocities such that $\beta \to 0$; the electromagnetic field vectors **E** and **B** take the form

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\,\frac{q\mathbf{R}}{R^3} \qquad \text{...(16)}$$

which is Coulomb's law. This shows that for non-relativistic velocities the field is *spherically symmetric* ; and

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 cq}{4\pi}\,\frac{\vec{\beta}\times\mathbf{R}}{R^3} = \frac{\mu_0 q}{4\pi}\,\frac{\mathbf{v}\times\mathbf{R}}{R^3} \qquad \left[\text{since } \vec{\beta} = \frac{\mathbf{v}}{c}\right]$$

$$= \frac{\mu_0 I\,dt}{4\pi}\,\frac{\mathbf{v}\times\mathbf{R}}{R^3} = \frac{\mu_0}{4\pi}\,\frac{I(\mathbf{v}\,dt)\times\mathbf{R}}{R^3}$$

i.e.

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\,\frac{I\,\vec{dl}\times\mathbf{R}}{R^3} \qquad \text{...(17)}$$

which is *Biot-Savart field.*

(ii) The electromagnetic fields vary inversely as the square of distance R.

(iii) For high velocities $\beta \to 1$ of $v \to c$, both field magnitudes depend on the angles θ between the direction of motion of charge and radius vector **R**. From equation (12) it is obvious that electric field is greater for greater values of θ, *i.e.* the field is not spherically symmetric as in the static case ; but it is stronger in the direction perpendicular to velocity. If $\mathbf{E}_\perp$ and $\mathbf{E}_\parallel$ are electric fields perpendicular and parallel to velocity, then

$$\left.\begin{aligned} E_{\perp}\ (\theta = 90^{\circ}) &= \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{R}}{\mathbf{R}^3\sqrt{(1-\beta^2)}} \\ \text{and} \qquad E_{\parallel}\ (\theta = 0^{\circ}) &= \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{R}\,(1-\beta^2)}{R^3} \end{aligned}\right] \qquad \text{...(18)}$$

This means that the electric field is increased in a direction perpendicular to direction of motion in the ratio of $1/\sqrt{(1-\beta^2)}$; while in the direction of motion the field is decreased in the ratio $(1-\beta^2)$. Thereby indicating that *the electric field of a moving charge is* $[1/(1-\beta^2)^{3/2}]$ *times stronger in the direction perpendicular to its motion than the field at the same distance along the direction of motion.* At very high velocities the field thus resembles more and more the field in a plane wave. For a short time, as a high velocity charge passes an observer, he sees a purely transverse electric and magnetic fields.

(iv) The magnetic field **B** is related to **E** through the relation*

$$\mathbf{B}\,(\mathbf{r}, t) = \frac{\vec{\beta} \times \mathbf{E}\,(\mathbf{r}, t)}{c} = \frac{\mathbf{v} \times \mathbf{E}\,(\mathbf{r}, t)}{c^2}.$$

This equation indicates that **B** is at right angles to both **v** and **E**.

(v) The Poynting vector is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

$$= \frac{\mathbf{E} \times (\mathbf{v} \times \mathbf{E})}{\mu_0 c^2} = \frac{(\mathbf{E}\cdot\mathbf{E})\,\mathbf{v} - (\mathbf{E}\cdot\mathbf{v})\,\mathbf{E}}{\mu_0 c^2}$$

When v approaches c ; we have seen that *electric field is perpendicular to* **v** and so $\mathbf{E}\cdot\mathbf{v} = 0$

Consequently

$$\mathbf{S} = \frac{E^2\mathbf{v}}{\mu_0 c^2} = \varepsilon_0 E^2 \mathbf{v}$$

Since $\mathbf{E} \propto \frac{1}{R^2}$; we note that

$$\mathbf{S} \propto \frac{1}{R^4},$$

In order to calculate the energy radiated by the charged particle, we must integrate the normal component of S over the surface of sphere. If we take the radius of sphere to be R, then the energy radiated will be proportional to $1/R^2$ since the element of surface area involves R^2. Therefore as $R \to \infty$, the energy radiated tends to zero. Thus we conclude that *the charged particles moving with constant velocity can not radiate energy.*

12.6. Radiation from an Accelerated Charge at Low Velocity – Larmor's Formula.

The electromagnetic fields of a point charge in motion are

$$\mathbf{E}\,(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\mathbf{R} - R\vec{\beta})(1-\beta^2)}{(R - \mathbf{R}\cdot\vec{\beta})^3} + \frac{\mathbf{R} \times (\mathbf{R} - R\vec{\beta}) \times \dot{\vec{\beta}}}{c\,(R - \mathbf{R}\cdot\vec{\beta})^3}\right] \qquad \text{...(1)}$$

---

*Comparing equation (12) and (15) and keeping in mind $\mu_0\varepsilon_0 = 1/c^2$, we immediately get $\mathbf{B} = (\vec{\beta} \times \mathbf{E})/c$.

and
$$\mathbf{B}(\mathbf{r}, t) = \left[\frac{\mathbf{n} \times \mathbf{E}}{c}\right] = \left[\frac{\mathbf{R} \times \mathbf{E}}{Rc}\right]. \qquad ...(2)$$

In equation (1) the first term in square bracket represents velocity field ; while the second term represents acceleration field. In preceding section, we have seen that velocity fields are essentially static fields and they are incapable of radiating energy. Hence for radiation we shall consider only *acceleration fields* given by

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R} \times (\mathbf{R} - R\vec{\beta}) \times \dot{\vec{\beta}}}{c\,(R - R\vec{\beta})^3}\right] \qquad ...(3)$$

with
$$\mathbf{B}_a(\mathbf{r}, t) = \left[\frac{\mathbf{n} \times \mathbf{E}_a}{c}\right] = \left[\frac{\mathbf{R} \times \mathbf{E}_a}{Rc}\right] \qquad ...(4)$$

where the square bracket notation reminds us that all quantities in square brackets must be evaluated at retarded time. If the velocity of charge is small compared with the velocity of light, then

$$\beta = \frac{v}{c} \ll 1 \text{ ; so that}$$

$$\mathbf{R} - R\vec{\beta} \to \mathbf{R}$$

Then the *acceleration field equation* (3) takes the form

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\vec{\mathbf{R}} \times (\mathbf{R} \times \dot{\vec{\beta}})}{R^3}\right] \qquad ...(3a)$$

$$= \frac{q}{4\pi\varepsilon_0 c}\,\frac{(\mathbf{R} \cdot \dot{\vec{\beta}})\,\mathbf{R} - (\mathbf{R} \cdot \mathbf{R})\,\dot{\vec{\beta}}}{R^3}$$

$$= \frac{q}{4\pi\varepsilon_0 c}\left\{\frac{(\mathbf{R} \cdot \dot{\vec{\beta}})\,\mathbf{R} - [R^2\,\dot{\vec{\beta}}]}{R^3}\right] \qquad ...(3b)$$

Using $\mathbf{n} = \mathbf{R}/R$ where $\mathbf{n}$ is a unit vector along $\mathbf{R}$, we get

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{(\mathbf{n} \cdot \dot{\vec{\beta}}\,\mathbf{n} - \dot{\vec{\beta}}}{R}\right] \qquad ...(4)$$

If $\mathbf{a}$ is the acceleration of the particle, then $\mathbf{a} = \dot{\mathbf{v}} = c\dot{\vec{\beta}}$ or $\dot{\vec{\beta}} = \mathbf{a}/c$ and we get

$$\mathbf{E}_a(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{(\mathbf{n} \cdot \mathbf{a})\,\mathbf{n} - \mathbf{a}}{R}\right] \qquad ...(5)$$

Now
$$\mathbf{B}_a = \left[\frac{\mathbf{n} \times \mathbf{E}_a}{\mu_0 c}\right] \qquad ...(6)$$

Therefore the *instantaneous energy flux given by Poynting vector* is

$$\mathbf{S}_a^{*} = \mathbf{E}_a \times \mathbf{H}_a = \frac{\mathbf{E}_a \times \mathbf{B}_a}{\mu_0}$$

---

*All values on R.H.S. correspond to retarded position of charge.

$$= \left[\frac{\mathbf{E}_a \times (\mathbf{n} \times \mathbf{E}_a)}{\mu_0 c}\right] = \frac{1}{\mu_0 c}\ [(\mathbf{E}_a \cdot \mathbf{E}_a) - (\mathbf{E}_a \cdot \mathbf{n})\,\mathbf{E}_\alpha]$$

But $\mathbf{E}_a$ is perpendicular to **R** or **n** as is obvious from (3), thus $\mathbf{E}_a \cdot \mathbf{n} = 0$; hence we get

$$\mathbf{S}_a = \frac{1}{\mu_0 c}\ |E_a|^2\,\mathbf{n}$$

Substituting value of E_a from (5) we get

$$\mathbf{S}_a = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0 c^2}\right)^2 \left|\left\{\frac{(\mathbf{n}\cdot\mathbf{a})\,\mathbf{n}-\mathbf{a}}{R}\right\}^2\right|\ \mathbf{n}$$

$$= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0 c^2}\right)^2 \left[\frac{(\mathbf{n}\cdot\mathbf{a})^2 - 2\,(\mathbf{n}\cdot\mathbf{a})\,(\mathbf{a}\cdot\mathbf{n}) + a^2}{R^2}\right]$$

Using $u_0\varepsilon_0 = 1/c^2$ and $\mathbf{n}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{n}$, we obtain

$$\mathbf{S}_a = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{4\pi c^3}\left[\frac{a^2 - (\mathbf{n}\cdot\mathbf{a})}{R}\right]\mathbf{n}$$

If θ is the angle between **n** and **a**, then $\mathbf{n}\cdot\mathbf{a} = a\cos\theta$; So that we get

$$\mathbf{S}_a = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c^3}\left[\frac{a^2 - a^2\cos^2\theta}{R^2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2a^2}{4\pi c^3 . R^2}\sin^2\theta\,\mathbf{n} \qquad \text{...(6)}$$

This expression represents the power flow per unit area. The total power radiated may be obtained by integrating this Poynting vector over a closed surface surrounding the charge. A convenient choice for such a surface is a sphere centred at the retarted position of the charge. It is convenient to choose z. axis along acceleration a (fig. 12.3). Then

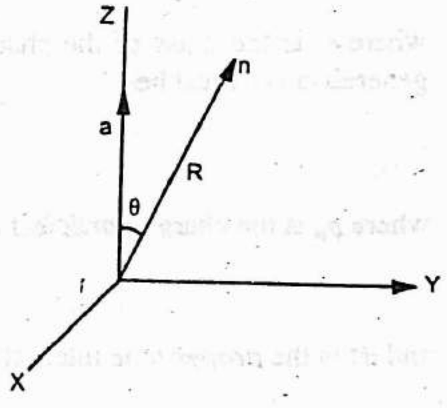

Fig. 12.3

$$P = \int_S \mathbf{S}_a \cdot \mathbf{n}\,da$$

(where *da* is an element of area)

$$= \int_S \frac{1}{4\pi\varepsilon_0}\,\frac{q^2a^2}{4\pi c^3 R}\sin^2\theta\ \mathbf{n}\cdot\mathbf{n}\,da$$

But $\mathbf{n}\cdot\mathbf{n} = 1$ and

$da = (R\,d\theta)\,(R\sin\theta\,d\phi)$; therefore

$$P = \int_0^\pi\int_0^{2\pi}\frac{1}{4\pi\varepsilon_0}\,\frac{q^2a^2\sin^2\theta}{4\pi c^3 R^2}\ (R\,d\theta)\quad(R\sin\theta\,d\phi)$$

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2a^2}{4\pi c^3}\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi} d\phi$$

But $\int_0^{2\pi} d\phi = 2\pi$ and $\int_0^\pi \sin^3\theta\,d\theta = \frac{4}{3}$; therefore we get

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2a^2}{4\pi c^3}\left(\frac{4}{3}\right)\cdot(2\pi) = \frac{1}{4\pi\varepsilon_0}\,\frac{2q^2a^2}{3c^3} \qquad \text{...(9)}$$

This is familiar *Larmor's formula* for the power radiated by a slowly moving (non-relativistic) accelerated charge.

12.7. Radiation from an Accelerated Charge at High Velocity

Relativistic Generalisation of Larmor's Formula.

Larmor's non-relativistic formula is

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2q^2a^2}{3c^3} \qquad \text{...(1)}$$

This formula can be generalised for arbitrary velocities of charge by using the fact that radiated electromagnetic energy behaves under Lorentz transformation like the fourth component of a four-vector*. Since $P = dE_{rad}/dt$ *i.e.* $dE_{rad} = Pdt$ and time t also behaves under Lorentz transformations like the fourth component of *space-time* four-vector; this means that the power P is a Lorentz invariant quantity. If we can find a Lorentz invariant which reduces to non-relativistic Larmor's formula (1) for $\beta = v/c \ll 1$, then we have the required relativistic generalisation of Larmor's formula. There are, of course, many Lorentz invariants which reduce to desired for $\beta \to 0$. But from the general form of electromagnetic fields for a moving charge, it is evident that the general result must involve only $\vec{\beta}$ and $\vec{\beta}$. With this restriction on the order of derivatives which can appear, we get unique result.

To find the appropriate generalisation of Larmor's formula let us use the fact that $\mathbf{a} = \mathbf{F}/m = \left(\dfrac{d\mathbf{p}/dt}{m}\right)$ and write Larmor's *non-relativistic formula* (1) in the form

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2q^2(\mathbf{a}\cdot\mathbf{a})}{3c^3} = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{m^2c^2} \left(\frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{p}}{dt}\right) \qquad \text{...(2)}$$

where m is the mass of the charge and $\mathbf{p}$ its momentum. Equation (2) suggests that its Lorentz invariant generalisation must be

$$P = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2}{3} \frac{q^2}{m^2c^2} \left(\frac{dp_\mu}{d\tau} \frac{dp_\mu}{d\tau}\right) \qquad \text{...(3)}$$

where p_μ is the charge *particle's momentum energy four vector* given by

$$p_\mu = \left(\mathbf{p}, \frac{iE}{c}\right) \qquad \text{...(4)}$$

and $d\tau$ is the *proper time* interval given by

$$\left.\begin{aligned} dt &= \frac{d\tau}{\sqrt{(1 - v^2/c^2)}} = \frac{d\tau}{\sqrt{(1-\beta^2)}} = \gamma\, d\tau \\ \text{where} \qquad \gamma &= \frac{1}{\sqrt{(1-\beta^2)}} \end{aligned}\right\} \qquad \text{...(5)}$$

To check that equations (3) reduces properly to equation (2) as $\beta \to 0$, we evaluate the four-vector scalar product.

$$\frac{dp_\mu}{d\tau} \frac{dp_\mu}{d\tau} = \left(\frac{dp_1}{d\tau}\right)^2 + \left(\frac{dp_2}{d\tau}\right)^2 + \left(\frac{dp_3}{d\tau}\right)^2 + \left(\frac{dp_4}{d\tau}\right)^2$$

---

*If a pulse of electromagnetic radiatioln of finite spatial extent exists in charge and current free space, then total electromagnetic momentum and energy transform like a four-vector.

$$= \left(\frac{d\mathbf{p}}{d\tau}\right)^2 + \left\{\frac{d}{d\tau}\left(\frac{iE}{c}\right)\right\}^2 = \left(\frac{d\mathbf{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2$$

But $d\tau = \dfrac{dt}{\gamma}$

$$\therefore \qquad \frac{dp_\mu}{d\tau}\,\frac{dp_\mu}{d\tau} = \gamma^2\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{\gamma^2}{c^2}\left(\frac{dE}{dt}\right)^2$$

Making this substitution in equation (3), we get

$$P = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\,\frac{q^2\gamma^2}{m^2c^3}\left[\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right] \qquad \text{...(6)}$$

Now in order to express above equation in terms of velocity and acceleration ; let us use relativistic formulae

$$E = \frac{mc^2}{\sqrt{(1-\beta^2)}} \quad \text{and} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{(1-\beta^2)}} \qquad \text{...(7)}$$

where m is the rest mass.

From (7), we have

$$\frac{dE}{dt} = \frac{d}{dt}\{mc^2(1-\beta^3)^{-1/2}\} = \frac{mc^2\,\vec{\beta}\cdot\dot{\vec{\beta}}}{(1-\beta^2)^{3/2}} = mc^2\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}} \qquad \text{...(8)}$$

and

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left\{\frac{m\mathbf{v}}{\sqrt{(1-\beta^2)}}\right\} = \frac{d}{dt}\left\{\frac{mc\,\vec{\beta}}{\sqrt{(1-\beta^2)}}\right\}$$

$$= \frac{mc\,[\dot{\vec{\beta}}(1-\beta^2) + \dot{\vec{\beta}}(\beta\cdot\dot{\vec{\beta}})]}{(1-\beta^2)^{3/2}}$$

i.e.

$$\frac{d\mathbf{p}}{dt} = mc\gamma^3\,[\dot{\vec{\beta}}\cdot(1-\beta^2) + \vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})] \qquad \text{...(9)}$$

Using equations (8) and (9), equation (6) becomes

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^3}{m^2c^3}\,[m^2c^2\gamma^6\{\dot{\vec{\beta}}(1-\beta^2) + \vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\}^2 - m^2c^2\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2]$$

on simplification, we get

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2}{c}\,\gamma^6\,[(\dot{\vec{\beta}})^2 - \beta^2\dot{\beta}^2 + (\vec{\beta}\cdot\dot{\vec{\beta}})^2]$$

But $(\vec{\beta}\times\dot{\vec{\beta}})^2 = (\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \beta^2\dot{\beta}^2$,

therefore

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^6}{c}\,[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2] \qquad \text{...(10)}$$

This expression represents the *relativistic generalisation of Larmor's formula.*

Now let us consider the following particular cases ;

Case (i). If the motion of the particle is non-relativistic, then $\beta \to 0$ and $\gamma = \frac{1}{\sqrt{(1-\beta^2)}} \to 1$, therefore equation (10) takes the form

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2}{c}\,(\vec{\dot{\beta}})^2$$

But $\vec{\dot{\beta}} = \mathbf{a}/c$; therefore we get

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\mathbf{a}^2}{c^3} \qquad \text{...(11)}$$

This well known *non-relativistic Larmor's formula.* This verifies the relativistic generalisation of Larmor's formula given by (10).

Case (ii). If the motion of the particle is such that its velocity and acceleration are parallel (or collinear) ; then

$$|\vec{\beta} \times \vec{\dot{\beta}}| = \beta\,\dot{\beta}\sin 0 = 0$$

So that from equation (10)

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^6}{c}\,\vec{\dot{\beta}}^2$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{2q^2\gamma^6\mathbf{a}^2}{3c^3} \qquad \text{...(12)}$$

This principal example of this type of radiations is ***Bremsstrahlung radiations*** which are produced from accelerated electrons with the assumption that the direction of motion of electrons does not change.

Case. (iii). If the motion of the particle in such that its velocity and acceleration are perpendicular ; then

$$|\vec{\beta} \times \vec{\dot{\beta}}| = \beta\,\dot{\beta}\sin 90 = \beta\,\dot{\beta}$$

So that from equation (10), we get

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^6}{c}\,(\dot{\beta}^2 - \beta^2\dot{\beta}^2)$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^6}{c}\,\dot{\beta}^2(1-\beta^2)$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^4}{c}\,\dot{\beta}^2$$

i.e.

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2\gamma^4 a^2}{c^3} \qquad \text{...(13)}$$

Example of this type of radiations occurs in circular accelerators like synchroton or cyclotron where energy loss is drastic.

12.8. Angular Distribution of Radiation Emitted by an Accelerated Charge.

For relativistic motion of charged particle the acceleration field depends on the velocity as well as acceleration and is given by

$$\left.\begin{aligned} \mathbf{E}_a &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}\times(\mathbf{R}-R\overrightarrow{\beta)}\times\overrightarrow{\dot{\beta}}}{c\,(R-\mathbf{R}\cdot\overrightarrow{\beta)}^3}\right] \\ \text{and}\qquad \mathbf{B}_a &= \frac{\mathbf{n}\times\mathbf{B}_a}{c} = \frac{\mathbf{R}\times\mathbf{E}_a}{Rc} \end{aligned}\right\} \qquad \text{...(1)}$$

The *Poynting Vector is*

$$\begin{aligned} [\mathbf{S}_a] &= [\mathbf{E}_a\times\mathbf{H}_a] = \left[\frac{\mathbf{E}_a\times\mathbf{B}_a}{\mu_0}\right] \\ &= \left[\frac{\mathbf{E}_a\times(\mathbf{n}\times\mathbf{E}_a)}{\mu_0 c}\right] = \left[\frac{(\mathbf{E}_a\cdot\mathbf{E}_a)\,\mathbf{n}-(\mathbf{E}_a\cdot\mathbf{n})\,\mathbf{E}_a}{\mu_0 c}\right] \\ &= \left[\frac{\mathbf{E}_a^2\mathbf{n}}{\mu_0 c}\right] \quad [\text{Since } \mathbf{E}_a\cdot\mathbf{n} = \mathbf{E}_a\cos 90^\circ = 0] \end{aligned} \qquad \text{...(2a)}$$

The *radial component of Poynting vector is*

$$[\mathbf{S}_a\cdot\mathbf{n}] = \left[\frac{\mathbf{E}_a^2}{\mu_0 c}\right] \qquad \text{...(2b)}$$

Here $\mathbf{S}_a\cdot\mathbf{n}$ is the energy emitted per unit area per unit time detected at an observation point at time t due to radiation emitted by the charge at time $t' = t - R(t')/c$. If we want to calculate the energy radiated during a finite period, say from $t' = T_1$ to $t' = T_2$, we would write

$$\begin{aligned} W &= \int_{t=T_1+[\{R(T_1)\}/c]}^{t=T_2+[\{R(T_2)\}/c]} [\mathbf{S}_a\cdot\mathbf{n}]\,dt \\ &= \int_{t'=T_1}^{t'=T_2} (\mathbf{S}_a\cdot\mathbf{n})\,\frac{dt}{dt'}\,dt' \end{aligned}$$

We thus note that the useful and meaningful quantity is $(\mathbf{S}_a\cdot\mathbf{n})\,(dt/dt')$, which represents the *power radiated per unit area is terms of charge's own time.*

The *power radiated per unit solid angle is defined as*

$$\begin{aligned} \frac{dP(t')}{d\Omega} &= (\text{Power radiated per unit area}) \times \frac{4\pi R^2}{4\pi} \\ &= R^2 \times (\text{Power radiated per unit area}) \\ &= R^2\,(\mathbf{S}_a\cdot\mathbf{n})\,\frac{dt}{dt'} \end{aligned} \qquad \text{...(3)}$$

As $t' = t - R/c$.

$$\therefore \qquad \frac{dt'}{dt} = 1 - \frac{1}{c}\frac{dR}{dt} = 1 - \frac{1}{c}\frac{dR}{dt}\frac{dt'}{dt}$$

i.e.

$$\left(1+\frac{1}{c}\frac{dR}{dt'}\right)\frac{dt'}{dt} = 1 \quad \text{or} \quad \frac{dt'}{dt} = \frac{1}{1+\dfrac{1}{c}\dfrac{dR}{dt'}}$$

But
$$\frac{dR}{dt'} = -\frac{\mathbf{v}\cdot\mathbf{R}}{R} = -\mathbf{v}\cdot\mathbf{n} \text{ since } \mathbf{n} = \frac{\mathbf{R}}{R}$$

∴
$$\frac{dt'}{dt} = \frac{1}{1-(\mathbf{v}\cdot\mathbf{n}/c)} = \frac{1}{1-\mathbf{n}\cdot\vec{\beta}}$$

∴
$$\frac{dt}{dt'} = 1-\mathbf{n}\cdot\vec{\beta} = \kappa \qquad \text{...(4)}$$

Using (4), equation (3) gives

$$\frac{dP(t')}{d\Omega} = \kappa R^2(\mathbf{S}_a\cdot\mathbf{n}) \qquad \text{...(5)}$$

$$= \kappa\mathbf{R}^2\,\frac{E_a^2}{\mu_0 c} \quad \text{[using } (2b)\text{]}$$

$$= \frac{\kappa R^2}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left[\frac{\mathbf{R}\times(\mathbf{R}-\mathbf{R}\vec{\beta})\times\dot{\vec{\beta}}}{c\,(R-R\cdot\vec{\beta})^3}\right]^2$$

$$= \frac{\kappa R^2}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{c\,(1-\mathbf{n}\cdot\vec{\beta})^3}\right]^2$$

Using $\kappa = 1-\mathbf{n}\cdot\vec{\beta}$ and $\mu_0\varepsilon_0 = \frac{1}{c^2}$, we get

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c}\,\frac{[\mathbf{n}\times\{(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}}\}]^2}{c\,(1-\mathbf{n}\cdot\vec{\beta})^5} \qquad \text{...(6)}$$

This equation represents the general expression for the angular distribution of power radiated by an accelerated charge. Let us now consider the following particular cases.

Case (i). When velocity and acceleration are parallel, then

$$\vec{\beta}\times\dot{\vec{\beta}} = 0$$

Therefore equation (6) reduces to

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c}\,\frac{[\mathbf{n}\times(\mathbf{n}\times\dot{\vec{\beta}})]^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c}\,\frac{[(\mathbf{n}\cdot\dot{\vec{\beta}})\,\mathbf{n}-(\mathbf{n}\cdot\mathbf{n})\,\dot{\vec{\beta}}]^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c}\,\frac{[\{(\mathbf{n}\cdot\dot{\vec{\beta}})\,\mathbf{n}-\dot{\vec{\beta}}\}]^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c}\,\frac{[(\mathbf{n}\cdot\dot{\vec{\beta}})^2-2\,(\mathbf{n}\cdot\dot{\vec{\beta}})^2]+\dot{\vec{\beta}}^2}{(1-\mathbf{n}\cdot\vec{\beta})^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{\vec{\dot{\beta}}^2 - (\mathbf{n}\cdot\vec{\dot{\beta}})^2}{(1-\mathbf{n}\cdot\vec{\beta})^5} \qquad \text{...(7)}$$

If θ is the angle of observation between **n** and common direction of $\vec{\dot{\beta}}$ and $\vec{\beta}$, then

$$\mathbf{n}\cdot\vec{\dot{\beta}} = \dot{\beta}\cos\theta \quad \text{and} \quad \mathbf{n}\cdot\vec{\beta} = \beta\cos\theta$$

So that equation (7) reduces to

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\beta}^2 - \dot{\beta}^2\cos^2\theta}{(1-\beta\cos\theta)^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}$$

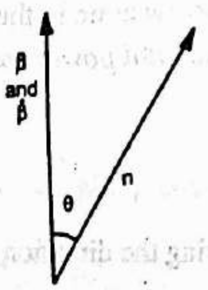

Fig. 12..4

But $\vec{\dot{\beta}} = \frac{\mathbf{a}}{c}$

$$\therefore \quad \frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad \text{...(8)}$$

This is required relativistic expression.

For non-relativistic case $\beta \ll 1$, so that expression (8) takes the form

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\sin^2\theta \qquad \text{...(9)}$$

Thus for an accelerated charge in non-relativistic motion the angular distribution shows a simple $\sin^2\theta$ behaviour. But as $\beta \to 1$, the angular distribution is tipped forward more and more and increase in magnitude as indicated schematically in fig. 11.5. Θ_{max} is the angle for which intensity is maximum.

The value of θ_{max} may be obtained by the condition

$$\frac{d}{d\theta}\left(\frac{dP}{d\Omega}\right) = 0 \quad \text{for} \quad \theta = \theta_{max}$$

i.e.

$$\frac{d}{d\theta}\left[\frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right]_{\theta=\theta_{max}} = 0$$

i.e.

$$\frac{d}{d\theta}\left\{\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right\}\Bigg|_{\theta=\theta_{max}} = 0$$

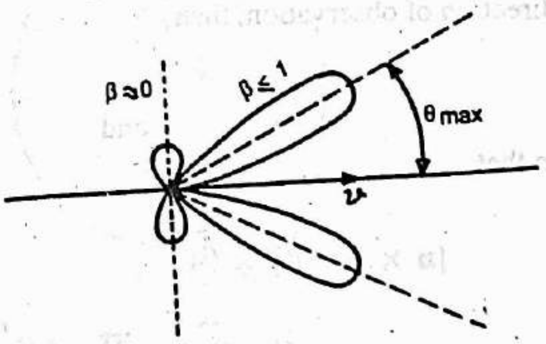

Fig. 12.5. Radiation pattern for charge accelerated along its direction of motion.

This gives $[3\beta\cos^2\theta + 2\cos\theta - 5\beta]_{\theta=\theta_{max}} = 0.$

or

$$\cos\theta_{max} = \frac{\sqrt{(1+15\beta^2)} - 1}{3\beta}$$

i.e.
$$\theta_{\max} = \cos^{-1}\left[\frac{\sqrt{(1+15\beta^2)}-1}{3\beta}\right] \qquad \text{...(10a)}$$

This in the limit $\beta \to 1$ reduces to

$$\theta_{\max} = \tfrac{1}{2}\sqrt{(1-\beta^2)} = \frac{1}{2\gamma} \qquad \text{...(10b)}$$

For relativistic particles $\theta_{\max}$ is very small, thereby indicating that the angular distribution is confined to a very narrow cone in the direction of motion.

The *total power radiated* can be obtained by integrating equation (8) over all angles *i.e.*

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{1}{4\pi\varepsilon_0}\, \frac{q^2 a^2}{4\pi c^3} \int \frac{1-\sin^2\theta}{(1-\beta\cos\theta)^5}\, d\Omega$$

Choosing the direction of $\vec{\beta}$ (or $\dot{\vec{\beta}}$) along Z-axis, we get

$$\begin{aligned}
P &= \frac{1}{4\pi\varepsilon_0}\, \frac{q^2 a^2}{4\pi c^3} \int_0^{\pi}\int_0^{2\pi} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\, \frac{(R\, d\theta)\,(R\sin\theta\, d\phi)}{R^2} \\
&= \frac{1}{4\pi\varepsilon_0}\, \frac{q^2 a^2}{4\pi c^3} \int_0^{\pi} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\, d\theta \int_0^{2\pi} d\phi \\
&= \frac{1}{4\pi\varepsilon_0}\, \frac{q^2 a^2}{4\pi c^3}\, \frac{4}{3\,(1-\beta^2)^3} \cdot (2\pi) \\
&= \frac{1}{4\pi\varepsilon_0}\, \frac{q^2 a^2}{4\pi c^3}\, \frac{8\pi}{3\,(1-\beta^2)^3}
\end{aligned}$$

i.e.
$$P = \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2 a^2 \gamma^6}{c^3} \qquad \text{...(11)}$$

This it exactly same result as equation (12) of section 12.7.

Case (ii). When velocity and acceleration are perpendicular, then we choose a coordinate system such that instantaneously β is along Z-axis and $\dot{\vec{\beta}}$ along X-axis. If (θ, ϕ) are polar coordinates defining the direction of observation, then

$$\left.\begin{aligned}
&\mathbf{n}\cdot\vec{\beta} = \beta\cos\theta \\
\text{and}\quad &\mathbf{n}\cdot\dot{\vec{\beta}} = \dot{\beta}\,\sin\theta\cos\phi
\end{aligned}\right] \qquad \text{...(12)}$$

so that

$$\begin{aligned}
&[\mathbf{n}\times(\mathbf{n}-\vec{\beta})\times\dot{\vec{\beta}}]^2 \\
&\quad = [(\mathbf{n}\cdot\dot{\vec{\beta}})(\mathbf{n}-\vec{\beta}) - \{\mathbf{n}\cdot(\mathbf{n}-\vec{\beta})\}\dot{\vec{\beta}}]^2 \\
&\quad = (\mathbf{n}\cdot\dot{\vec{\beta}})^2(\mathbf{n}-\vec{\beta})^2 - 2(\mathbf{n}\cdot\dot{\vec{\beta}})(\mathbf{n}-\vec{\beta}). \\
&\qquad\quad \{\mathbf{n}\cdot(\mathbf{n}-\vec{\beta})\}\,\dot{\vec{\beta}} + \{\mathbf{n}\cdot(\mathbf{n}-\beta)\}^2\,\dot{\vec{\beta}}^2
\end{aligned}$$

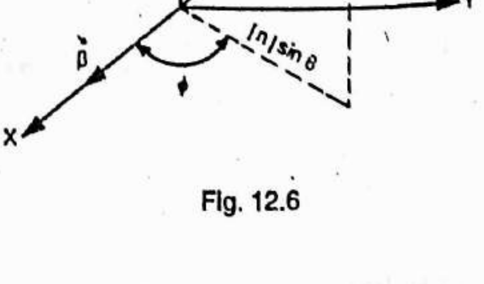

Fig. 12.6

$$= (\mathbf{n}\cdot\overrightarrow{\dot{\beta}})^2 (1 - 2\beta\cdot\mathbf{n} + \beta^2) - 2(\mathbf{n}\cdot\overrightarrow{\dot{\beta}})(\mathbf{n} - \overrightarrow{\beta})\cdot(1 - \mathbf{n}\cdot\overrightarrow{\beta})\overrightarrow{\dot{\beta}} + \dot{\beta}^2 (1 - \mathbf{n}\cdot\overrightarrow{\beta})$$

$$= (\mathbf{n}\cdot\overrightarrow{\dot{\beta}})^2 (1 - 2\overrightarrow{\beta}\cdot\mathbf{n} + \beta^2) - 2(\mathbf{n}\cdot\overrightarrow{\dot{\beta}})^2 (1 - \mathbf{n}\cdot\overrightarrow{\beta}) + \dot{\beta}^2 (1 - \mathbf{n}\cdot\overrightarrow{\beta})^2$$

$$\left(\text{since } \beta\cdot\overrightarrow{\dot{\beta}} = \beta\,\overrightarrow{\dot{\beta}}\cos 90 = 0\right)$$

Using (12), we get

$$[\mathbf{n}\times(\mathbf{n} - \overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]^2 = (\dot{\beta}\sin\theta\cos\phi)^2 (1 - 2\beta\cos\theta + \beta^2)$$

$$- 2(\dot{\beta}\sin\theta\cos\phi)^2 (1 - \beta\cos\theta) + \dot{\beta}^2 (1 - \beta\cos\theta)^2$$

$$= (\dot{\beta}\sin\theta\cos\phi)^2 [1 - 2\beta\cos\theta + \beta^2 - 2 + 2\beta\cos\theta] + \dot{\beta}^2 (1 - \beta\cos\theta)^2$$

$$= \dot{\beta}^2 [(1 - \beta\cos\theta)^2 - \sin^2\theta\cos^2\phi\,(1 - \beta^2)] \qquad \text{...(13)}$$

Making these substitutions in equation (6), we get

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4\pi c}\left[\frac{\dot{\beta}^2\,\{(1 - \beta\cos\theta)^2 - \sin^2\theta\cos^2\phi\,(1 - \beta^2)\}}{(1 - \beta\cos\theta)^5}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2\dot{\beta}^2}{4\pi c\,(1 - \beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi\,(1 - \beta^2)}{(1 - \beta\cos\theta)^2}\right]$$

Using $\dot{\beta} = \frac{\mathbf{a}}{c}$ and $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$, we obtain

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3 (1 - \beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2 (1 - \beta\cos\theta)^2}\right] \qquad \text{...(14)}$$

This is required expression for the present case. The formula indicates that for large velocities the radiation is very much more intense in the forward direction than in any other direction as shown in fig. 12.7. As velocity v approaches c *i.e.* β approaches 1, then the radiation becomes a sharp forward ray.

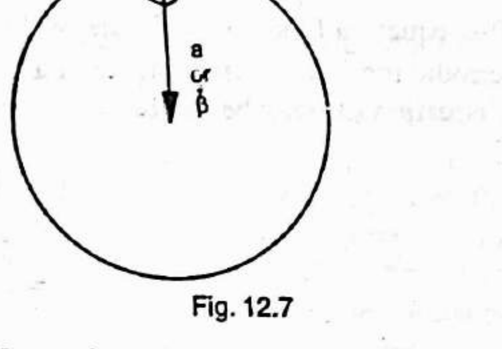

Fig. 12.7

The total power radiated can be found by integrating (14) over all angles

i.e.
$$P(t') = \int \frac{dP}{d\Omega}\,d\Omega$$

i.e.
$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^2}\int_0^{\pi}\int_0^{2\pi}\frac{1}{(1 - \beta\cos\theta)^2}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2 (1 - \beta\cos\theta)^2}\right]\sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3}\int_0^{\pi}\int_0^{2\pi}\frac{1}{(1 - \beta\cos\theta)^2}\left[1 - \frac{(1 - \beta^2)\sin^2\theta\cos^2\phi}{(1 - \beta\cos\theta)^2}\right]\sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q^2 a^2}{4\pi c^3}\cdot\frac{8\pi}{3(1 - \beta^2)^2}$$

$$\therefore \qquad P(t') = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}\gamma^4$$

which is result as equation (13) of section 12.7.

12.9. Radiation Damping. The Abraham-Lorentz Formula

According to the laws of classical electrodynamics an accelerated charge radiates electromagnetic energy, which comes at the expense of particle's kinetic energy. Consequently under the influence of a given force, a charged particle accelerates less than a neutral particle of the same mass. This implies that the radiation exerts a force back on the charge. This force is called the **radiation reaction** or **radiation damping.** It may be derived as follows :

If F_{ext} is the external force acting on a particle of mass m, then Newton's equation of motion is

$F_{ext} = ma$, where a is the acceleration.

If the particle under consideration a charged accelerated particle, it will emit radiation, which it turn will damp the motion of the particle. If F_{rad} is the damping force, then we can write

$$F_{ext} + F_{rad} = ma \qquad \text{...(1)}$$

According to Larmor's formula; the total power radiated by a non-relativisitic particle ($v << c$) is given by

$$P = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{q^2a^2}{c^3} \qquad \text{...(2)}$$

i.e.

$$\vec{\mathbf{F}}_{rad}\cdot\vec{\mathbf{v}} = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3} \qquad \text{...(3)}$$

For the conservation of energy, the energy lost by the particle in any interval of time must be equal to energy carried away by the radiation plus whatever extra energy has been supplied into the velocity fields. Now if we assume that at the end of the specified time interval the system returns to its initial state, then the net energy supplied into velocity fields is zero; so that equation (2), in the form of energy conservation becomes

$$\int_{t_1}^{t_2}\vec{\mathbf{F}}_{rad}\cdot\vec{\mathbf{v}}\,dt = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\int_{t_1}^{t_2}a^2\,dt \qquad \text{...(3)}$$

This equation holds only if state of the system is identical at instants t_1 and t_2. For example in the case of periodic motion the time interval must be equal to an integral number of full cycles. The integral on R.H.S. of equation (3) may be evaluated by parts *i.e.*

$$\int_{t_1}^{t_2}a^2\,dt = \int_{t_1}^{t_2}\frac{d\vec{\mathbf{v}}}{dt}\cdot\frac{d\vec{\mathbf{v}}}{dt}\,dt \qquad \left(\text{since } \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}\right)$$

$$= \left[\vec{\mathbf{v}}\cdot\frac{d\vec{\mathbf{v}}}{dt}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d^2\vec{\mathbf{v}}}{dt^2}\cdot\vec{\mathbf{v}}\,dt$$

First term is zero because velocities and accelerations at t_1 and t_2 are identical. So

$$\int_{t_1}^{t_2}a^2\,dt = -\int_{t_1}^{t_2}\frac{d^2\vec{\mathbf{v}}}{dt^2}\cdot\vec{\mathbf{v}}\,dt \qquad \text{...(4)}$$

Using this equation (3) takes the form

$$\int_{t_1}^{t_2} \vec{\mathbf{F}}_{rad} \cdot \vec{\mathbf{v}}\, dt = \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2}{c^3} \int_{t_1}^{t_2} \frac{d^2\vec{\mathbf{v}}}{dt^2} \cdot \vec{\mathbf{v}}\, dt$$

or

$$\int_{t_1}^{t_2} \left[\vec{\mathbf{F}}_{rad} - \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2}{c^3}\, \frac{d^2\vec{\mathbf{v}}}{dt^2} \right] \cdot \vec{\mathbf{v}}\, dt = 0$$

This equations is satisfied only if

$$\vec{\mathbf{F}}_{rad} = \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2}{c^3}\, \frac{d^2\vec{\mathbf{v}}}{dt^2}$$

Writing

$$\frac{d^2\vec{\mathbf{v}}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{\mathbf{v}}}{dt}\right) = \frac{d}{dt}\vec{\mathbf{a}} = \dot{\vec{\mathbf{a}}} = \ddot{\mathbf{v}}$$

∴

$$\mathbf{F}_{rad} = \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2}{c^3}\, \dot{\vec{\mathbf{a}}} \qquad \text{...(5)}$$

This equation represents the radiation reaction or radiation damping force and is called the **Abraham-Lorentz Formula.**

$$F_{rad} = \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2}{c^3}\, \dot{a} = m\tau\dot{a}$$

where for convenience damping factor $\tau = \frac{1}{4\pi\varepsilon_0}\, \frac{2}{3}\, \frac{q^2}{mc^3}$

Now equation (1) may be expressed as

$$\mathbf{F}_{ext} + m\tau\dot{\mathbf{a}} = m\dot{\mathbf{v}}$$

$$\mathbf{a} = \dot{\mathbf{v}}, \ \dot{\mathbf{a}} = \ddot{\mathbf{v}}$$

$$\mathbf{F}_{ext} + m\tau\ddot{\mathbf{v}} = m\dot{\mathbf{v}}$$

$$m(\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}}) = \mathbf{F}_{ext}$$

This equation is called **Abraham-Lorentz equation of motion.**

12.10. Cherenkov Radiation.

- A charged particle in uniform motion in a straight line does not radiate. But charged particle moving with constant velocity in a material medium emits, in cetain circumstances an unusual type of radiation; first observed by P.A. Cherenkov. Hence such radiation is called *Cherenkov radiation.* The emission of Cherenkov radiation is a cooperative phenomenon involving a large number of atoms of the medium whose electrons are accelerated by fields of the passing particle and so emit radiation. It must be emphasised that Cherenkov radiation is entirely unrelated to bremssrahlung which is almost always emitted by the moving (accelerated) electron itself; while the Cherenkov radiation is emitted by the medium under the action of the passing particle. The distinction between the two types of radiation appears with particular clarity when the particle has a very large mass ; the bremsstrahlung disappears and the cherenkov radiation is unaffected.

The wave vector and frequency of an electromagnetic wave propagated in an isotropic, non-permeable medium ($K_m = 1$) are related by

$$k = \frac{\omega}{v} = \frac{n\omega}{c} \quad \text{where} \quad n = \frac{c}{v} = \sqrt{K_e} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

is the refractive index, which is real. We know that the frequency of the Fourier component of the field of a particle moving uniformly in the *X*-direction in a medium is related to the *x*-component of the wave vector by $\omega = k_x v$. If this component is a freely propagated wave, these two relations must be consistent. Since $k > k_x$, therefore it follows that we must have

$$v = \frac{\omega}{k_x} > \frac{\omega}{k}$$

i.e.
$$v > \frac{c}{n} \left(\text{Since } \frac{\omega}{k} = \frac{c}{n} \right) \quad \text{...(A)}$$

where n is function of ω. Thus *Cherenkov radiation of frequency ω occurs if the velocity of the particle exceeds the phase velocity of waves of that frequency in the medium concerned.*

If θ is the angle between the direction of motion of the particle and the direction of emission. We have

$$k_x = k \cos\theta = \left(\frac{n\omega}{c}\right) \cos\theta \text{ and since } k_x = \frac{\omega}{v}$$

we find that

$$\frac{\omega}{v} = \left(\frac{n\omega}{c}\right)\cos\theta \;\; i.e. \;\; \cos\theta = \frac{c}{nv} \quad \text{...(B)}$$

This relation indicates that a definite value of angle θ corresponds to radiation of given frequency. That is the radiation of each frequency is emitted forwards and is distributed over the surface of a cone of vertical angle 2θ, where θ is given by equation (B) [fig. 12.8]. The distribution of radiation in angle and frequency are thus related in a definite manner.

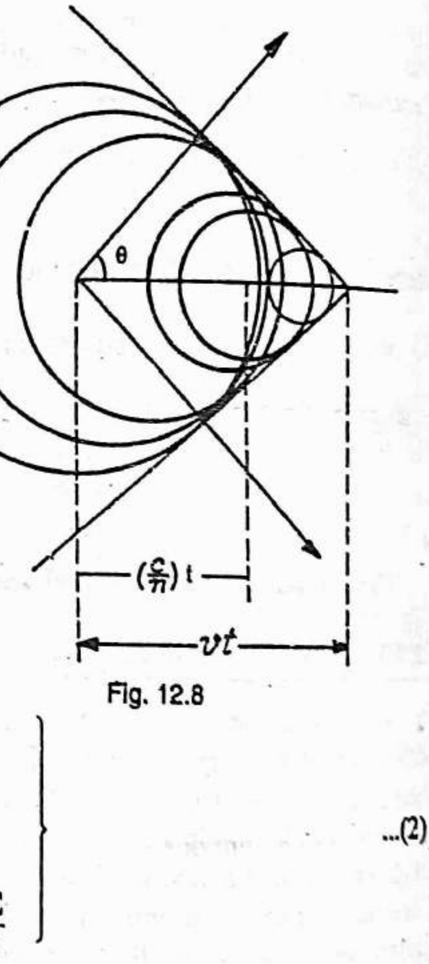

Fig. 12.8

Maxwell's equations are

$$\left.\begin{aligned} &\text{div } \mathbf{D} = \rho, && \text{div } \mathbf{B} = 0 \\ &\text{Curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{Curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right] \quad \text{...(1)}$$

These equations for isotropic non-permeable medium of permittivity ε (*i.e.* where $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$) take the form

$$\left.\begin{aligned} \text{div } \mathbf{E} &= \frac{\rho}{\varepsilon} \\ \text{div } \mathbf{B} &= 0 \\ \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \quad \text{curl } \mathbf{H} &= \mu_0\mathbf{J} + \mu_0\varepsilon\frac{\partial \mathbf{E}}{\partial t} \end{aligned}\right\} \quad \text{...(2)}$$

For a non-permeable medium the refractive index n is given by

$$n = \sqrt{(K_e K_m)} = \sqrt{\left(\frac{\varepsilon}{\varepsilon_0}\cdot\frac{\mu_0}{\mu_0}\right)} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = K_e$$

In the present case the extraneous charge and current distributions are given by

$$\rho = e\delta(\mathbf{r} - \mathbf{v}t) \text{ and } \mathbf{J} = e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t) \quad \text{...(3a)}$$

If ϕ and $\mathbf{A}$ are scalar and vector potentials, then

$$\mathbf{B} = \text{curl}\,\mathbf{A},\ \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\,\phi. \qquad \text{...(3b)}$$

So that Maxwell's equations (2) are satisfied identically. The additional condition imposed on potentials **A** and ϕ is the Lorentz condition

$$\text{div}\,\mathbf{A} + \mu_0\,\varepsilon\,\frac{\partial \phi}{\partial t} = 0 \text{ or } \text{div}\,\mathbf{A} + \frac{n^2}{c^2}\frac{\partial \phi}{\partial t} = 0. \qquad \text{...(4)}$$

$$\left(\text{since } \mu_0\varepsilon_0 = 1/c^2 \text{ and } \frac{\varepsilon}{\varepsilon_0} = K_\varepsilon = n^2\right)$$

Then substituting equation (3) in equation (2), we note that the Maxwell's equations are equivalent to the following two inhomogeneous equations for the potentials

$$\left.\begin{aligned} \nabla^2 \mathbf{A} - \mu_0\,\varepsilon\,\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\,\mu_0 \mathbf{J} \\ \text{and} \qquad \nabla^2 \phi - \mu_0\,\varepsilon\,\frac{\partial^2 \phi}{\partial t^2} &= -\,\frac{\rho}{\varepsilon} \end{aligned}\right\} \qquad \text{...(5)}$$

These equations may be equivalently written as

$$\left.\begin{aligned} \nabla^2 \mathbf{A} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\,\mu_0\, e\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t) \\ \nabla^2 \phi - \frac{n^2}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -\,\frac{e}{\varepsilon}\,\delta(\mathbf{r} - \mathbf{v}t) \end{aligned}\right] \qquad \text{...(6)}$$

Expanding A and ϕ as Fourier space integral *i.e.*

$$\left.\begin{aligned} \mathbf{A} &= \int_{-\infty}^{+\infty} A_k\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \\ \phi &= \int_{-\infty}^{+\infty} \phi_k\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \end{aligned}\right] \qquad \text{...(7)}$$

Taking the Laplacian of these equations, we get

$$\left.\begin{aligned} \nabla^2 \mathbf{A} &= -\int_{-\infty}^{+\infty} \mathbf{A}_k k^2\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \\ \text{and} \qquad \nabla^2 \phi &= -\int_{-\infty}^{+\infty} \phi_k k^2\, e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned}\right\} \qquad \text{...(8)}$$

and the Fourier components of $\nabla^2\mathbf{A}$ and $\nabla^2\phi$ are

$$\left.\begin{aligned} (\nabla^2 \mathbf{A})_k &= -\,k^2\,\mathbf{A}_k \\ (\nabla^2 \phi)_k &= -\,k^2\,\phi_k \end{aligned}\right\} \qquad \text{...(9)}$$

Hence the Fourier components of equations (6) are

$$\begin{aligned} -\,\mathbf{A}^2\,\phi_k - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{A}_k}{\partial t^2} &= -\,\frac{1}{(2\pi)^3}\int \mu_0 e\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV \\ &= -\,\frac{1}{(2\pi)^3}\,\mu_0\, e\mathbf{v}\, e^{-i\mathbf{k}\cdot\mathbf{v}t} \end{aligned} \qquad \text{...(10a)}$$

and
$$-k^2\phi_k - \frac{n^2}{c^2}\frac{\partial^2\phi_k}{\partial t^2} = -\frac{1}{(2\pi)^3}\int\frac{\delta(\mathbf{r}-\mathbf{v}t)}{\varepsilon}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,dV$$
$$= \frac{1}{(2\pi)^3}\cdot\frac{e}{\varepsilon}\,e^{-i\mathbf{k}\cdot\mathbf{v}t} \qquad ...(10b)$$

Equations (10) may be written as
$$k^2\mathbf{A}_k + \frac{n^2}{c^2}\frac{\partial^2\mathbf{A}_k}{\partial t^2} = \frac{\mu_0 e}{(2\pi)^3}\,\mathbf{v}\,e^{-i\mathbf{k}\cdot\mathbf{v}t} \qquad ...(11a)$$
and
$$k^2\phi_k + \frac{n^2}{c^2}\frac{\partial^2\phi_k}{\partial t^2} = \frac{e}{(2\pi)^2\varepsilon}\,e^{-i\mathbf{k}\cdot\mathbf{v}t} \qquad ...(11b)$$

This means that **A** and ϕ depend on time through the factor $e^{-i\mathbf{k}\cdot\mathbf{v}t}$. If we put $\omega = \mathbf{k}\cdot\mathbf{v} = k_x v$, with the assumption that the particle of charge e is moving along X-axis. Thus the time factor becomes $e^{-i\omega t}$, so that
$$\frac{\partial^2}{\partial t^2} \rightarrow -\omega^2$$
keeping this in mind and using $\varepsilon = K_e\varepsilon_0 = n^2\varepsilon_0$, we may write equations (11) as
$$\left.\begin{aligned} k^2\mathbf{A}_k - \frac{n^2\omega^2}{c^2}\mathbf{A}_k &= \frac{\mu_0 e}{(2\pi)^3}\,v e^{-i\omega t} \\ \text{and}\qquad k^2\phi_k - \frac{n^2\omega^2}{c^2}\phi_k &= \frac{e}{(2\pi)^3\cdot n^2}\,e^{-i\omega t}\end{aligned}\right\} \qquad ...(12)$$

Hence the Fourier components of **A** and ϕ are given by
$$\left.\begin{aligned} \mathbf{A}_k &= \frac{\mu_0}{4\pi}\frac{e}{2\pi^2}\frac{v}{k^2-\dfrac{\omega^2n^2}{c^2}}\,e^{-i\omega t} \\ \phi_k &= \frac{1}{4\pi\varepsilon_0}\frac{e}{2\pi^2n^2}\frac{1}{\left(k^2-\dfrac{\omega^2n^2}{c^2}\right)}\,e^{-i\omega t}\end{aligned}\right\} \qquad ...(13)$$

From equation (3), the Fourier component of electric field is given by
$$\mathbf{E}_k = i\omega\cdot\mathbf{A}_k - i\mathbf{k}\,\phi_k. \qquad ...(14)$$

Substituting values of $\mathbf{A}_k$ and ϕ_k, we get
$$E_k = \frac{\mu_0}{2\pi}\frac{i\omega e}{2\pi^2}\frac{\mathbf{v}}{k^2-\left(\dfrac{\omega^2n^2}{c^2}\right)}\,e^{-i\omega t} - \frac{1}{4\pi\varepsilon_0}\frac{i\mathbf{k}e}{2\pi^2n^2}\cdot\frac{1}{\left(k^2-\dfrac{\omega^2n^2}{c^2}\right)}\,e^{-i\omega t}$$

Writing $\mu_0\varepsilon_0 = \dfrac{1}{c^2}$ or $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$, we get

$$\mathbf{E}_k = \frac{1}{4\pi\varepsilon_0}\cdot\frac{ie\omega}{2\pi^2 n^2}\left[\frac{\dfrac{n^2}{c^2}\,\mathbf{v} - \dfrac{\mathbf{k}}{\omega}}{\left(k^2 - \dfrac{\omega^2 n^2}{c^2}\right)}\right] e^{-i\omega t} \qquad \text{...(15)}$$

The total electric field is obtained by inverting the Fourier transform

$$\mathbf{E} = \int_{-\infty}^{+\infty} \mathbf{E}_k\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \qquad \text{...(16a)}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{ie}{2\pi^2 n^2}\int_{-\infty}^{+\infty} \frac{\left(\dfrac{n^2}{c^2}\,\mathbf{v} - \dfrac{\mathbf{k}}{\omega}\right)\omega}{\left(k^2 - \dfrac{n^2\omega^2}{c^2}\right)}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega t}\, d\mathbf{k} \qquad \text{...(16b)}$$

Therefore the *force acting on a particle* is

$$\mathbf{F} = e\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\,\frac{ie^2}{2\pi^2 n^2}\int_{-\infty}^{+\infty} \frac{\left(\dfrac{n^2}{c^2}\,\mathbf{v} - \dfrac{\mathbf{k}}{\omega}\right)\omega}{\left(k^2 - \dfrac{n^2}{c^2}\,\omega^2\right)}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega t}\, d\mathbf{k}$$

Now let us find the value of the force at the point occupied by the particle namely $\mathbf{r} = \mathbf{v}t$. Then the factor $e^{i\mathbf{k}\cdot\mathbf{r}}$ in the integrand of (17) takes the form $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i\mathbf{k}\cdot\mathbf{v}t} = e^{i\omega t}$ and hence cancels with $e^{-i\omega t}$. Substituting $k_x v = \omega$, $q = \sqrt{(k_y^2 + k_z^2)}$; so that

$$k^2 = k_x^2 + k_y^2 + k_z^2 = (\omega^2/v^2) + q^2$$

and replacing $dk_y\, dk_z = 2\pi q\, dq$, we can write the magnitude of E as

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{ie}{\pi}\int_{-\infty}^{+\infty}\int_0^{q_0} \frac{\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\omega\, q\, dq\, d\omega}{n^2\left[q^2 + \omega^2\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\right]}$$

The energy loss by the moving particle is just the work done by the force* $e\mathbf{E}$ exerted on the particle by the field which it produces. Therefore the magnitude of force acting at the point occupied by the particle *i.e.* $\mathbf{r} = vt$ is

$$F = eE = \frac{1}{4\pi\varepsilon_0}\,\frac{ie^2}{\pi}\int_{-\infty}^{+\infty}\int_0^{q_0} \frac{\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\omega\, q\, dq\, d\omega}{n^2\left[q^2 + \omega^2\left(\dfrac{1}{v^2} - \dfrac{n^2}{c^2}\right)\right]}. \qquad \text{...(19)}$$

The *energy loss per unit length* is the work done by the force over that distance which is just $U = F$; it is called the *stopping power* of the substance with respect to the particle. The *energy loss in the frequency interval* $d\omega$ *is*

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*The magnetic force $q\mathbf{v} \times \mathbf{B}$ is seen by symmetry to be zero and in any case is perpendicular to velocity of the particle and hence does no work on it.

$$dU = dF = -d\omega \cdot \frac{1}{4\pi\varepsilon_0} \frac{ie^2}{\pi} \Sigma\omega \left(\frac{1}{c^2} - \frac{1}{n^2 v^2}\right) \int \frac{q\,dq}{q^2 - \omega^2\left(\frac{n^2}{c^2} - \frac{1}{v^2}\right)}$$

where the summation is over terms with $\pm\,\omega = |\,\omega\,|$. Let us introduce a new variable

$$p = q^2 - \omega^2 \left(\frac{n^2}{c^2} - \frac{1}{v^2}\right).$$

Then

$$dU = dF = -d\omega \left(\frac{1}{c^2} - \frac{1}{n^2 v^2}\right) \int \frac{dp}{p}.$$

The integration is performed by the method of contour integration. The singular point is given by $p = 0$ (for which $q^2 + k_x^2 = k^2$). In integration along the real p-axis we must pass, through the singular point $p = 0$ in some manner which is determined by the fact that, although we suppose that $n^2(\omega)$ is real, the medium being transparent, it actually has a small imaginary part, which is positive for $\omega > 0$ and negative for $\omega < 0$. Accordingly p has a small negative or positive imaginary part, and the path of integration ought to pass below or above the real axis respectively. This means that, when the path of integration is displaced to the real axis we must pass below or above the singular point respectively. This gives a contribution to dU and the real parts cancel in the sum. Identing the path of integration with infinitesimal semicircles, we find

$$\Sigma\,\omega \int \frac{dp}{p} = 2\pi i\omega.$$

Thus the final expression for the energy loss per unit length in the frequency range $d\omega$ is

$$dU = dF = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{c^2} \left(1 - \frac{c^2}{n^2 v^2}\right) \omega\, d\omega. \qquad ...(20)$$

This expression represents for the intensity of radiation in a freqeuncy range $d\omega$.

From expression (20) it is obvious that Cherenkov radiation is emitted only when

$$1 > \frac{c^2}{n^2 v^2} \quad \text{or} \quad v > \frac{c}{n} \qquad ...(21)$$

which is in agreement with equation (A).

According to equation (B), the radiation is emitted in an angle interval

$$d\theta = \frac{c}{vn^2 \sin\theta} \frac{dn}{d\omega}\, d\omega.$$

The total intensity of Cherenkov radiation is obtained by integrating (20) over all frequencies for which the medium is transparent.

12.11. Radiation due to an Oscillating Electric Dipole

An electric dipole consists of two equal and opposite charges separated by a finite distance. If q is the magnitude of each charge and $d\mathbf{l}$ is the separation. then *dipole moment* **p** is defined as

$$\mathbf{p} = q\, dl \qquad ...(1)$$

where direction of **p** or dl is from $-q$ to $+q$.

If the charge varies sinusoidally, the dipole moment will vary accordingly, *i.e.* if $q = q_0 e^{-i\omega t}$ then

$$\mathbf{p} = (q_0 e^{-i\omega t})\, dl$$

i.e.

$$\mathbf{p} = \mathbf{p}_0 e^{-i\omega t}, \quad \text{where } \mathbf{p}_0 = q_0\, dl \qquad ...(2)$$

A dipole whose moment varies sinusoidally is known as *oscillating dipole. An oscillating dipole may be assumed to be accelerated charge equivalent to distribution* whose dipole-moment is equal to the dipole moment of oscillating dipole

$$\mathbf{p} = \int \rho \mathbf{r}\; dV'. \qquad ...(3)$$

The time rate of change of *dipole moment is given by*

$$\dot{\mathbf{p}} = \frac{dp}{dt} = \frac{d}{dt}(q\, dl) = \frac{dq}{dt}\, dl = I\, dl \qquad ...(4)$$

This shows that the *time rate of change of dipole moment of an oscillating dipole is equal to a current element.** *As oscillating charges of electric dipole are undergoing acceleration* and an accelerated charge always radiates energy, *therefore an electric dipole radiates energy.*

Now in order to calculate the radiation emitted by an electric dipole, let us assume that the dipole consists of two charged spheres located at $z = \pm\, dl/2$ connected by a wire of negligible capacitance as shown in fig. 12.9.

Let us further assume that the length of the dipole $d\mathbf{l}$ is small compared to wavelength λ of radiation emitted due to periodic oscillations of charges so that all source points have the same retarded time and no phase difference occurs at observation point P between waves originating from different parts of the source.

Let polar coordinates of the observation point P be (r, θ) only due to axial symmetry. Now let us calculate the following parameters of interest.

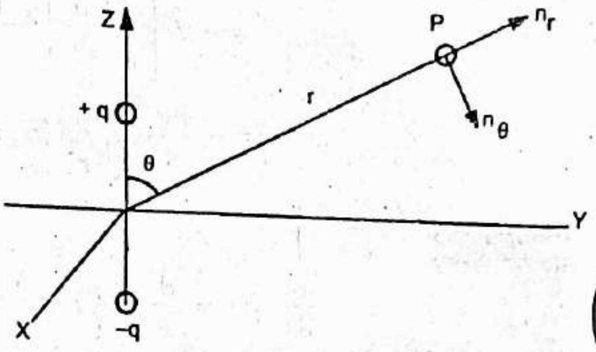

Fig. 12.9. An oscillating dipole.

(i) Vector Potential A. The vector potential due to current distribution is given by the relation

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}]}{r}\, dV' = \frac{\mu_0}{4\pi r}\,[I]\, dl \qquad \left[\text{since} \int \mathbf{J}\, dV' = I\, dl\right].$$

Using equation (4), we get

$$\mathbf{A} = \frac{\mu_0}{4\pi r}\,[\dot{\mathbf{p}}]. \qquad ...(5)$$

From this equation it is clear that

(i) The vector potential **A** depends only on r and varies as r^{-1}.

(ii) The direction of **A** is always that of $\dot{\mathbf{p}}$; this indicates that the vector potential is always parallel to $d\mathbf{l}$ (*i.e.* z-axis or polar axis).

(iii) If **n** is a unit vector along the polar axis, then we may write

$$\mathbf{n} = \mathbf{n}_r \cos\theta - \mathbf{n}_\theta \sin\theta \qquad ...(6)$$

then equation (5) may be expressed as

$$\mathbf{A} = \frac{\mu_0}{4\pi r}\,[\dot{\mathbf{p}}]\,(\cos\theta\, \mathbf{n}_r - \sin\theta\, \mathbf{n}_\theta)$$

or

$$(A_r\mathbf{n}_r + A_\theta\mathbf{n}_\theta + A_\phi\mathbf{n}_\phi) = \frac{\mu_0}{4\pi r}\,[\dot{\mathbf{p}}]\,(\cos\theta\, \mathbf{n}_r - \sin\theta\, \mathbf{n}_\theta) \qquad ...(7)$$

Therefore the components of **A** may be expressed as

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*This equivalence is used to discuss the radiation due to antenna (oscillating current elements).

$$\left.\begin{aligned} A_r &= \frac{\mu_0}{4\pi}\,[\dot{\mathbf{p}}]\,\frac{\cos\theta}{r} \\ A_\theta &= -\frac{\mu_0}{4\pi}\,[\dot{\mathbf{p}}]\,\frac{\sin\theta}{r} \\ A_\phi &= 0. \end{aligned}\right\} \quad \text{...(8)}$$

(ii) Scalar Potential ϕ. The Lorentz condition for free space is

$$\operatorname{div}\mathbf{A} + \mu_0\varepsilon_0\frac{\partial\phi}{\partial t} = 0 \text{ or } \operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0. \quad \text{...(9)}$$

i.e.

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= -c^2\operatorname{div}\mathbf{A} \\ &= -c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\ &= -c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\mu_0[\dot{p}]\cos\theta}{4\pi r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-\mu_0[\dot{p}]\sin^2\theta}{4\pi r}\right)\right] \\ &= -\frac{\mu_0c^2}{4\pi}\left[\left(\frac{1}{r^2}[\dot{p}]\cos\theta + \frac{r\cos\theta}{r^2}\frac{\partial}{\partial r}[\dot{p}]\right) - \left(\frac{2\sin\theta\cos\theta}{r^2\sin\theta}[\dot{p}]\right)\right] \quad \text{...(10)} \end{aligned}$$

But $[\dot{p}] = \dot{p}\left(t - \frac{r}{c}\right)$

$$\therefore \quad \frac{\partial}{\partial r}[\dot{p}] = \frac{\partial}{\partial r}\dot{p}\left(t-\frac{r}{c}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\dot{p}\left(t-\frac{r}{c}\right) = -\frac{1}{c}[\ddot{p}]$$

Making this substitution in equation (10), we get

$$\frac{\partial\phi}{\partial t} = -\frac{\mu_0c^2}{4\pi}\left[-\frac{\cos\theta}{rc}[\ddot{p}] - \frac{\cos\theta}{r^2}[\dot{p}]\right]$$

Using

$$\mu_0\varepsilon_0 = \frac{1}{c^2}.$$

and

$$\frac{\partial}{\partial t}\left\{\frac{[\dot{p}]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right\} = +\left\{\frac{\cos\theta}{r^2}[\dot{p}] + \frac{\cos\theta[\ddot{p}]}{rc}\right\}$$

We get

$$\frac{\partial\phi}{\partial t} = \frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\left\{\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right]$$

This gives

$$\phi = \frac{1}{4\pi\varepsilon_0}\left\{\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right\} \quad \text{...(11)}$$

From equation (11) it is clear that :

(i) The scalar potential varies as $\cos\theta$. This indicates that ϕ is zero in the equatroial plane where the potentials due to two charges cancel exactly.

(ii) The scalar potential consists of two parts, one varying as r^{-2} and the other varying as r^{-1}. The first term dominates for small values of r and hence this term is responsible for induction field in the nearer zone, while the second term dominantes for large values of r and hence this term is responsible for radiation field in the distant zone.

From the knowledge of **A** and ϕ we may calculate field vectors **B** and **E** using the relations.

$$\mathbf{B} = \text{curl }\mathbf{A} \text{ and } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad }\phi. \qquad \text{...(12)}$$

(iii) Magnetic field Induction. B. We have **B** = curl **A**.

$$= \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta - \frac{\partial A_\theta}{\partial\phi}\right)\right\}\mathbf{n}_r + \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right\}\mathbf{n}_\theta + \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\mathbf{n}_\phi$$

Here $A_\phi = 0$ and A_θ, A_r are independent of co-ordinate ϕ, hence we may write

$$\mathbf{B} = \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\mathbf{n}_\phi$$

Substituting values of A_r and A_θ, we obtain

$$\mathbf{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left\{-\frac{\mu_0 r}{4\pi r}[\dot{p}]\sin\theta\right\} - \frac{\partial}{\partial\theta}\left\{\frac{\mu_0[\dot{p}]\cos\theta}{4\pi r}\right\}\right)\mathbf{n}_\phi$$

$$= \frac{1}{r}\left(-\frac{\mu_0}{4\pi}\sin\theta\left(-\frac{\ddot{p}}{c}\right) + \frac{\mu_0}{4\pi r}[\dot{p}]\sin\theta\right)\mathbf{n}_\phi$$

This gives

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left\{\frac{[\dot{p}]\sin\theta}{r^2} + \frac{[\ddot{p}]\sin\theta}{rc}\right\}\mathbf{n}_\phi$$

From equation (13) it is clear that :

(i) The magnetic induction varies as $\sin\theta$. This indicates that the magnetic induction **B** is minimum along the axis of the dipole and maximum along the equatorial plane.

(ii) The magnetic induction **B** contains two terms, one varying as r^{-2} and the other varying as r^{-1}. The first term dominates for small values of r and hence this term is responsible for induction field; while the second term dominates for large values of r and hence this term is responsible for radiation field.

thus

$$\mathbf{B}_{\text{induction}} = \frac{\mu_0}{4\pi}\frac{[\dot{p}]\sin\theta}{r}\mathbf{n}_\phi = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}\mathbf{n}_\phi = \frac{\mu_0}{4\pi}\frac{I\,\mathbf{dl}\times\mathbf{r}}{r^3} \qquad \text{...(14)}$$

which is exactly *Biot-Savart law*

and

$$\mathbf{B}_{\text{radiation}} = \frac{\mu_0}{4\pi}\frac{[\ddot{p}]\sin\theta}{rc}\mathbf{n}_\phi \qquad \text{...(15)}$$

(iv) Electric field strength E. We know that electric field strength is given by

$$\mathbf{E} = -\text{grad }\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$= -\left(\frac{\partial\Phi}{\partial r}\mathbf{n}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{n}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{n}_\phi\right) - \frac{\partial}{\partial t}[A_r\mathbf{n}_r + A_\theta\mathbf{n}_\theta + A_\phi\mathbf{n}_\phi]$$

Using equations (8) and (11), we get

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\left\{\frac{\partial}{\partial r}\left(\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right)\right\}\mathbf{n}_r + \frac{1}{r}\left\{\frac{\partial}{\partial\theta}\left(\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right)\right\}\mathbf{n}_\theta\right]$$

$$- \frac{\mu_0}{4\pi}\left[\frac{\partial}{\partial t}\left\{\frac{[\dot{p}]\cos\theta}{r}\mathbf{n}_r - \frac{[\dot{p}]\sin\theta}{r}\mathbf{n}_\theta\right\}\right]$$

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\left\{-\frac{2[p]\cos\theta}{r^3} + \frac{\cos\theta}{r^2}\frac{\partial[p]}{\partial r} - \frac{[\dot{p}]\cos\theta}{r^2 c} + \frac{\cos\theta}{rc}\frac{\partial[\dot{p}]}{\partial r}\right\}\mathbf{n}_r + \left\{-\frac{[p]\sin\theta}{r^3} - \frac{[\dot{p}]\sin\theta}{r^2 c}\right\}\mathbf{n}_\theta\right] - \frac{\mu_0}{4\pi}\left[\frac{[\ddot{p}]\cos\theta}{r}\mathbf{n}_r - \frac{[\ddot{p}]\sin\theta}{r}\mathbf{n}_\theta\right] \quad ...(16)$$

But $$\frac{\partial}{\partial r}[\dot{p}] = \frac{\partial}{\partial r}p\left(t - \frac{r}{c}\right) = -\frac{1}{c}\frac{\partial}{\partial t}p\left(t - \frac{r}{c}\right) = -\frac{1}{c}[\dot{p}]$$

and $$\frac{\partial}{\partial r}[\dot{p}] = -\frac{1}{c}[\ddot{p}].$$

Also $$\mu_0\varepsilon_0 = \frac{1}{c^2} \text{ or } \mu_0 = \frac{1}{c^2\varepsilon_0}.$$

Making these substitution in equation (16), we get

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\left\{-\frac{2[p]\cos\theta}{r^3} - \frac{[\dot{p}]\cos\theta}{r^2 c} - \frac{[\dot{p}]\cos\theta}{r^2 c} - \frac{[\ddot{p}]\cos\theta}{rc^2}\right\}\mathbf{n}_r + \left[-\frac{[p]\sin\theta}{r^3} - \frac{[\dot{p}]\sin\theta}{r^2 c}\right\}\mathbf{n}_\theta\right] - \frac{1}{4\pi\varepsilon_0 c^2}[\ddot{p}]\frac{\cos\theta}{r}\mathbf{n}_r - \frac{[\ddot{p}]\sin\theta}{r}\mathbf{n}_\theta\Bigg]$$

Now collecting terms containing $[p]$, $[\dot{p}]$ and $[\ddot{p}]$, we get

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2[p]\cos\theta}{r^3}\mathbf{n}_r + \frac{[p]\sin\theta}{r^3}\mathbf{n}_\theta\right] + \frac{1}{4\pi\varepsilon_0}\left[\frac{2[\dot{p}]\cos\theta}{r^2 c}\mathbf{n}_r + \frac{[\dot{p}]\sin\theta}{r^2 c.}\mathbf{n}_\theta\right] + \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{c^2 r}\mathbf{n}_r. \quad ...(17)$$

It is clear from equation (17) that the electric field consists of three parts :

(i) The first part falls off as r^{-3} and dominates over other parts only at small distances. This part represents only the static field of an electric dipole. Thus

$$\mathbf{E}_{\text{static}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2[p]\cos\theta}{r^3}\mathbf{n}_r + \frac{[p]\sin\theta}{r^3}\mathbf{n}_\theta\right] \quad ...(17a)$$

(ii) The second part depends on $[\dot{p}]$ or current I and arises due to steady currents. This part represents the *induced field* and falls of as r^{-2}. Thus

$$\mathbf{E}_{\text{induction}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2[\dot{p}]\cos\theta}{r^2 c}\mathbf{n}_r + \frac{[\dot{p}]\sin\theta}{r^2 c}\mathbf{n}_\theta\right]. \quad ...(17b)$$

(iii) The third part depends on $[\ddot{p}]$ and falls off r^{-1}. This part dominates at large distances and represents the *radiation field.* Thus

$$\mathbf{E}_{\text{radiation}} = \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{rc^2}\mathbf{n}_\theta. \quad ...(17c)$$

(5) **Poynting Vector and Radiated Power.** The energy flow takes place due to radiation fields given by equations (15) and (17c). Therefore the *Poynting vector* will be given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E}_{\text{radiation}} \times \left(\frac{\mathbf{B}_{\text{radiation}}}{\mu_0}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{c^2 r}\,\mathbf{n}_\theta \times \frac{\mu_0}{4\pi}\frac{[\ddot{p}]\sin\theta}{rc^2\mu_0}\,\mathbf{n}_\phi \text{ Using (15) and (17c)}$$

$$= \frac{[\ddot{p}]^2\sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}(\mathbf{n}_\theta \times \mathbf{n}_\phi) = \frac{[\ddot{p}]^2\sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}\,\mathbf{n}_r \qquad \text{...(18)}$$

But $[p] = p_0\cos\omega(t - r/c)$, we get

$$[\dot{p}] = -p_0\omega\sin\omega\left(t - \frac{r}{c}\right) \text{ and } [\ddot{p}] = -p_0\,\omega^2\cos\omega\left(t - \frac{r}{c}\right)$$

$$[\ddot{p}]^2 = p_0^2\,\omega^4\cos^2\omega\,(t - r/c).$$

Making this substituting in equation (18), we get

$$\mathbf{S} = \frac{p_0^2\omega^4\cos^2\omega\,(t - r/c)\sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}\,\mathbf{n}_r \qquad \text{...(18a)}$$

∴ The time averaged poynting vector is given by

$$<\mathbf{S}> = \frac{p_0^2\omega^4 < \cos^2\omega\,(t - r/c) > \sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}\,\mathbf{n}_r$$

$$= \frac{p_0^2\omega^4\sin^2\theta}{32\pi^2\varepsilon_0 c^3 r^2}\,\mathbf{n}_r. \qquad \text{...(19)}$$

[Since average value of $\cos^2\omega\,(t - r/c)$ *over a complete cycle is* $\frac{1}{2}$].

This expression represents the average energy radiated per second per unit area in a normal direction by an oscillating dipole.

From equation (19) it is clear that the *time averaged* poynting vector varies as $\sin^2\theta$. This means that the flow of energy is zero along the dipole axis and maximum along the equitorial axis as shown in fig. 12·10.

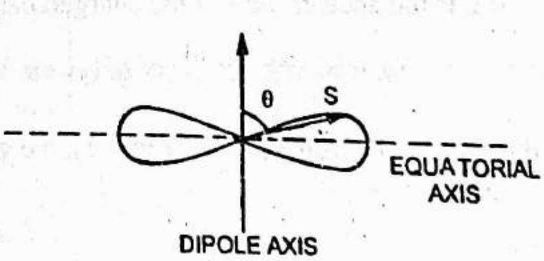


Fig. 12.10

∴ *The averaged power radiated per unit solid angle is given by*

$$\frac{dP}{d\Omega} = |<\mathbf{S}>|\frac{4\pi r^2}{4\pi} = <S>r^2 = \frac{p_0^2\omega^4\sin^2\theta}{32\pi^2\varepsilon_0 c^3} \qquad \text{...(20)}$$

The *total power radiated* may be obtained by integrating equation (20) for all angles *i.e.*

$$P = \int\frac{dP}{d\Omega}\,d\Omega = \frac{p_0^2\omega^4}{32\pi^2\varepsilon_0 c^3}\int_0^\pi\int_0^{2\pi}\sin^2\theta\,(d\theta)\,(\sin\theta\,d\phi)$$

$$= \frac{p_0^2\omega^4}{32\pi^2\varepsilon_0 c^3}\int_0^{\pi}\sin^3\theta\,d\theta\int_0^{2\pi}d\phi = \frac{p_0^2\omega^4}{32\pi^2\varepsilon_0 c^3}\cdot\left(\frac{4}{3}\right)\cdot 2\pi$$

$$= \frac{p_0^2\omega^4}{12\pi\,\varepsilon_0\,c^3} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{p_0^2\omega^4}{3c^3}. \qquad ...(21)$$

From this equation it is clear that the total power radiated by an oscillating dipole is proportional to the fourth power of the frequency. The examples of electric dipole radiators are the emission of radiation by an excited atom and electromagnetic waves transmitted by an antenna.

Ex. 1. *Treating electric dipole to be equivalent to an accelerated charge, calculate :*

(i) *The dipole moment amplitude p_0 in terms of charge* **q** *and acceleration* **a** *of accelerated charge.*

(ii) *Instantaneous rate of radiation from the charge.*

Solution. (i) Equation of motion of accelerated charge in sinusoidally varying electric field **E** is given by

$$m\mathbf{a} = q\mathbf{E} \qquad \text{or} \qquad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{q\mathbf{E}}{m}.$$

As $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$ $\qquad \therefore \dfrac{d^2\mathbf{r}}{dr^2} = \dfrac{q\mathbf{E}}{m}e^{-i\omega t}.$

This gives $\qquad \mathbf{r} = -\dfrac{q\mathbf{E}_0}{m\omega^2}e^{-i\omega t}.$

But dipole moment $\mathbf{p} = q\,\mathbf{r} = -\dfrac{q_0\mathbf{E}_0}{m\omega^2}e^{-i\omega t}$

or $\qquad |\mathbf{p}| = \left|-\dfrac{q^2\,\mathbf{E}_0}{m\omega^2}\right|e^{-i\omega t} \qquad$ But $\qquad p = p_0 e^{-i\omega t}.$

This gives

$$p_0 = \frac{q^2\mathbf{E}_0}{n\omega^2}. \qquad ...(1)$$

If **a** is the acceleration of the charged particle under an electric field of frequency ω at $t = 0$, then

$$m\,\mathbf{a} = q\,\mathbf{E}_0\,e^{-i\omega t} = q\mathbf{E}_0\Big|_{t=0} \qquad \text{or} \quad \mathbf{a} = \frac{q\,\mathbf{E}_0}{m}\Big|_{t=0}$$

Making this substitution in equation (1), we get

$$p_0 = \frac{qa}{\omega^2}. \qquad ...(2)$$

(ii) The time averaged power radiated is given by

$$<P> = \frac{1}{4\pi\varepsilon_0}\frac{(q^2a^2/\omega^4)\,\omega^4}{3c^3} = \frac{1}{4\pi\varepsilon_0}\frac{q^2a^2}{3c^3} \qquad ...(3)$$

If P is the instantaneous power, then

$$<P> = <P\cos^2\omega\,(t - r/c)> = P<\cos^2\omega\,(t - r/c)> = P/2$$

$$P = 2<P> = \frac{1}{4\pi\varepsilon_0}\frac{2q^2a^2}{3c^3}, \qquad ...(4)$$

which is exactly Larmor's non-relativistic formula.

12.12. Electric Quadrupole Radiation

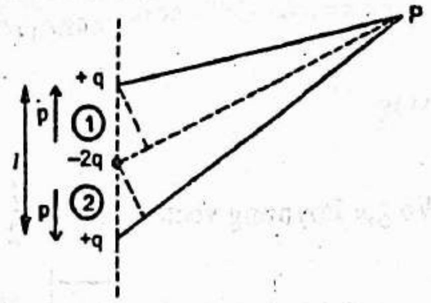


Fig. 12.10

A quadrupole is a combination of two electric dipoles. Consider a quadrupole containing two electric dipoles, each of dipole moment p, aligned along a given straight line (fig.). This system possesses quadrupole moment which varies with time and hence it will emit radiation, called quadrupole radiation. The quadrupole field may be derived by considering it to be the superposition of two dipole fields. The electric field due to a single dipole located at the origin, at any point P in the radiation zone is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{rc^2}\,\mathbf{n}_\theta \qquad \text{...(1)}$$

But $\quad [p] = p_0\, e^{-i\omega\left(t-\frac{r}{c}\right)}$, we get $\;[\dot{p}\,] = -p_0\, i\omega\, e^{-i\omega\left(t-\frac{r}{c}\right)}$

and $\quad [\ddot{p}\,] = -p_0\,(i\omega)^2\, e^{-i\omega\left(t-\frac{r}{c}\right)} = p_0\omega^2\, e^{-i\omega\left(t-\frac{r}{c}\right)}$

$$\therefore \qquad \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2\sin\theta}{rc^2}\, e^{-i\omega\left(t-\frac{r}{c}\right)}\mathbf{n}_\theta$$

i.e.
$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2\sin\theta}{rc^2}\, e^{-i\omega\left(t-\frac{r}{c}\right)} \qquad \text{...(3)}$$

The centres of dipoles are at distances $\frac{l}{4}$ and $-\frac{l}{4}$ from the origin. From fig it is clear that the dipole (1) is at a distance $\frac{l}{4}$ cone nearer to point P than from its distance from origin and (2) is farther by the same amount. The phases of two dipoles relative to origin are therefore $\frac{kl}{4}\cos\theta$ and $-\frac{kl}{4}\cos\theta - \pi$

where $k = \frac{\omega}{c}$ and $-\pi$ is introduced because two dipoles are oscillating out of phase. Therefore the electric field component due to quadrupole is

$$E_\theta = +\frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2\sin\theta}{rc^2}\, e^{-i\left\{\omega\left(t-\frac{r}{c}\right)-\frac{kl}{4}\cos\theta\right\}} + \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\, e^{-i\left\{\omega\left(t-\frac{r}{c}\right)-\left(-\frac{kl}{4}\cos\theta-\pi\right)\right\}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\left\{e^{\frac{ikl}{4}\cos\theta} + e^{-i\pi}\, e^{-\frac{ikl}{4}\cos\theta}\right\} e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\left\{e^{\frac{ikl}{4}\cos\theta} - e^{-\frac{ikl}{4}\cos\theta}\right\} e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p_0\omega^2}{rc^2}\sin\theta\; 2\,i\sin\left(\frac{kl}{4}\cos\theta\right) e^{-i\omega\left(t-\frac{r}{c}\right)}$$

$$= \frac{2i\, p_0\omega^2 \sin\theta}{4\pi\varepsilon_0\, rc^2} \sin\left(\frac{kl}{4}\cos\theta\right) e^{-i\omega\left(t-\frac{r}{c}\right)} \quad ...(4)$$

using $$H_\phi = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_\theta \quad ...(5)$$

We get Poynting vector $<\vec{S}> = \frac{1}{2}\,\mathrm{Re}\,(\vec{E} \times \vec{H}^*) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\; |E_\theta|^2\, \vec{n}_r$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \left| \frac{2i\, p_0\omega^2}{4\pi\,\varepsilon_0\, rc^2} \sin\theta \sin\left(\frac{kl}{4}\cos\theta\right) e^{-i\omega\left(t-\frac{r}{c}\right)} \right|^2 \vec{n}_r$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{4p_0^2\,\omega^4 \sin^2\theta}{(4\pi\varepsilon_0)^2\, r^2 c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right) \vec{n}_r$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{p_0^2\omega^4 \sin^2\theta}{8\pi^2\,\varepsilon_0^2\, r^2\, c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right) \vec{n}_r$$

$$= \frac{1}{\sqrt{\mu_0\varepsilon_0}}\, \frac{p_0^2\omega^4 \sin^2\theta}{8\pi^2\,\varepsilon_0\, r^2 c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right) \vec{n}_r \qquad \left(\because\ k = \frac{\omega}{c}\right)$$

$$= \frac{c\, p_0^2\omega^4 \sin^2\theta}{8\pi^2\,\varepsilon_0\, r^2 c^4} \sin^2\left(\frac{kl}{4}\cos\theta\right) \vec{n}_r$$

$$= \frac{p_0^2\,\omega^4 \sin^2\theta}{8\pi^2\,\varepsilon_0\, c^3 r^2} \sin^2\left(\frac{kl}{4}\cos\theta\right) \vec{n}_r \quad ...(6)$$

Average power radiated per unit solid angle $<\frac{dP}{d\Omega}> = <S>\, r^2\, \vec{n}_r$

$$= \frac{p_0^2\omega^4 \sin^2\theta}{8\pi^2\,\varepsilon_0 c^3} \sin^2\left(\frac{kl}{4}\cos\theta\right) \vec{n}_r \quad ...(7)$$

Special cases

case (i) If $l = \lambda$ *i.e.* $kl = 2\pi$;

then $$<\frac{dP}{d\Omega}> = \frac{\omega^4 p_0^2}{8\pi^2\,\varepsilon_0\, c^3} \sin^2\theta \sin^2\left(\frac{\pi}{2}\cos\theta\right)$$

The quadrupole moment

$$Q = 2p_0 l$$

$$\therefore \quad <\frac{dP}{d\Omega}> = \frac{\omega^4 Q^2}{8\pi^2\,\varepsilon_0\, c^3\, 4l^2} \sin^2 \sin^2\left(\frac{\pi}{2}\cos\theta\right)$$

using $$kl = 2\pi \Rightarrow l = \frac{2\pi}{k} = \frac{2\pi\, c}{\omega},$$

we get $$<\frac{dP}{d\Omega}> = \frac{\omega^6\, Q^2}{128\,\pi^2\,\varepsilon_0\, c^5} \sin^2\theta \sin^2\left(\frac{\pi}{2}\cos\theta\right)$$

case (ii) If $l \to 0$; then as p_0 increases; Q remains constant; In this case the dipole pair reduces to a point quadrupole located at the origin; then $\sin\left(\frac{kl}{4}\cos\theta\right) = \frac{kl}{4}\cos\theta$, so that equation (2) is written as

$$E_\theta = \frac{ik^2 p_0}{2\pi\,\varepsilon_0 r}\sin\theta\cdot\left(\frac{kl}{4}\cos\theta\right)e^{i\omega t}$$

Hence poynting vector $<\vec{\mathbf{S}}> = \frac{1}{2}\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}^*\right) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,|E_\theta|^2\,\vec{\mathbf{n}_r}$

$$= \frac{\omega^6 p_0^2 l^2}{128\pi^2\varepsilon_0 c^5 r^2}\sin^2\theta\cos^2\theta \text{ and } <\frac{dP}{d\Omega}> = \frac{\omega^6 p_0^2 l^2}{128\pi^2\varepsilon_0 c^5}\sin^2\theta\cos^2\theta$$

It is obvious from this expression that the maximum power is radiated at $\theta = 45°$.

12.13. Radiation due to a Small Current Element

An oscillating dipole may be treated as a small current element since dipole moment $\mathbf{p} = q\,d\mathbf{l}$.

$$\therefore \dot{\mathbf{p}} = I d\,l = \dot{q}\,d\mathbf{l}\ (\textit{for } I = \dot{q} = dq/dt),$$

i.e. the time rate of variation of dipole moment is equal to small current element.

Also
$$\ddot{\mathbf{p}} = \frac{dI}{dt}d\,l.$$

Now taking current variations as sinusoidal *i.e.*

$$I = I_0 e^{-i\omega t},\ \text{then } \frac{dI}{dt} = -i\omega I_0 e^{-i\omega t}$$

So that
$$\dot{\mathbf{p}} = I\,d\mathbf{l} = I_0 e^{i\omega t}\,d\,l$$

$$\ddot{\mathbf{p}} = \frac{dI}{dt}d\,l = -i\omega I_0 e^{-i\omega t}\,d\,l \qquad \text{...(1)}$$

This implies that radiation field equations developed for a dipole may be applied to the case of small linear current element. The electric and magnetic radiation fields for an oscillating dipole are given by

$$\mathbf{E}_{\text{radiation}} = \frac{1}{4\pi\varepsilon_0}\frac{[\ddot{p}]\sin\theta}{c^2 r}\mathbf{n}_\theta \qquad \text{...(2)}$$

and
$$\mathbf{B}_{\text{radiation}} = \frac{\mu_0}{4\pi}\frac{[\ddot{p}]\sin\theta}{cr}\mathbf{n}_\phi$$

$\therefore$ Therefore in this case *i.e. for a small current element*, using (1)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{[-i\omega I_0 e^{-i\omega t}dl]\sin\theta}{c^2 r}\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0\,dl\cdot\sin\theta}{c^2 r}e^{-i\omega(t-r/c)}\mathbf{n}_\theta \qquad \text{...(3)}$$

and
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{[-i\omega I_0 e^{-i\omega t}dl\sin\theta]}{cr}\mathbf{n}_\phi = -\frac{\mu_0}{4\pi}\frac{i\omega I_0\,dl\sin\theta}{cr}e^{-i\omega(t-r/c)}\mathbf{n}_\phi.$$

or
$$\mathbf{B} = \frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0\, dl \sin\theta}{c^3 r} e^{-i\omega(t-r/c)}\,\mathbf{n}_\phi \qquad ...(4)$$

Comparing (3) and (4), we note that $\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi$ ...(5)

where $|\mathbf{E}|$ is the modulus of electric field given by (3).

The Poynting vector for complex fields is given by

$$\mathbf{S} = \frac{1}{2}\,\mathrm{Re}\,[\mathbf{E}\times\mathbf{H}^*] = \frac{1}{2}\,\mathrm{Re}\left(\frac{\mathbf{E}\times\mathbf{B}^*}{\mu_0}\right)$$

Making substitutions from (3) and (4), we get

$$<\mathbf{S}> = \frac{1}{2\mu_0}\,\mathrm{Re}\left[\left\{-\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0\, dl\sin\theta}{c^2 r}e^{-i\omega(t-r/c)}\,\mathbf{n}_\theta\right\}\right.$$
$$\left.\times\left\{-\frac{\mu_0}{4\pi}\frac{(-i\omega I_0\, dl\sin\theta)}{cr}e^{-i\omega(t-r/c)}\,\mathbf{n}_\phi\right\}\right]$$
$$= \frac{1}{2}\frac{\omega^2 I_0^2\, dl^2\sin^2\theta}{16\pi^2\varepsilon_0 r^2 c^3}\,\mathbf{n}_\theta\times\mathbf{n}_\phi\,.$$

or
$$<\mathbf{S}> = \frac{\omega^2 I_0^2\, dl^2\sin^2\theta}{32\pi^2\varepsilon_0 r^2 c^3}\,\mathbf{n}_r \qquad ...(6)$$

The **averaged power radiated per unit solid angle** is given by

$$\frac{dP}{d\Omega} = <\mathbf{S}>\cdot\frac{4\pi r^2}{4\pi} = <\mathbf{S}>r^2 = \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\sin^2\theta \qquad ...(7)$$

The total power radiated may be obtained by integrating (7) over all angles. Thus

$$P = \int\frac{dP}{d\Omega}\,d\Omega = \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\int_0^{\pi}\int_0^{2\pi}\sin^2\theta\, d\theta\,(\sin\theta\, d\phi)$$
$$= \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\int_0^{\pi}\sin^3\theta\, d\theta\int_0^{2\pi}d\phi = \frac{\omega^2 I_0^2\, dl^2}{32\pi^2\varepsilon_0 c^3}\left(\frac{4}{3}\right)\cdot(2\pi)$$
$$\left[\text{since}\int_0^{\pi}\sin^3\theta\, d\theta = \tfrac{4}{3}\ \text{and}\int_0^{2\pi}d\phi = 2\pi\right]$$

i.e.
$$P = \frac{\omega^2 I_0^2 dl^2}{12\pi\varepsilon_0 c^2}. \qquad ...(8)$$

A more conventional form of above equation is obtained by introducing

$$\lambda = \frac{c}{v} = \frac{2\pi c}{2\pi v} = \frac{2\pi c}{\omega}\ \textit{i.e.}\ \ \omega = \left(\frac{2\pi c}{\lambda}\right);\ I_{rms} = \frac{I_0}{\sqrt{2}}.$$

The result is
$$P = \frac{\left(\frac{2\pi c}{\lambda}\right)^2(\sqrt{2}\, I_{rms})^2\, dl^2}{12\pi\,\varepsilon_0 c^3}$$
$$= \frac{2}{3}\frac{\pi}{\varepsilon_0 c}\left(\frac{dl}{\lambda}\right)^2 I_{rms}^2 \qquad (\text{since } \mu_0\varepsilon_0 = 1/c^2\ \textit{i.e.}\ 1/\varepsilon_0 c = \mu_0 c)$$

Making substitution for μ_0 and c, we get

$$P = \frac{2}{3}(4\pi \times 10^{-7} \times 3 \times 10^{8})\,\pi\left(\frac{dl}{\lambda}\right)^2 I_{rms}^2 = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 I_{rms}^2 \qquad \text{...(9)}$$

This is the required expression for power radiated by a short linear current element.

The power loss in a resistance R carrying current I_{rms} is given by

$$P = I_{rms}^2 R \qquad \text{...(10)}$$

Comparing equations (9) and (10), we see that it is sensible to define the *radiation resistance* of a dipole or a small current *element in free space** by

$$R_r = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 787\left(\frac{dl}{\lambda}\right)^2 \textit{ ohms.} \qquad \text{...(11)}$$

The radiation resistance may be defined as the resistance in which the power dissipated in the form of heat is equal to that (power) radiated by the current element carrying the same current, in the form of electromagnetic waves.

A centre-fed short linear antenna for which $dl \ll \lambda$ (shown in fig. 12.12) is an example of *linear current element radiator*. From equation (11) it is clear that radiation resistance for such an antenna is quite low. For example if $dl/d\lambda = 0{\cdot}01$, then radiation resistance is only $\{787 \times (0{\cdot}01)^2\} = 0{\cdot}0787$ ohms, a value much smaller than the ohmic resistance of such a conductor. Thus such an antenna dissipates most of the input power in the form of heat and radiates a very little power in the form of electromagnetic energy. Therefore a short linear antenna is, in general, an inefficient radiator. For efficient radiator the radiation resistance must be high and consequently the length (dl) of the antenna must be comparable with the wavelength λ, for which the dipole approximation does not hold. A half wave antenna (*i.e.* $dl = \lambda/2$) is commonly used to radiate electromagnetic energy into space.

12.14. Radiation From Linear Antenna

The small current element is not a practical antenna. The *practical antenna* is a centre-fed thin linear antenna having a length comparable with free space wavelength. The current ampiitude of such an antenna decreases uniformly from a maximum at the centre to minimum at the ends. The antenna of length 'd' is assumed to be oriented along Z-axis with its gap at the origin as shown in fig. 12.12.

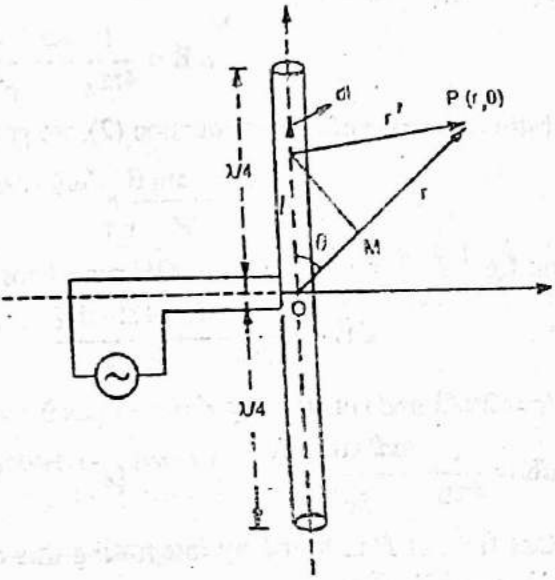


Fig. 12.12: Centre-fed linear antenna

If damping due to emission of radiation is neglected, the current along the antenna may be taken sinusoidal in time. Let $I = \text{Re}\,(I_0 e^{i\omega t}) = I_0 \cos \omega t$ be the current fed to the centre-fed linear antenna by means of a suitable electronic circuit [Here Re denotes "real part of"]. It is obvious that at $t = 0$, the amplitude of current at the centre is I_0. If dl is an element of antenna at a distance l from the centre, then amplitude of current-element dl will be

---

* The radiation-resistance in a medium of permittivity and permeabilityis defined as $B = \dfrac{2\pi}{3c}\sqrt{\dfrac{\mu}{\varepsilon}}\left(\dfrac{dl}{\lambda}\right)^2 I_{rms}^2$.

$$I_0 \cos \omega \left(\frac{l}{c} \right) = I_0 \cos \frac{2\pi l}{\lambda} \left[\text{since at } dl\,,\ t = \frac{l}{c} \text{ and } \frac{\omega}{c} = \frac{2\pi}{\lambda} \right]$$

Therefore *instantaneous current at dl* will be $I = \left(I_0 \cos \frac{2\pi l}{\lambda} \right) \cos \omega t$

This may be expressed as $I = \frac{I_0}{2}\left[\cos\left(\omega t - \frac{2\pi l}{\lambda} \right) + \cos\left(\omega t + \frac{2\pi l}{\lambda} \right) \right]$

In exponential form, this may be expressed as

$$I = \text{Re}\left[\frac{I_0}{2}\left\{ e^{-i(\omega t - 2\pi l/\lambda)} + e^{-i(\omega t + 2\pi l/\lambda)} \right\} \right] \qquad \text{...(1a)}$$

i.e.

$$I = \frac{I_0}{2} e^{-i\omega t} \{ e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)} \} \qquad \text{...(1b)}$$

where the real part of complex exponentials is implied.

∴ The current amplitude at d :

$$I_l = \frac{I_0}{2} \{ e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)} \} \qquad \text{...(2)}$$

1. Electric field Strength : From preceding section we know that the electric and magnetic fields due to a small current element dl carrying current amplitude I_0 at a distance r from the element is given by

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{i\omega I_0\, dl \sin\theta}{c^2 r} e^{-i\omega(t - r/c)}\, \mathbf{n}_\theta$$

and

$$\mathbf{B} = \frac{|\mathbf{E}|}{c} \mathbf{n}_\phi \qquad \text{...(3)}$$

In this case the field point P lies at a distance r' from the element dl and the current amplitude at a distance l from the centre is I_l given by (2) : therefore the electric field at point P due to small element dl is given by

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{i\omega I_l\, dl \sin\theta}{c^2 r'} e^{-i\omega(t - r'/c)}\, \mathbf{n}_\theta \qquad \text{...(4)}$$

Substituting value of I_l from equation (2), we get

$$d\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{i\omega\, dl \sin\theta}{c^2 r'} e^{-i\omega(t - r'/c)} \cdot \left[\frac{I_0}{2}\left\{ e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)} \right\} \right] \mathbf{n}_\theta$$

Now from fig. 12.12, we have $r' = r - OM = r - l\cos\theta$

therefore

$$d\mathbf{E} = -\frac{1}{2\pi\varepsilon_0} \frac{i\omega\, dl \sin\theta\, I_0}{2c^2 r'} e^{-i\omega(t - r/c)}\, e^{-i(\omega/c)\cos\theta} \left\{ e^{i(2\pi l/\lambda)} + e^{-i(2\pi l/\lambda)} \right\} \mathbf{n}_\theta$$

Using $\omega/c = 2\pi/\lambda$ and substituting $r' = r - l\cos\theta \approx r$ in the denominator, we get

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{\omega dl \sin\theta\, I_0}{2c^2 r} e^{-i\omega(t - r/c)} [e^{-i(2\pi l/\lambda)(\cos\theta - 1)} + e^{-i(2\pi l/\lambda)(\cos\theta + 1)}]\, \mathbf{n}_\theta$$

The resultant field at P is found by integrating this equation for the whole length of antenna *i.e.* between limits $-d/2$ to $+d/2$. Thus

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{i\omega I_0 \sin\theta}{2c^2 r} e^{-i\omega(t - r/c)} \int_{-d/2}^{+d/2} [e^{-i(2\pi l/\lambda)(\cos\theta - 1)} + e^{-i(2\pi l/\lambda)(\cos\theta + 1)}]\, dl\, \mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i\omega I_0 \sin\theta}{2c^2 r}e^{-i\omega(t-r/c)}\times\left[\frac{e^{-i(2\pi l/\lambda)(\cos\theta-1)}}{\left(-\frac{2\pi i}{\lambda}(\cos\theta-1)\right)}+\frac{e^{-i(2\pi l/\lambda)(\cos\theta+1)}}{\left(-\frac{2\pi i}{\lambda}(\cos\theta+1)\right)}\right]_{-d/2}^{+d/2}\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{2cr}\left(\frac{\omega}{c}\right)\left(\frac{\lambda}{\pi}\right)e^{-i\omega(t-r/c)}\left(\frac{e^{-i(\pi d/\lambda)(\cos\theta-1)}-e^{i(\pi d/\lambda)(\cos\theta-1)}}{-2i(\cos\theta-1)}\right)$$

$$+\left(\frac{e^{-i(\pi d/\lambda)(\cos\theta+1)}-e^{i(\pi d/\lambda)\cos\theta+1}}{-2i(\cos\theta+1)}\right)\Bigg]\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{2cr}\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{\pi}\right)e^{-i\omega(t-r/c)}\times\left[\frac{\sin\left\{\frac{\pi d}{\lambda}(\cos\theta-1)\right\}}{(\cos\theta-1)}+\frac{\sin\left(\frac{\pi d}{\lambda}\right)(\cos\theta+1)}{(\cos\theta+1)}\right]\mathbf{n}_\theta$$

Now using trigonometric relations

$$\sin\left\{\frac{\pi d}{\lambda}(\cos\theta-1)\right\}=\sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)-\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}$$

and

$$\sin\left\{\frac{\pi d}{\lambda}(\cos\theta+1)\right\}=\sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}+\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}$$

we get $\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{cr}e^{-i\omega(t-r/c)}$

$$\left[\frac{\sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}-\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}}{\cos\theta-1}\right.$$

$$\left.+\frac{\sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}+\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda}}{\cos\theta+1}\right]\mathbf{n}_\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{cr}e^{-i\omega(t-r/c)}$$

$$\left[\frac{\left[\sin\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\cos\frac{\pi d}{\lambda}-\cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}\right](\cos\theta+1)+\left[\sin\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\cos\frac{\pi d}{\lambda}+\cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}\right](\cos\theta-1)}{(\cos^2\theta-1)}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{i I_0 \sin\theta}{cr}e^{-i\omega(t-r/c)}$$

$$\frac{\left[2\sin\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\cos\frac{\pi d}{\lambda}\cos\theta-2\cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}\right]}{\cos^2\theta-1}$$

$$= -i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{cr}e^{-i(t-r/c)}\frac{\left[\cos\left\{\frac{\pi d}{\lambda}\cos\theta\right\}\sin\frac{\pi d}{\lambda}-\sin\left\{\cos\theta\frac{\pi d}{\lambda}\right\}\cos\frac{\pi d}{\lambda}\cos\theta\right]}{\sin\theta} \qquad ...(5)$$

This equation represents the required expression for electric field.

2. Magnetic field Induction B. We have $\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi$

where | **E** | is the numerical value of **E**.
Substituting | **E** | from equation (5), we get

$$\mathbf{B} = -i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{c^2 r}e^{-i\omega(t-r/c)}\left[\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}\cdot\cos\theta}{\sin\theta}\right]\mathbf{n}_\phi$$

$$= -i\frac{\mu_0}{4\pi}\frac{2I_0}{r}e^{-i\omega(t-r/c)}\cdot\left[\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\frac{\pi d}{\lambda}\cdot\cos\theta}{\sin\theta}\right]\mathbf{n}_\theta \quad ...(6)$$

This is the required expression for *magnetic field* induction.

8. Poynting Vector and Radiated Power. The Poynting vector *i.e.* the flow of energy per unit area per second, for complex fields is

$$\mathbf{S} = \tfrac{1}{2}\,\mathrm{Re}\,(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2\mu_0}\,\mathrm{Re}\,(\mathbf{E}\times\mathbf{B}^*)$$

Substituting values of **E** and **B** from (5) and (6), we get

$$\mathbf{S} = \frac{1}{2\mu_0}Re\left[\left\{-i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{cr}e^{-i\omega(t-r/c)}\right.\right.$$

$$\left.\cdot\left[\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda} - \cos\theta\right)\cos\frac{\pi d}{\lambda}\cos\theta}{\sin\theta}\right]\mathbf{n}_\theta\right]\times\left[\frac{i\mu_0}{4\pi}\frac{2I_0}{r}e^{i\omega(t-r/c)}\right.$$

$$\left.\left.\left(\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}\cos\theta.\right)\cos\left(\frac{\pi d}{\lambda}\right)\cdot\cos\theta}{\sin\theta}\right)\mathbf{n}_\phi\right]\right]$$

$$= \frac{1}{2\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\mu_0}{4\pi}\right)\left(\frac{2I_0}{cr}\right)\left\{\frac{2I_0}{r}\right\}$$

$$\left|\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cdot\cos\theta}{\sin\theta}\right|^2\mathbf{n}_r$$

Again using $\mu_0\varepsilon_0 = 1/c^2$, we get

$$\mathbf{S} = \frac{\mu_0 c}{8\pi^2}\frac{I_0^2}{r^2}\left|\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cos\theta}{\sin\theta}\right|^2\mathbf{n}_r \quad ...(7a)$$

or

$$\mathbf{S} = \frac{\mu_0 c}{4\pi^2}\frac{I_{rms}^2}{r^2}\cdot\left|\frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\left(\frac{\pi d}{\lambda}\right) - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cos\theta}{\sin\theta}\right|^2\mathbf{n}_r \quad ...(7b)$$

The power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \mathbf{n_r} \frac{4\pi r^2}{4\pi} = (\mathbf{S} \cdot \mathbf{n}_r r^2)$$

$$= \frac{\mu_0 c}{4\pi^2} I_{rms}^2 \left| \frac{\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\frac{\pi d}{\lambda} - \sin\left(\frac{\pi d}{\lambda}\cos\theta\right)\cos\left(\frac{\pi d}{\lambda}\right)\cos\theta}{\sin\theta} \right|^2 \quad ...(8)$$

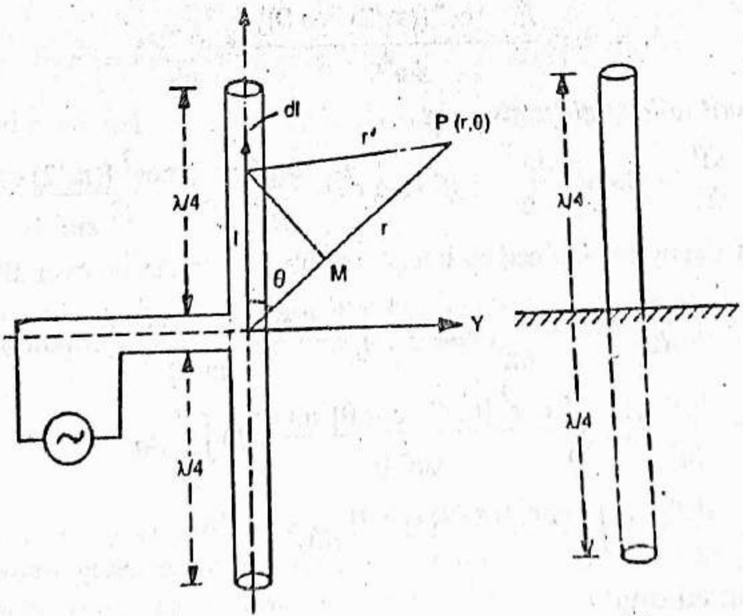
The total power radiated may be obtained by integrating above expression over all angles. As the integration of expression (8) is complicated, therefore we shall consider only the particuiar case of half wave antenna which is of common use for radio broadcast.

12.15. Radiation from a Linear Half-wave Antenna

A linear half-wave antenna is simply a straight conductor of length equal to half a free space wavelength (fig. 12.13a). In many cases the purpose of a half wave antenna is served by a linear conductor of length equal to a quarter free space wavelength set vertically on the ground, which acts as a mirror and so an additional quarter wavelength is provided by ground reflection (fig. 12.13b). To achieve good conductivity ground in the neighbourhood of antenna may be covered with a conducting screen. Such antennas are commonly used for radio broadcast purposes. The *electric and magnetic fields* may be obtained directly as in the preceding section by substituting $\lambda/2$ for d. Then we get

$$\text{Electric field. } \mathbf{E} = -i\frac{1}{4\pi\varepsilon_0}\frac{2I_0}{cr}e^{-i\omega(t-r/c)} \times \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\theta. \quad ...(1)$$

From equation (1) it is clear that : (i) The electric field strength due to a half-wave antenna for a given current $I = I_0 e^{-i\omega t}$ fed at the centre is independent of frequency. (ii) At $\theta = 0$. $\mathbf{E}$ becomes indeterminate; but in the limit $\theta \to 0$. $\mathbf{B}$ becomes zero which may be seen as follows :



(a) (b)

Fig. 12.13

$$\lim_{\theta \to 0} \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta} = \lim_{\theta \to 0} \frac{\sin[(\pi/2)\cos\theta]\,[(\pi/2)\sin\theta]}{\cos\theta} = 0$$

and magnetic field $\mathbf{B} = \frac{|\mathbf{E}|}{c}\mathbf{n}_\phi = -\frac{i}{4\pi\varepsilon_0}\frac{2I_0}{c^2 r} e^{-i\omega(t-r/c)} \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\phi$

Using $\mu_0\varepsilon_0 = 1/c^2$ or $\varepsilon_0 c^2 = 1/\mu_0$, we get

$$\mathbf{B} = -i\frac{\mu_0}{4\pi}\frac{2I_0}{r} e^{-i\omega(t-r/c)} \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\phi. \qquad ...(2)$$

Poynting Vector and Radiated Power

The Poynting vector, *i.e.* flow of energy per unit area per second, for complex field is

$$\mathbf{S} = \tfrac{1}{2}\,\mathrm{Re}\,(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2\mu_0}\,\mathrm{Re}\,(\mathbf{E} \times \mathbf{B}^*)$$

Substituting values of **E** and **B*** from (1) and (2), we get

$$\mathbf{S} = \frac{1}{2\mu_0}\,\mathrm{Re}\left[\left\{-\frac{i}{4\pi\varepsilon_0}\frac{2I_0}{cr} e^{-i\omega(t-r/c)} \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\theta\right\} \times \left\{\frac{i\mu_0}{4\pi}\frac{2I_0}{cr} e^{i\omega(t-r/c)} \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\mathbf{n}_\phi\right\}\right]$$

Using $\mu_0\varepsilon_0 = 1/c^2$ and rearranging, we get $\mathbf{S} = \frac{\mu_0 c}{8\pi^2}\frac{I_0^2}{r^2}\frac{\cos^2(\pi/2)\cos\theta}{\sin^2\theta}\mathbf{n_r}$ ...(3a)

or

$$\mathbf{S} = \frac{\mu_0 c}{4\pi^2}\frac{I_{rms}^2}{r^2}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}\mathbf{n_r} \qquad ...(3b)$$

Substituting values of μ_0 and c, we get

$$\mathbf{S} = \frac{(4\pi \times 10^{-7})(3 \times 10^8)}{4\pi^2}\frac{I_{rms}^2}{r^2}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}\mathbf{n_r}$$

$$\mathbf{S} = 9.55\,\frac{I_{rms}^2\cos^2[(\pi/2)\cos 0)]}{\sin^2\theta}\;n_r \qquad ...(3c)$$

The power flow per unit solid angle is given by

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\mathbf{n}_r\,\frac{4\pi r^2}{4\pi} = (\mathbf{S}\cdot\mathbf{n}_r)\,r^2 = \frac{\mu_0 c}{4\pi^2}\,I_{rms}^2\,\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta} \qquad ...(4)$$

The total power radiated may be obtained by integrating above expression over all angles *i.e.*

$$P = \int \frac{dP}{d\Omega}\,d\Omega = \frac{\mu_0 c}{4\pi^2}\,I_{rms}^2\int_0^\pi\int_0^{2\pi}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}\,(\sin\theta\,d\theta\,d\phi)$$

$$= \frac{\mu_0 c}{4\pi^2}\,I_{rms}^2\int_0^\pi\frac{\cos^2[(\pi/2)\cos\theta]\sin\theta}{\sin^2\theta}\,d\theta\int_0^{2\pi}d\phi$$

$$= \frac{\mu_0 c}{2\pi}\,I_{rms}^2\int_0^\pi\frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}\,d\theta. \qquad ...(5)$$

To solve the integral, we substitute

$$(\pi/2)\cos\theta = \pi/2 - \alpha/2 \quad i.e. \quad \cos\theta - 1 = \alpha/\pi.$$

So that

$$\cos^2[(\pi/2)\cos\theta] = \cos^2(\pi/2-\alpha/2) = \sin^2(\alpha/2) = \frac{1-\cos\alpha}{2}$$

and

$$-\sin\theta\, d\theta = -\frac{d\alpha}{\pi} \quad i.e. \quad d\theta = \frac{d\alpha}{\pi\sin\theta}.$$

Also

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(1-\frac{\alpha}{\pi}\right)^2 = \left(\frac{2\alpha}{\pi}-\frac{\alpha^2}{\pi^2}\right)$$

Making these substitutions in equation (5), we get

$$P = \frac{\mu_0 c I_{rms}^2}{2\pi}\int_0^{\pi}\frac{(1-\cos\alpha)/2}{\sin\theta}\cdot\frac{d\alpha}{\pi\sin\theta}$$

$$= \frac{\mu_0 c I_{rms}^2}{2\pi}\int_0^{2\pi}\frac{(1-\cos\alpha)}{2\pi\left(\dfrac{2\alpha}{\pi}-\dfrac{\alpha^2}{\pi^2}\right)}d\alpha$$

$$= \frac{\mu_0 c I_{rms}^2}{2\pi}\int_0^{2\pi}\frac{(1-\cos\alpha)}{(4\pi\alpha-2\alpha^2)}\,d\alpha$$

Using $\dfrac{1}{4\pi\alpha-2\alpha^2} = \dfrac{1}{4\pi}\left[\dfrac{1}{\alpha}+\dfrac{1}{(2\pi-\alpha)}\right]$, we get

$$P = \frac{\mu_0 c\, I_{rms}^2}{8\pi}\left[\int_0^{2\pi}\frac{1-\cos\alpha}{\alpha}\,d\alpha + \int_0^{2\pi}\frac{(1-\cos\alpha)}{(2\pi-\alpha)}\,d\alpha\right] \qquad ...(6)$$

From tables of cosine integral, we get

$$\int_0^{2\pi}\frac{1-\cos\alpha}{\alpha}\,d\alpha = \int_0^{2\pi}\frac{(1-\cos\alpha)}{(2\pi-\alpha)}\,d\alpha = 2{\cdot}44 \qquad ...(7)$$

Thus we get

$$P = \frac{\mu_0 c\, I_{rms}^2}{8\pi}\times 2\times 2{\cdot}44 = \frac{4{\cdot}88}{8\pi}\,\mu_0 c\, I^2_{rms} \qquad ...(8)$$

Substituting value of μ_0 and c, we get

$$P = \frac{4{\cdot}88}{8\pi}\,(4\pi\times 10^{-7})(3\times 10^{8})\,I_{rms}^2 = 73{\cdot}2\, I_{rms}^2 \qquad ...(9)$$

Comparing this with ohmic power loss equation $P = I_{rms}^2 R$, we note that the *radiation resistance for half wave antenna is* 73·2 *ohms*. This value is much greater as compared with dipole antenna for which radiation resistance R_r = 0·08 ohms. This implies that a half wave antenna is much more efficient than a dipole antenna.

12.16. Antenna Array

An antenna array is a set of suitably spaced, similarly oriented antennas (or aerials) to radiate energy predominently in some given direction. Antenna arrays make use of interference phenomenon that occur between the radiations from the different elements of the array. Accordingly the resultant field developed by an antenna array at a distant point will be the superposition of field contributions from each consituent antenna of the array. There are two types of common antenna arrays :

1. Broad side array. It exhibits the marked directional properties in the horizontal plane and in a direction normal to the line of the array (*i.e.* for which $\phi = \pi/2$ and $\theta = \pi/2$).

2. End fire array. It exhibits the marked directional properties in the horizontal plane and in a direction along the line of the array. (*i.e.* for which $\phi = 0$ and $\theta = \pi/2$).

Consider a linear* antenna array consisting of n antennas equally spaced along a straight line with common spacing d between two consecutive antennas fig. 12.14. If $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, ...$ are the fields due to antennas 1, 2, 3..., respectively, then the resultant field at P will be given by

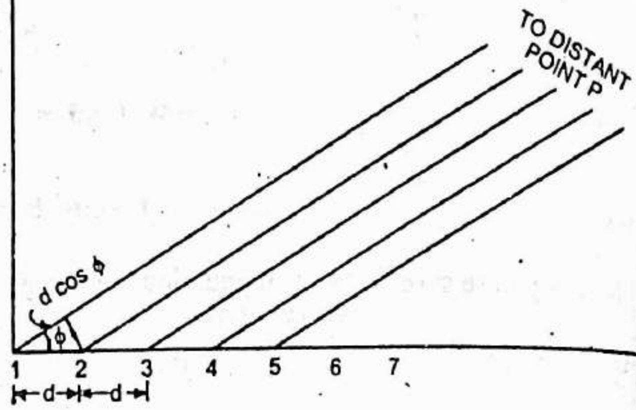


Fig. 12.14. Antenna array

$$\mathbf{E}_r = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \ldots + \mathbf{E}_n = \sum_{i=1}^{n} \mathbf{E}_i$$

If ψ is the phase angle between radiations from two consecutive antennas, and if $E_i = E$, then we may write

$$E_2 = Ee^{i\psi},\ E_3 = Ee^{2i\psi},\ E_4 = Ee^{3i\psi} ....., E_n = Ee^{i(n-1)\psi}$$

$$E_r = E + Ee^{i\psi} + Ee^{2i\psi} + Ee^{3i\psi} + \ldots + Ee^{i(n-1)\psi}$$

or

$$E_r = E[1 + e^{i\psi} + e^{2i\psi} + \ldots + e^{i(n-1)\psi}] \qquad ...(1)$$

Multiplying equation (1) throughout by $e^{i\psi}$, we get

$$E_r e^{i\psi} = E[e^{i\psi} + e^{2i\psi} + e^{3i\psi} + \ldots + e^{in\psi}] \qquad ...(2)$$

Subtracting equation (2) from equation (1), we get

$$(1 - e^{i\psi})E_r = E(1 - e^{in\psi})$$

$$\therefore \quad E_r = E\left[\frac{1 - e^{in\psi}}{1 - e^{i\psi}}\right] \qquad ...(3)$$

$$= \frac{e^{in\psi/2}[e^{in\psi/2} - e^{-in\psi/2}]}{e^{i\psi/2}[e^{i\psi/2} - e^{-i\psi/2}]}$$

$$= Ee^{i(n-1)\psi/2}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]$$

$$= E\frac{\sin n\psi/2}{\sin \psi/2}\, e^{i\xi} \qquad ...(4)$$

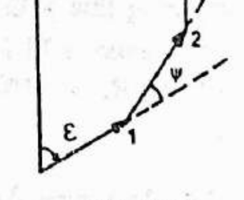


Fig. 12.14

where value of ξ is given by

$$\xi = \frac{(n-1)\psi}{2} \qquad ...(5)$$

and refers to the field from antenna-1.

If the phase is referred to the central point of the array, then $\xi = 0$ and equation (4) takes the form

$$E_r = E\frac{\sin n\psi/2}{\sin \psi/2} \qquad ...(6)$$

When $\psi = 0$, equation (6) becomes indeterminate, so that for this case E_r must be obtained in the limit $\psi = 0$; then

---

*An antenna array is said to the linear when the elements of the array are spaced equally along a straight line.

$$E_r = \lim_{\psi \to 0} E \frac{\left(\cos \frac{n\psi}{2}\right)\frac{n}{2}}{\left(\cos \frac{\psi}{2}\right)\frac{1}{2}} = nE \qquad \text{...(7)}$$

This is the maximum value of the field which E can attain and this happens when radiations from n antennas reach P in the same phase. Thus

$$E_{\max} = nE \qquad \text{...(8)}$$

Using equation (8) equation (6) may be expressed as

$$E_r = E_{\max} \frac{\sin n\psi/2}{n \sin \psi/2} \qquad \text{...(9)}$$

From fig. 12.14 it is clear that the path difference between radiations from the consecutive antennas at distant point P is $d \cos \phi$. Therefore the phase difference between fields at P due to this path difference is

$$\delta = \frac{2\pi}{\lambda}(d \cos \phi)$$

If α is the progressive phase shift between elements (*i.e.* α is) net angle by which the current in any element leads the current in the preceding element), then the resultant phase difference ψ between fields from two consecutive antennas will be given by

$$\psi = \frac{2\pi}{\lambda}(d \cos d) + \alpha \qquad \text{...(10)}$$

We know that the power radiated is proportional to the square of field vector therefore it will be maximum where field E_r is maximum and minimum where field E_r is minimum. Hence the conditions of principal maxima and minima may be defined by :

(a) Principal Maxima. The directions of principal maxima are given by

$$\frac{E_r}{E_{\max}} = \frac{1}{n}\frac{\sin (n\psi/2)}{\sin (\psi/2)} = 0 \qquad \text{...(11a)}$$

The principal maximum of the array occurs at $\psi \to 0$.

Since $\psi = \frac{2\pi}{\lambda} \cdot (d \cos \phi + \alpha)$,

therefore principal maximum occurs when

$$\frac{2\pi}{\lambda}(d \cos \phi + \alpha) = 0 \ \textit{i.e.} \ \cos \phi = -\frac{\alpha\lambda}{2\pi d} \qquad \text{...(11b)}$$

(a) Null or Minima. The directions of minima are given by

$$\frac{E_r}{E_{\max}} = \frac{1}{n}\frac{\sin\left(\frac{n\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} = 0 \qquad \text{...(12)}$$

i.e. when $\frac{n\psi}{2} = \pm p\pi$; $p = 1, 2, 3...$

(b) Secondary Maxima. The directions for secondary maxima are given by

$$\frac{d}{d\phi}\left(\frac{E_r}{E_{\max}}\right) = \frac{d}{d\phi}\left\{\frac{1 \sin (n\psi/2)}{n (\sin \psi/2)}\right\} = 0 \qquad \text{...(13)}$$

where ψ is given by equation (10).

Alternatively we may say that the secondary maxima occur when the numerator of equation (9) is a maximum,* that is when

$$\frac{n\psi}{2} = \pm (2m + 1)\frac{\pi}{2}, \ m = 1, 2, 3,...$$

---

*It may be noted that $n\psi/2 = \pi/2$ does not give a maximum.

These secondary maxima occur approximately midway between the minima.

(d) The width of principal maxima measured between first minima is twice the angle between the principal maximum and first minimum.

The latter angle $\Delta\phi = \psi_1$ is given by

$$\frac{n\psi_1}{2} = \pi \quad i.e. \quad \Delta\phi = \pm\frac{2\pi}{n} \qquad \text{...(14a)}$$

Therefore width of *principal maximum* (or *principal lobe*)

$$2\,\Delta\phi = 2\psi_1 = \frac{4\pi}{n} \qquad \text{...(14b)}$$

A wide variety of radiation patterns may be obtained by different geometrical shaping and phase relationships between constitutent antennas. Let us now consider the particular cases of interest already specified.

(i) Broad side array. For *broad side array* the maximum radiations occurs normal to the line of the array at $\phi = \pi/2$ and so from equation (11b), we get

$$\cos\frac{\pi}{2} = -\frac{\alpha\lambda}{2\pi d}$$

This gives $\alpha = 0$.

Hence *for broad side array the current in all the antennas must be fed in the same phase.*

Hence in this case ϕ is given by

$$\psi = \frac{2\pi}{\lambda}\,d\cos\phi \quad i.e. \quad \cos\phi = \frac{\lambda\psi}{2\pi d} \qquad \text{...(15)}$$

Since in this case the principal maximum occurs at $\pi/2$, therefore let us say that the *first null* occurs at an angle $(\pi/2 + \Delta\phi)$; where

$$\cos\left(\frac{\pi}{2} + \Delta\phi\right) = \frac{\lambda\psi_1}{2\pi d}$$

But from (14a), $\psi_1 = \frac{2\pi}{n}$; therefore

$$\cos\left(\frac{\pi}{2} + \Delta\phi\right) = \frac{(2\pi/n)\,\lambda}{2\pi d} = \frac{\lambda}{nd}$$

If $\Delta\phi$ is small, it is approximately given by

$$\Delta\phi = \frac{\lambda}{nd}$$

and the width of principal lobe is

$$2\,\Delta\phi = \frac{2\lambda}{nd} \qquad \text{...(16)}$$

Hence ***for a uniform broad side array the width of the principal lobe (in radians) is approximately twice the reciprocal of array length in wavelengths.***

Case (ii). End-fire array. For an end-fire array the maximum radiation is along the line of the array at $\phi = 0$, so the condition

$$\cos\phi = -\frac{\alpha\lambda}{2\pi d} \text{ gives}$$

$$\cos 0 = -\frac{\alpha\lambda}{2\pi d} \quad \text{or} \quad \alpha = -\frac{2\pi d}{\lambda} \qquad \text{...(17)}$$

This indicates that *for an end-fire array the phase between consecutive antennas must be retarded progressively by the same amount as the spacing between elements in radians.* Thus the condition of phase difference for this case takes the form

$$\psi = \frac{2\pi}{\lambda}(d\cos\phi) - \frac{2\pi d}{\lambda}$$

i.e.

$$\psi = \frac{2\pi d}{\lambda}(\cos\phi - 1) \qquad \text{...(18)}$$

The principal maximum is at $\phi = 0$ and the first null is at $\phi_1 = \Delta\phi$ where

$$\psi_1 = \frac{2\pi d}{\lambda}(\cos\phi_1 - 1) = -\frac{2\pi}{n} \qquad \text{using (14a)}$$

or

$$(\cos\Delta\phi - 1) = \frac{\lambda}{nd}$$

For $\Delta\phi$ small, the above equation gives approximately

$$\frac{(\Delta\phi)^2}{2} = \frac{\lambda}{nd}$$

∴ *The width of principal lobe,*

$$2\,\Delta\phi = 2\sqrt{\frac{2\lambda}{nd}} \qquad \text{...(19)}$$

i.e. the width of the principal lobe of a uniform end-fire array, as given approximately by (19), is greater than that for a uniform broad side array of the same length.

QUESTIONS AND PROBLEMS

1. (a) Derive expression for the inhomogeneous wave equations of scalar and vector potentials.
   (b) Using Fourier analysis, derive the solutions of scalar and vector potentials in terms of charge and current distributions respectively. **(Agra 1972)**
2. (a) What is the significance of retarded potentials ? Derive expressions for Lienard-Wiechert potentials.
   (b) Show that the total power radiated by an accelerated charge at low velocity is

$$P = \frac{e^2a^2}{6\pi\varepsilon_0 c^3}$$

   where symbols have their usual meanings. **(Lucknow 1998; Agra 96; Rohilkhand 77)**
3. Discuss electromagnetic scalar and vector potentials. Obtain an expression of the field radiated by an accelerated charge. **(Rohilkhand 1984)**
4. What are Lienard-Wiechert potentials ? Calculate the electric and magnetic field vectors for a uniformly moving electron using *L.W.* potentials. Discuss the results obtained. **(Rohilkhand 1986; Meerut 81)**
5. What are Lienard-Wiechart potentials ? Calculate the electric and magnetic field vector for a uniformly moving point charge using these potentials. Discuss the results obtained. **(Rohilkhand 1987)**
6. Write down the Lienard Wiechert potentials produced by a moving point charge. Obtain the corresponding electric and magnetic fields and identify their respective induction and radiation parts. Show that there is no radiation of charge if its acceleration is zero. **(Rohilkhand 1986)**
7. Starting from the expression for the retarded potentials find the electromagnetic field of a point charge in arbitrary motion. **(Meerut 1989, 83)**
8. Derive the expression for the rate of radiatioin of energy from an accelerated charge at low velocity. **(Rohilkhand 1980)**
9. (a) Obtain expressions for the Lienard Wiechert potentials of a moving point charge. Discuss what do you mean by the retarded time. **(Meerut 1998, 71)**
   (b) Calculate the electric and magnetic fields of a point charge moving in a straight line with constant velocity. **(Meerut 1991)**
10. Explain the term retarded potential and obtain the electromagnetic field of a moving point charge in space. Discuss the result when the distances are large. **(Kanpur 1986)**
11. Derive expressions for the field radiated by an accelerated charge at low velocity.
    What are Bremsstrahung and Cherenkov radiations. **(Rohilkhand 1993, Agra 92)**
12. Find the power radiated by an accelerated charged particle in the non-relativistic motion and describe its angular distribution. **(Meerut 1988)**

13. Obtain the power radiated and angular distribution of radiation emitted by an accelerated charged particle in non-relativistic motion.

    Does a charge moving with uniform velocity radiate ? **(Meerut 1997)**

14. Discuss the effect of radiation damping on the motion of an accelerated charge. Give a simple account of the theory of line width. **(Meerut 1998, 97)**
15. Derive an expression for the radiated energy from a high velocity electron moving with an acceleration parallel to velocity. How do you explain the constancy of the energy spectrum throughout the frequency scale. **(Meerut 1976)**
16. Obtain the electric and magnetic fields from an oscillating electric dipole and calculate the time averaged power radiated per unit solid angle and the total power radiated. **(Meerut 1980, 71)**
17. Find expression for the radiation field of an occillating electric dipole, Calculate the power radiated. Enumerate other multiple sources of electromagnetic radiation. **(Meerut 1989, 87)**
18. Deduce expressions for electric and magnetic fields of an oscillating electric dipole. **(Rohilkhand 1983)**
19. Give an account of Lienard and Wiechert potentials and find an expression for the field of a charge in uniform motion. **(Rohilkhand 1993, 83)**
20. Give expression for the field produced by an accelerated charge in space. **(Meerut 1992)**
21. Find the power radiated by an oscillating electric dipole and describe its angular distribution. **(Meerut 1982; Rohilkhand 92, 89, 85, 81)**
22. Find the Poynting vector and the power radiated by electric quadrupole. **(Delhi 1996)**
23. Calculate the radiation resistance of centre fed half wave antenna. Hence explain why it is efficient radiator while dipole antenna is not ?
24. What is an antenna array ? Discuss its directional properties analytically.
25. An antenna of length L carries alternating current of angular frequency ω. Treating it as an oscillating dipole, find the total power radiated. Show that the equivalent resitance to produce the same power loss (the radiation resistance) is

$$R = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{L\omega}{2\pi c}\right)^2$$

(Meerut 1975)

26. Discuss the principle of working of an antenna array. Derive an expression for the resultant field in an direction making an angle ϕ with the line of an array consisting of N equally spaced antennas carrying currents of equal magnitude and having successive phase difference. **(Meerut 1991, 89, 81)**
27. Define antenna array. What is meant by gain of an antenna array ? Show that the resultant field in a direction making an angle ϕ with the line of an array consisting of N aerials equally spaced d apart and carrying equal aerial currents and phase in given by

$$F = F_0 \frac{\sin\left(\frac{N\phi}{2}\right)}{N\sin\frac{\phi}{2}}$$

where F_0 is the value of maximum field and

$$\phi = \frac{2\pi}{\lambda}(d\cos\theta)$$

(Meerut 1980, 79)

28. Discuss the phenomenon of electric multipcle radiation and give the complete analysis for the dipole radiation. **(Agra 1983)**
29. (a) Discuss physically the basis of production of electromagnetic radiation by moving charges.

    (b) Give the theory of radiation emitted by an accelerated charge. **(Agra 1981)**

30. Write short notes on
    - (i) Retarded potentials or Lienard and Wiechert potentials, **(Meerut97; Rohilkhand 1996, 92)**
    - (ii) Virtual photon, **(Rohilkhand 1983; Agra 72)**
    - (iii) Cherenkov radiation, **(Meerut 97; Rohilkhand 1984; Agra 82)**
    - (iv) Bremsstrahlung;

9

Applications of Electromagnetic Waves I : Reflection and Refraction

[Including Wave Guides and Resonant Cavities]

9.1 Introduction

So far we have discussed the behaviour of a plane electromagnetic wave in free space and in a homogeneous medium and showed that it is propagated with speed of light $c = 1/\sqrt{(\mu_0 \varepsilon_0)}$ in free space and with a velocity $u = c/n$ (speed of light divided by refractive index) in a homogeneous medium. Now we shall consider the macroscopic behaviour(particularly optical) of a plane electromagnetic wave when it is incident an infinite plane boundary between two semi-infinite media of different indices of refraction,such for instance as free space and a metallic conductor. We shall be led to the familiar laws of reflection and refraction, laws that were established, before the advent of electromagnetic theory merely from general wave theory. In order to discuss the behaviour of electromagnetic waves at the boundary, we must find the boundary conditions holding at a surface of discontinuity between two media

9.2 Boundary Conditions at the Surface of Discontinuity

In each of the two media, let us assume that the solution of Maxwell's equations that we desire is a plane wave, just as in an infinite media. At the boundary beetween the two-media, however, certain boundary conditions are to be met and these demand that there be a definite relations between the boundary conditions which must be satisfied by field vectors **D, B, E** and **H** at the interface between media.

(i) Boundary condition for electric displacement vector D : According to Maxwell's first equation

$$\text{div}\,\mathbf{D} = \rho \qquad \dots(1)$$

At any interface between two media, let us construct a pillbox-like surface S composed of S_1, S_2, S_3 and S_4 shown in fig. 9.1 Integrating equation (1) over the pill-box-shaped volume V, we obtain

$$\int_V \text{div}\,\mathbf{D}\, dv = \int_V \rho\, dv \qquad \dots(2)$$

Changing volume integral on L.H.S of above equation into surface integral by usual Gauss's divergence theorem, we get

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dv$$

i.e.

$$\int_{S_1} \mathbf{D}_1.\mathbf{n}_1\, dS + \int_{S_2} \mathbf{D}_2.\mathbf{n}_2\, dS + \int_{S_3} \mathbf{D}'_1.\mathbf{n}'_1\, dS + \int_{S_4} \mathbf{D}'_2.\mathbf{n}'_2\, dS = \int_V \rho\, dV \qquad \dots(3)$$

where **n**'s represent unit vectors normal to concerned surface.

In above equation the third and fourth terms given the contribution to the surface integral from the walls of the pillbox. If **D** is bounded, letting the height of the pillbox h, approach zero, then the third and fourth terms vanish and S_1 approaches S geometrically and the entire surface takes the form of A as shown in fig. 9.1.

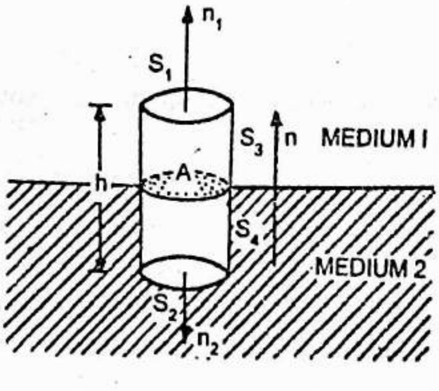


Fig. 9.1

Thus in the limit $h \to 0$ equation (3) takes the form

$$\lim_{h\to 0}\left[\int_S \mathbf{D}_1.\mathbf{n}_1\, dS + \int_S \mathbf{D}_2.\mathbf{n}_2\, dS\right] = \lim_{h\to 0}\int_V \rho\, dV.$$

If σ is the surface charge density at the interface, then above equation takes the form

$$\mathbf{D}_1.\mathbf{n}_1 S_1 + \mathbf{D}_2.\mathbf{n}_2 S_2 = \sigma A$$

i.e.

$$(\mathbf{D}_1 \cdot \mathbf{n}_1 + \mathbf{D}_2 \cdot \mathbf{n}_2)\, A = \sigma A \qquad \text{(since } S_1 = S_2 = A).$$

Since area is arbitrary and $\mathbf{n}_1 = \mathbf{n},\ \mathbf{n}_2 = -\mathbf{n}$; we get

$$\mathbf{D}_1.n - \mathbf{D}_2.\mathbf{n} = \sigma$$

i.e.

$$\mathbf{D}_{1n} - \mathbf{D}_{2n} = \sigma \qquad ...(4)$$

where D_{1n} and D_{2n} are the normal components of electric displacement vector in the two media, the normal being chosen along **n**. Thus we conclude that ***the normal component of electric displacement is not continuous at the interface but changes by an amount equal to the free surface charge density of charge at the interface.***

(ii) Boundary conditions for magnetic induction B :

Maxwell's second equation is

$$\text{div}\,\mathbf{B} = 0.$$

Taking volume integral of above equation over the entire volume of pillbox, we get

$$\int_V \text{div}\,\mathbf{B}\, dv = 0. \qquad ...(5)$$

Using Gauss's divergence theorem to change volume integral into surface integral

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

i.e.

$$\int_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1\, dS + \int_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2\, dS + \int_{S_3} \mathbf{B}'_1 \cdot \mathbf{n}_1\, dS + \int_{S_4} \mathbf{B}'_2 \cdot \mathbf{n}_2\, dS = 0.$$

Letting $h \to 0$ as in part (i), we note that third and fourth terms vanish while S_1 and S_2 approach each other ; so that in the limit $h \to 0$, above equation takes the form

$$\int_A (\mathbf{B}_1 \cdot \mathbf{n}_1 + \mathbf{B}_2 \cdot \mathbf{n}_2)\, dS = 0 \qquad \text{(since } S_1 = S_2 = A)$$

Since surface is arbitrary, we get

$$\mathbf{B}_1 \cdot \mathbf{n}_1 + \mathbf{B}_2 \cdot \mathbf{n}_2 = 0.$$

Now taking into account the opposite direction of $\mathbf{n}_1$ and $\mathbf{n}_2$, *viz* $\mathbf{n_1} = \mathbf{n}$ and $\mathbf{n_2} = -\mathbf{n}$, we get

$$\mathbf{B}_1 \cdot \mathbf{n} - \mathbf{B}_2 \cdot \mathbf{n} = 0 \quad i.e. \quad \mathbf{B}_{1n} = \mathbf{B}_{2n}, \qquad ...(6)$$

i.e. the normal component of magnetic induction (B) is continuous across the interface.

(iii) Boundary condition for electric intensity vector E :

Maxwell's third equation is

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad ...(7)$$

At the interface between two media, we now construct a rectangular loop *abcd* bounding a surface *S* as shown in fig 9.2.

Integrating equation (7) over the loop *abcd*, we get

$$\int_S \text{curl } \mathbf{E}\cdot\mathbf{n}\, dS = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}\, dS.$$

Now transforming surface integral on L.H.S. into line integral over the path *abcd* by using Stroke's theorem we get

$$\oint_{abcd} \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}\, dS. \qquad ...(8a)$$

or

$$\int_{ab} \mathbf{E}_1\cdot d\mathbf{l} + \int_{cd} \mathbf{E}_2\cdot d\mathbf{l} + \int_{bc \text{ and } da} \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot n\, dS. \qquad ...(8b)$$

If the loop is now shrunk by letting $h \to 0$, then the contribution to the integral from sides *bc* and *da* will vanish

i.e.

$$\lim_{h\to 0} \int_{bc \text{ and } da} \mathbf{E}\cdot d\mathbf{l} \to 0$$

and also the surface integral on R.H.S. tends to zero provided $(\partial \mathbf{B}/\partial t)$ is everywhere finite. Thus in the limit $h \to 0$, equation (8b) takes the form

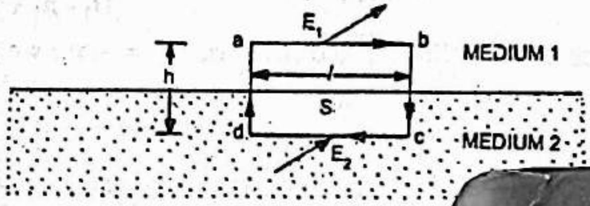


Fig. 9.2

$$\left[\int_{ab} \mathbf{E}_1\cdot d\mathbf{l} + \int_{cd} \mathbf{E}_2\cdot d\mathbf{l}\right] = 0$$

or

$$E_{1t}\, l - E_{2t}\, l = 0 \text{ i.e. } E_{2t} = E_{1t}, \qquad ...(9)$$

where E_{1t} and E_{2t} are the tangential components of the electric field in the two media. Equation (9) represents that *tangential components of* E *must be continuous across the interface.*

(iv) Boundary condition for magnetic field intensity H :

Maxwell's fourth equation is

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad ...(10)$$

Again taking the surface integral over rectangular loop *abcd*, we get

$$\int_S \text{curl } \mathbf{H}\cdot\mathbf{n}\, dS = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot\mathbf{n}\, dS. \qquad ...(11)$$

Using Stoke's theorem to change surface integral on L.H.S. into line integral over the path *abcd*, we get

$$\oint_{abcd} \mathbf{H}\cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot\mathbf{n}\, dS$$

or

$$\int_{ab} \mathbf{H}_1\cdot d\mathbf{l} + \int_{cd} \mathbf{H}_2\cdot d\mathbf{l} + \int_{bc \text{ and } da} \mathbf{H}\cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot\mathbf{n}\, dS \qquad ...(12)$$

If the loop is shrunk again in the limit $h \to 0$, we note that the contribution to line integral along sides *bc* and *da* vanishes

i.e.
$$\lim_{h \to 0} \int_{bc + da} \mathbf{H} \cdot dl \to 0 \qquad \text{...(13a)}$$

$$\lim_{h \to 0} \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n}\, dS \to 0$$

i.e.
$$\frac{\partial \mathbf{D}}{\partial t} \text{ is bounded everywhere} \qquad \text{....(13b)}$$

and
$$\lim_{h \to 0} \int_S \mathbf{J} \cdot d\boldsymbol{S} \to \mathbf{J}_{S\perp}\, l \qquad \text{...(13c)}$$

where $J_{S\perp}$ represents the components of surface current density perpendicular to the direction of H-component which is being matched. The idea of surface current density is closely analogous to that of a surface charge density-it represents a finite current in an infinitesimal layer. Then in the limit $h \to 0$, equation (12) takes the form

$$\int_{ab} \mathbf{H} \cdot dl + \int_{cd} \mathbf{H} \cdot dl = \mathbf{J}_{S\perp}\, l$$

or
$$H_{1t} l - H_{2t} l = J_{S\perp}\, l$$

i.e.
$$H_{1t} - H_{2t} = J_{S\perp}\, l. \qquad \text{...(14)}$$

Thus the *tangential component of magnetic field intensity is not continuous at the interface* ; but changes by an amount equal to the component of the surface curent density perpendicular to tangential component of $\vec{H}$.

The surface current density is zero unless the conductivity is infinite ; hence for finite conductivity $J_S = 0$; so

$$H_{1t} - H_{2t} = 0 \text{ i.e. } H_{1t} = H_{2t}.$$

That is unless one medium has infinite conductivity the tangential component of magnetic field intensity is continuous.

All the above boundary conditions may be summarised as :

1. The normal component of electric displacement (D) is discontinuous across the interface by an amount equal to free surface charge density of charge at the interface.

i.e.
$$D_{1n} - D_{2n} = \sigma. \qquad \text{...(8.1)}$$

2. The normal component of magnetic induction **B** is continuous across the interface

i.e.
$$B_{1n} = B_{2n}. \qquad \text{...(8.2)}$$

3. The tangential component of electric field intensity **E** is continuous across the interface,

i.e.
$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \qquad \text{...(8.3)}$$

4. (a) If either of the two media has infinite conductivity, the tangential component of magnetic field intensity is discontinuous at the interface by an amount equal to surface current density at the interface.

i.e.
$$H_{1t} = H_{3t} = J_{S\perp}. \qquad \text{...(7.4a)}$$

(b) If both media have finite conductivity, the tangential component of magnetic field intensity is continuous across at the interface

i.e.
$$H_{1t} = H_{2t}. \qquad \text{...(7.4b)}$$

9.2. Reflection and Refraction of Electromagnetic Waves at the Interface of Non-conducting Media.

The reflection and refraction of light at a plane surface between two media of different dielectric properties are familiar phenomenon. The various aspects of the phenomenon divide themselves into two categories :

1. Kinematic Properties.

(i) Law of reflection. The angle of reflection is equal to the angle of incidence *i.e.* $\theta_r' = \theta_i$ where θ_i is angle of incidence and θ'_r is angle of reflection.

(ii) Snell's law of refraction states.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1}$$

where θ_i and θ_r are the angles of incidence and refraction ; while n_1 and n_2 are corresponding indices of refraction.

(iii) Law of frequency. The incident, reflected and refracted waves all have the same frequency.

2. Dynamic Properties. These properties are concerned with :

(i) intensities of reflected and refracted waves.

(ii) phase changes and polarisation.

The kinematic properties follow immediately from the wave nature of the phenomenon and the fact that there are certain boundary conditions imposed on field vectors to be satisfied. But they do not depend on the nature of waves or the boundary conditions. On the other hand, the dynamic properties depend entirely on the specific nature of electromagnetic field and their boundary conditions.

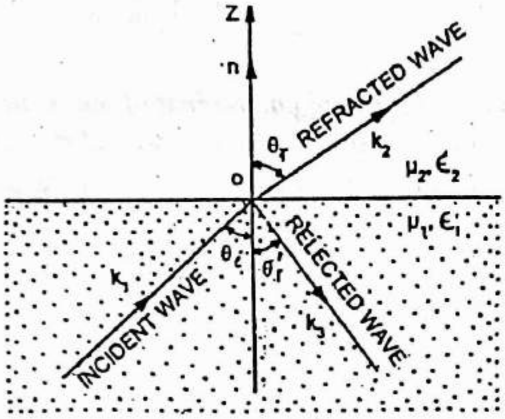


Fig. 9.3 Reflection and refraction

Let us consider a plane interface at $Z = 0$ separating two homogeneous, chargefree and non-conducting isotropic media, characterised by permittivities ε_1 and ε_2 ; permeabilities μ_1 and μ_2 respectively as shown in fig. 9.3. Let a plane wave with wave vector $\mathbf{k}_1$ and frequency ω_1 be incident from medium 'I' at point O on the interface. This wave is partly reflected and partly transmitted (or refracted). Let the reflected and refracted waves have wave vectors $\mathbf{k}'_1$ and $\mathbf{k}_2$, frequencies ω'_1 and ω_2 respectively. Also let $\mathbf{n}$ be the unit vector normal to the interface and directed from medium '1' into medium '2'. The field vectors for incident reflected and refracted waves may be expressed as

$$\left.\begin{aligned} \textit{For incident wave}\quad \mathbf{E}_1 &= \mathbf{E}_{01}\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1 t} \\ \mathbf{B}_1 &= \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\omega_1} \text{ or } \mathbf{H}_1 = \frac{\mathbf{k}_1 \times \mathbf{E}}{\mu_1\omega_1} \end{aligned}\right\} \quad \text{...(1)}$$

$$\left.\begin{aligned} \textit{For reflected wave}\quad \mathbf{E}'_1 &= \mathbf{E}'_{01}\, e^{i\,\mathbf{k}_1'\cdot\mathbf{r} - i\omega t} \\ \mathbf{B}'_1 &= \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\omega_1'} \text{ or } \mathbf{H}_1' = \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\mu_1\omega_1'} \end{aligned}\right\} \quad \text{...(2)}$$

$$\left.\begin{aligned} \textit{For refracted wave}\quad \mathbf{E}_2 &= \mathbf{E}_{02}\, e^{i\,\mathbf{k}_2\cdot\mathbf{r} - i\omega t} \\ \mathbf{B}_2 &= \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\omega_2} \text{ or } \mathbf{H}_2 = \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2\omega_2} \end{aligned}\right\} \quad \text{...(3)}$$

Note. The cause of notation used above may be understood as : Incident and reflected wave are in medium 'I', hence incident wave has been characterised by symbol I (unprimed), while reflected wave has been represented by symbol I (primed). The refracted wave is in medium '2', hence it has been represented by symbol 2.

We can apply the boundary condition (iii) derived in preceding section namely that the tangential component of the electric field is continuous across the interface between the media (*i.e.* at $z = 0$). In this case at every point in the interface

$$(E_1)_{tangential} + (E_1)'_{tangential} = (E_2)_{tangential} \qquad \text{...(4)}$$

or

$$(E_{01})_{tangential}\, e^{i\,\mathbf{k}_1\cdot\mathbf{r} - i\omega t} + (E_{01}')_{tangential}\, e^{i\,\mathbf{k}_1'\cdot\mathbf{r} - i\omega_1' t}$$

$$= (E_{02})_{tangential}\, e^{i\,\mathbf{k}_2\cdot\mathbf{r}}\ c^{-i\omega_2 t}$$

i.e.

$$(E_{01})_{tangential}\, e^{i\,\mathbf{k}_1\cdot\mathbf{r}}\ e^{-i\omega t} + (E_{01}')_{tangential}\, e^{i\,\mathbf{k}_1\cdot\mathbf{r}}\ e^{i\omega_1' t}$$

$$= (E_{02})_{tangential}\, e^{i\,\mathbf{k}_2\cdot\mathbf{r}}\ e^{-i\omega_2 t} \qquad \text{...(5)}$$

Since this equality is independent of time, it immediately follows that

$$\omega_1 = \omega_1' = \omega_2 = \omega \text{ (say)}.$$

That is the incident, reflected and refracted waves all have the same frequency.

Since equation (5) holds for all points of the interface ($z = 0$), we msut have

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_1'\cdot\mathbf{r})_{z=0} = (\mathbf{k}_2\cdot\mathbf{r})_{z=0}\,. \qquad \text{...(7)}$$

This equation is independent of the nature of the boundary conditions and contain the kinematic aspects of reflection and refraction.

Writting equation (7) is somewhat expanded from *i.e.*

$$k_{1x}x + k_{1y}y = k'_{1x}x + k'_{1y}y = k_{2x}x + k_{2y}y \qquad \text{...(8)}$$

we get

$$\left.\begin{aligned} k_{1x} &= k'_{1x} = k_{2x} \quad \text{...}(a) \\ k_{1y} &= k'_{1y} = k_{2y} \quad \text{...}(b) \end{aligned}\right\} \qquad \text{...(9)}$$

Since incident beam is in XZ plane $k_{1y} = 0$, therefore equation (9) implies $k'_{1y} = k_{2y} = 0$, that is both $\mathbf{k}_1'$ and $\mathbf{k}_2$ also lie in XZ plane. As normal **n** is along Z axis, thus we conclude that all the ***three wave vectors and normal to the interface* n *all lie in the same plane.*** In other words ***the incident, reflected; refracted waves and the normal to the interface all lie in the same plane.***

Furthermore fig. 9.3, we get

$$\left.\begin{aligned} \mathbf{k}_1\cdot\mathbf{r} &= k_1\,(x\sin\theta_i + z\cos\theta_i) && \text{...(a)} \\ \mathbf{k}_1'\cdot\mathbf{r} &= k_1'\,(x\sin\theta_r' - z\cos\theta_r') && \text{...(b)} \\ \mathbf{k}_2\cdot\mathbf{r} &= k_1\,(x\sin\theta_r - z\cos\theta_r) && \text{...(c)} \end{aligned}\right\} \qquad \text{...(10)}$$

Substitution values from (10a) and (10b) in equation (7) viz.

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_1'\cdot\mathbf{r})_{z=0}$$

we get

$$k_1 x\sin\theta_i = k_1'\,x\sin\theta_r'$$

or

$$k_1\sin\theta_i = k_1'\sin\theta_r' \qquad \text{...(11)}$$

Since wave vector $\mathbf{k}_1$ and $\mathbf{k}_1'$ lie in the same medium, hence $k_1 = k_1' = \omega\sqrt{(\mu_1\varepsilon_1)} = (\omega/v_1)\,v_1$, being phase velocity of electromagnetic wave in medium 1. Therefore equation (11) gives

$$\sin\theta_i = \sin\theta_r' \text{ or } \theta_i = \theta_r' \qquad \text{...(12)}$$

i.e. the angle of incident (θ_i), *is equal to the angle of reflection* (θ_r'),

Now substituting values of $\mathbf{k}_1\cdot\mathbf{r}$ and $\mathbf{k}_2\cdot\mathbf{r}$ from equation (10) in (7) viz

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_2\cdot\mathbf{r})_{z=0}$$

we get

$$k_1 x\sin\theta_i = k\,x\sin\theta_r$$

i.e.
$$k_1 \sin \theta_i = k_2 \sin \theta_r$$

or
$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{k_2}{k_1} = \frac{\omega_2 \sqrt{(\mu_2 \varepsilon_2)}}{\omega_1 \sqrt{(\mu_1 \varepsilon_1)}} = \frac{\sqrt{(\mu_2 \varepsilon_2)}}{\sqrt{(\mu_1 \varepsilon_1)}} \quad (\text{since } \omega_1 = \omega_2 = \omega)$$

This gives
$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{n_2}{n_1}$$

(since refractive index, $n = \sqrt{(\mu\, \varepsilon)} = (c/v)$)

This is well known *Snell's law of refraction.*

9.4. Fresnel's Equations (Dynamic Properties of Reflection and Refraction).

The equations relating the amplitudes of the reflected and transmitted waves with that of incident wave are known as Fresnel's equations (or Fresnel's formulae).

We know that field vectors **E** and **B** in a plane electromagnetic wave are mutually perpendicular and also they are perpendicular to the direction of propagation of the wave. Though the field vector **E** is always at right angles to the direction of propagation, but that does not fix the direction uniquely, and is said that the wave is polarised in a particular direction of its electric (or according to an alternative convention, its magnetic vector) points in that direction.

It is convenient to consider two extreme cases (i) in which incident wave is polarised such that electric field vector **E** is normal to the plane of incidence, and (ii) in which field vector **E** is parallel to the plane of incidence. The general result may be obtained by a suitable linear combination of these two extreme results.

Case (i) When E-vector is perpendicular to the plane of incidence : Thc electric and magnetic field vector $\mathbf{E}_1$ and $\mathbf{B}_1$ of the incident wave are perpendicular to direction of propagation $\mathbf{k}_1$ as shown in fig. 9.4. The electric field vector $\mathbf{E}_1$ is perpendicular to the plane of incidence. Since the media are isotropic, the electric field vectors of the reflected and refracted waves are also perpendicular to the plane of incidence. All the electric field vectors are shown directed away from the observer. The orientations of the **B**-vectors are chosen to give a positive flow of energy in the direction of wave vectors. The field vectors of incident, reflected and refracted waves are given by equations (1), (2) and (3) of section 9.3.

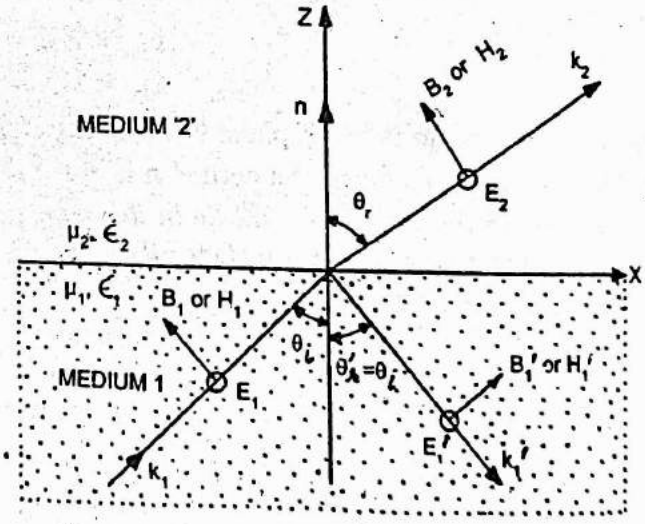


Fig. 9.4 Reflection and refraction with polarization perpendicular to plane of incidence,

Since electric field vectors are parallel to the boundary surface, the *continuity of the tangential component of electric field vectors* **E** *at the interface* requires that

$$E_{01} + E'_{01} = E_{02}, \qquad \text{..(1)}$$

Similarly, *the continuity of the tangential component of the magnetic field at the interface* requires that

$$H_{1t} + H'_{2t} = H_{2t}.$$

i.e.
$$-H_{01} \cos \theta_i + H'_{01} \cos \theta'_r = -H_{02} \cos \theta_r$$

i.e.
$$(H_{01} - H'_{01}) \cos \theta_i = H_{02} \cos \theta_r \qquad \text{...(2)}$$

(since from law of reflection $\theta_i = \theta'_r$)

But $\mathbf{B}_1 = \dfrac{\mathbf{k}_1 \times \mathbf{E}_1}{\omega_1}$; therefore $\mathbf{H}_1 = \dfrac{\mathbf{k}_1 \times \mathbf{E}_1}{\mu_1 \omega_1}$ (since $\mathbf{B}_1 = \mu_1 \mathbf{H}_1$)

or

$$\mathbf{H}_1 = \frac{k_1 n_1 \times \mathbf{E}_1}{\mu_1 \omega_1} \qquad \text{(where } n \text{ is vector along } \mathbf{k}_1\text{)}$$

$$= \frac{\sqrt{(\mu_1 \varepsilon_1)}\, n_1 \times \mathbf{E}_1}{\mu_1}$$

$$\left(\text{since } k_1 = \frac{\omega_1}{v_1} = \omega_1 \sqrt{(\mu_1 \varepsilon_1)} \text{ i.e. } \frac{k_1}{\omega_1} = \mu_{1-1}\right)$$

This gives,

$$\left.\begin{aligned} H_0 &= \sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{01} \\ H'_{01} &= \sqrt{\frac{\varepsilon_1}{\mu_1}}\, E'_{01} \\ H_{02} &= \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \end{aligned}\right\} \qquad \text{...(3)}$$

Using these results equation (2), gives

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_{01} - E'_{01}) \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \cos \theta_r. \qquad \text{...(4)}$$

Eliminating E_{02} from equations (1) and (2), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_{01} - E'_{01}) \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,(E_{01} + E'_{01}) \cos \theta_r.$$

On simplification, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp} = \frac{\sqrt{\dfrac{\varepsilon_1}{\mu_1}} \cos \theta_i - \sqrt{\dfrac{\varepsilon_2}{\mu_2}} \cos \theta_r}{\sqrt{\dfrac{\varepsilon_1}{\mu_1}} \cos \theta_i + \sqrt{\dfrac{\varepsilon_2}{\mu_2}} \cos \theta_r} \qquad \text{...(5)}$$

where the symbol $\perp$ denotes that in the case under consideration electric field is perpendicular to the plane of incidence. Equation (5), gives *the ratio of amplitudes of electric field vectors in reflected and incident waves i.e. relative electric field amplitudes of reflected wave with respect to incident wave.*

Similarly, eliminating E'_{01} from equations (1) and (4), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,\{(E_{01} - (E_{\theta 2} - E_{01}))\} \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \cos \theta_r$$

or

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(2E_{01} - E_{02}) \cos \theta_i = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02} \cos \theta_r.$$

On simplification, we get

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\sqrt{(\varepsilon_1/\mu_1)} \cos \theta_r}{\sqrt{(\varepsilon_1/\mu_1)} \cos \theta_i + \sqrt{(\varepsilon_2/\mu_2)} \cos \theta_r}. \qquad \text{...(6)}$$

This equation gives *relative electric field amplitude of refracted wave with respected to that of incident wave*. Equations (5), and (6) are known as *Fresnel's equations.* Now let us find the form of these equations in the particular case of non-conducting medium, where $\mu_1 = \mu_2 = \mu_0$, so that

$$n_1 = \sqrt{(K_{1m}K_{1e})} = \sqrt{\left(\frac{\mu_1}{\varepsilon_0}\frac{\varepsilon_1}{\varepsilon_0}\right)} = \sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$$

$$n_2 = \sqrt{(K_{2m}K_{2e})} = \sqrt{\left(\frac{\mu_2}{\mu_0}\frac{\varepsilon_2}{\varepsilon_0}\right)} = \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}$$

so that

$$\frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2/\varepsilon_0}{\varepsilon_1/\varepsilon_0}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{...(7)}$$

In this case equation (5) becomes

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \frac{\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_r}{\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_r}$$

$$= \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_3}{\varepsilon_1}}\cos\theta_r}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_r}$$

$$= \frac{\cos\theta_i - \sqrt{\frac{n_2}{n_1}}\cos\theta_r}{\cos\theta_i + \sqrt{\frac{n_2}{n_1}}\cos\theta_r} \qquad \text{...(8a)}$$

$$\left(\text{since } \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

Now using Snell's law $\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$, we obtain

$$\left(\frac{E'_1}{E_{01}}\right)_\perp = \frac{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_r}\cos\theta_r}{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_r}\cos\theta_r}$$

$$= \frac{\sin\theta_r\cos\theta_i - \sin\theta_i\cos\theta_r}{\sin\theta_r\cos\theta_i + \sin\theta_i\cos\theta_r}$$

i.e.

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} \qquad \text{...(8b)}$$

Similarly, equation (6) becomes

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_0}}\cos\theta_i}{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\mu_0}}\cos\theta_r}$$

$$= \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_r}$$

$$= \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{n_2}{n_1}}\cos\theta_r} \qquad \text{...(9a)}$$

Using Snell's law

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_r}\cos\theta_r} = \frac{2\sin\theta_r\cos\theta_i}{\sin\theta_r\cos\theta_i + \sin\theta_i\cos\theta_r}$$

i.e.

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)} \qquad \text{...(9b)}$$

Equations (8) and (9) represents *Fresnel's equations* for nonconducting media when electric field vector **E** is perpendicular to the plane of incidence.

Fresnel's equations (8) and (2) lead to following results :

(a) When $(n_1/n_2) < 1$ *i.e.* $n_1 < n_2$ or in words *when an electromagnetic wave in incident on the interface of two dielectrics from a rare medium,* then from Snell's law $\sin\theta_i/\sin\theta_r = n_2/n_1 > 1$; $\sin\theta_i > \sin\theta_r$ or $\theta_i > \theta_r$; *i.e.* refracted rays are deviated towards the normal hence from equation (8b), $\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}$ is negative thereby indicating that *the reflected and the incident waves are in a opposite phases at the interface.* In other words, *when an electromagnetic wave is reflected from a denser medium ; it suffers a phase change of π radians.*

(b) When $(n_1/n_2) > 1$ *i.e.* $n_1 > n_2$ or in words, when an electromagnetic wave is incident on the interface of two dielectric from a denser medium, then from Snell's law

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} < 1 \text{ or } \theta_i < \theta_r$$

i.e. the refracted ray is deviated away from the normal and hence from equation (8a), $\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}$ is positive, thereby indicating that reflected and incident waves are in same phase.

(c) In both the above cases (a) and (b), $\left[\frac{E_{02}}{E_{01}}\right]_{\perp}$ in equation (9b) is always positive; there by indicating that the refracted or transmitted wave does not suffer any phase change.

Case (ii). When E vector is parallel to the plane of incidence : In this case electric field vectors of ail in the three waves are parallel to the plane of incidence as shown fig. 9.5.

The directions of **B** (or **H**) vectors are chosen so as to give a position flow of energy in the direction of wave vectors. In this situation the magnetic field vectors **H** are all parallel to the boundary surface, hence the continuity of the tangential component of the magnetic field vectors **H** at the interface requires that

$$H_{01} + H'_{01} = H_{02}. \qquad \text{...(10)}$$

Similarly, the continuity of the tangential component of electric field vector **E** requires that

$$E_{01} \cos \theta_i - E'_{01} \cos \theta'_i = E_{02} \cos \theta_r$$

or $$(E_{01} - E'_{01}) \cos \theta_i = E_{02} \cos \theta_r$$

$$(\text{since } \theta'_r = \theta_i) \qquad \text{...(11)}$$

Now using equation (3), equation (10) takes the form

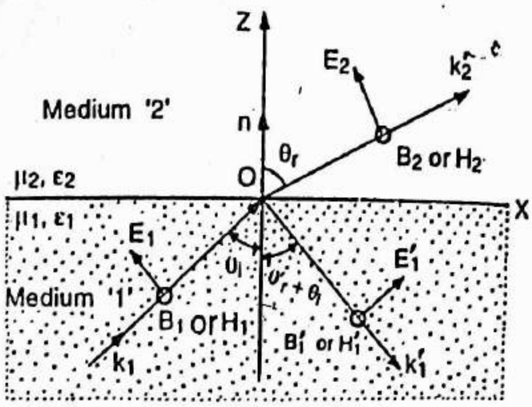


Fig. 9.5. Reflection and refraction with polarization parallel to the plane of incidence.

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{01} + \sqrt{\frac{\varepsilon_1}{\mu_1}}\, E'_{01} = \sqrt{\frac{\varepsilon_0}{\mu_2}}\, E_{01}. \qquad \text{...(12)}$$

Substituting E_{02} from (11) in eqn. (12), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_{01} + E'_{01}) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,(E_{01} - E'_{01})\,\frac{\cos\theta_i}{\cos\theta_r}$$

on simplification, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{-\left(\dfrac{\varepsilon_1}{\mu_1}\right) + \sqrt{\dfrac{\varepsilon_2}{\mu_2}}\,\dfrac{\cos\theta_i}{\cos\theta_r}}{\left(\dfrac{\varepsilon_1}{\mu_1}\right) + \left(\dfrac{\varepsilon_2}{\mu_2}\right)\dfrac{\cos\theta_i}{\cos\theta_r}} = \frac{\sqrt{\dfrac{\varepsilon_2}{\mu_2}}\cos\theta_i - \sqrt{\dfrac{\varepsilon_1}{\mu_1}}\cos\theta_r}{\sqrt{\dfrac{\varepsilon_1}{\mu_2}}\cos\theta_i + \sqrt{\dfrac{\varepsilon_1}{\mu_1}}\cos\theta_r} \qquad \text{...(13)}$$

where the symbol parallel (||) denotes that in the case under consideration electric field vector is parallel to the plane of incidence. Equation (13) gives the *ratio of amplitudes of reflected and incident waves.*

Now substituting E'_{01} from (11) in (12), we get

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{01} + \sqrt{\frac{\varepsilon_1}{\mu_1}}\left(E_{01} - E_{02}\,\frac{\cos\theta_r}{\cos\theta_i}\right) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\, E_{02}.$$

On simplification, we get

$$\sqrt{\left(\frac{E_{02}}{E_{01}}\right)_{\parallel}} = \frac{2\sqrt{\dfrac{\varepsilon_1}{\mu_1}}}{\sqrt{\dfrac{\varepsilon_2}{\mu_2}} + \sqrt{\dfrac{\varepsilon_1}{\mu_1}}\,\dfrac{\cos\theta_r}{\cos\theta_i}} = \frac{2\sqrt{\dfrac{\varepsilon_1}{\mu_1}}\cos\theta_i}{\sqrt{\dfrac{\varepsilon_2}{\mu_2}}\cos\theta_i + \sqrt{\dfrac{\varepsilon_1}{\mu_1}}\cos\theta_r}. \qquad \text{...(14)}$$

This equation gives the *ratio of amplitudes of refracted and incident waves.* Equations (13) and (14) represent. *Fresnel's equations* for the case when **E** vector is parallel to the plane of incidence.

For non-conducting media $\mu_1 = \mu_2 = \mu_0$ so

$$n_1 = \sqrt{(\mu_1\varepsilon_1/\mu_0\varepsilon_0)} = \sqrt{(\varepsilon_1/\varepsilon_0)}\ ;\ n_2 = \sqrt{(\mu_2\varepsilon_2/\mu_0\varepsilon_0)} = \sqrt{(\varepsilon_2/\varepsilon_0)}$$

and

$$\frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2/\varepsilon_0}{\varepsilon_1/\varepsilon_0}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{...(15)}$$

In this case equation (13) takes the form

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\sqrt{(\varepsilon_2/\mu_0)}\cos\theta_i - \sqrt{(\varepsilon_1/\mu_0)}\cos\theta_r}{\sqrt{(\varepsilon_2/\mu_0)}\cos\theta_i + \sqrt{(\varepsilon_2/\mu_0)}\cos\theta_r} = \frac{\sqrt{(\varepsilon_2/\varepsilon_1)}\cos\theta_i - \cos\theta_r}{\sqrt{(\varepsilon_2/\varepsilon_1)}\cos\theta_i + \cos\theta_r}$$

$$= \frac{\dfrac{n_2}{n_1}\cos\theta_i - \cos\theta_r}{\dfrac{n_2}{n_1}\cos\theta_r + \cos\theta_r}. \qquad \text{...(16 a)}$$

Using Snell's law $\dfrac{n_2}{n_1} = \dfrac{\sin\theta_i}{\sin\theta_r}$, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\left(\dfrac{\sin\theta_i}{\sin\theta_r}\right)\cos\theta_i - \cos\rho_r}{\left(\dfrac{\sin\theta_i}{\cos\theta_r}\right)\cos\theta_i + \cos\theta_r}$$

$$= \frac{\sin\theta_i\cos\theta_i - \sin\theta_r\cos\theta_r}{\sin\theta_i\cos\theta_i + \sin\theta_i\cos\theta_r} = \frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)} \qquad \text{...(16 b)}$$

Similarly equation (14) becomes

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\sqrt{\dfrac{\varepsilon_1}{\mu_0}}\cos\theta_i}{\sqrt{\dfrac{\varepsilon_2}{\mu_0}}\cos\theta_i + \sqrt{\dfrac{\varepsilon_1}{\mu_0}}\cos\theta_r} = \frac{2\cos\theta_1}{\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i + \cos\theta_r}$$

$$= \frac{2\cos\theta_i}{\dfrac{n_2}{n_1}\cos\theta_i + \cos\theta_r}. \qquad \text{...(17a)}$$

Using Snell's law, we get

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} \frac{2\cos\theta_i}{\left(\dfrac{\sin\theta_i}{\sin\theta_r}\right)\cos\theta_i + \cos\theta_r} = \frac{2\cos\theta_i\sin\theta_r}{\sin\theta_i\cos\theta_i + \sin\theta_r\cos\theta_r}$$

$$= \frac{2\cos\theta_i\sin\theta_r}{\frac{1}{2}(\sin 2\theta_i + \sin 2\theta_r)} = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)}. \qquad \text{...(17 b)}$$

Equations (16) and (17) represents *Fresnel's equations for nonconducting media when* **E** *vector is parallel to the plane of incidence.*

It is obvious that $(E_{02}/E_{01})_{\parallel}$ is always positive as in case (i), hence *the incidence and refracted beams are always in the same phase at the interface.* In this case there is an additional interesting feature if $\theta_i + \theta_r = 90°$, a perfectly possible situation, the denominator of (16b) is infinite, so that the reflection coefficient is zero. This angle of incidence $\theta_i = (90^\circ - \theta_r)$ is called the *Brewster's angle or polarising angle* and is denoted by θ_p. This if an unpolarised radiation consisting of a mixture of both types of radiation falls on the interface at this angle, only the radiation with its electric vector at right angles to the plane of incidence will be reflected, and the reflected radiation will be polarised.

From equation (16b) it is clear that the ratio $(E'_{01}/E_{01})_{\parallel}$ can be either positive or negative depending upon the value of $\dfrac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$. This is positive only if either $\theta_i > \theta_r$ and $\theta_i + \theta_r < \dfrac{\pi}{2}$ or $\theta_i < \theta_r$ and $\theta_i + \theta_r > \dfrac{\pi}{2}$. In all other cases $\dfrac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$ is negative. When $\dfrac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$ is positive, then $\left(\dfrac{E'_{01}}{E_{01}}\right)_{\parallel}$ is positive *i.e.* the reflected and incident waves will be in the phase at the boundary and when $\dfrac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$ is netgative $\left(\dfrac{E'_{01}}{E_{01}}\right)_{\parallel}$ is negative *i.e.* the reflected wave will suffer a phase change of π radian. Thus in this case the phase of reflected wave does not depend the ratio (n_2/n_1), but on relative values of θ_i and θ_r and their sum.

Fig. 9.6 shows the amplitudes of reflected and refracted disturbances as function of incident amplitudes for $n_2/n_1 = 1{\cdot}5$ and corresponds to light incident from air to glass.

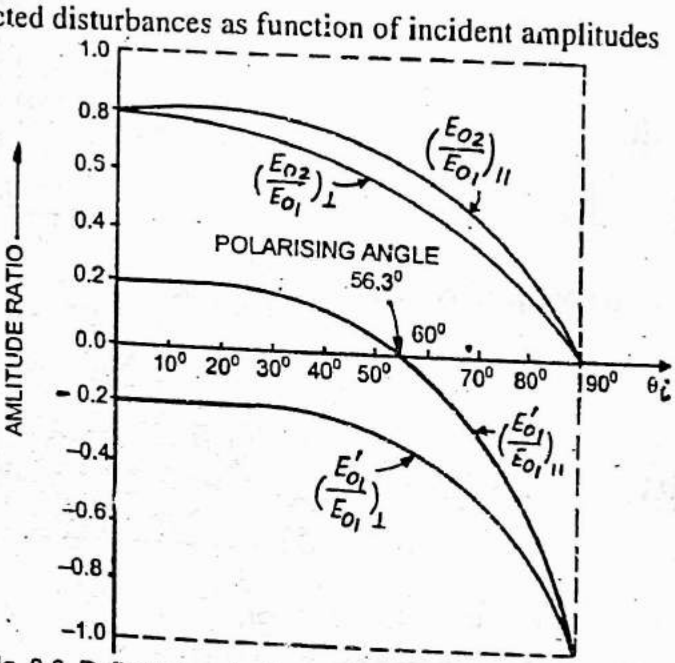


Fig. 9.6. Reflected and refracted amplitudes for $n_2/n_1 = 1{\cdot}5$.

For normal incidence $\theta_i = 0^\circ, \theta_r = 0^\circ$ Fresnel's equations for non-conducting media for the two cases becomes identical as is expected from fig. since then the plane of incidence is indeterminate

$$\left(\frac{E'_{01}}{E_{01}}\right) = \frac{n_2 - n_1}{n_2 + n_1} \qquad \text{...(18)}$$

$$\left(\frac{E_{02}}{E_{01}}\right) = \frac{2n_1}{n_2 + n_1}. \qquad \text{...(19)}$$

Note. The amplitudes of electric field vector **E** for incident. reflected and transmitted (refracted) waves are also denoted by E_i, E_r and E_t respectively. In these notation.

$$E_{01} \to E_i,\ E'_{01} \to E_r \text{ and } E_{02} \to E_t.$$

Summary. Fresnel's equations for non conducting media (where $\mu_1 = \mu_2 = \mu_0$) relating amplitudes of reflected and refracted waves to that of incident wave are :

Case. (i) E perpendicular to the plane of incidence.

$$\left.\begin{aligned} \left(\frac{E'_{01}}{E_{01}}\right)_{\perp} &= \left(\frac{E_r}{E_i}\right)_{\perp} = -\frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} && \text{...(a)} \\ \left(\frac{E_{02}}{E_{01}}\right)_{\perp} &= \left(\frac{E_t}{E_i}\right)_{\perp} = \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)} && \text{...(b)} \end{aligned}\right] \qquad \text{...(8.5)}$$

Case (ii) E parallel to plane of incidence.

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \left(\frac{E_r}{E_i}\right)_\perp = -\frac{\tan(\theta_i-\theta_r)}{\tan(\theta_i+\theta_r)} \qquad \text{...}(a)$$

$$\left(\frac{E_{02}}{E_{01}}\right)_\perp = \left(\frac{E_t}{E_i}\right)_\perp = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)\cos(\theta_i-\theta_r)} \qquad \text{...}(b) \qquad \text{...(8.6)}$$

At normal incidence ($\theta_i = 0$, $\theta_r = 0$), both above equations reduce to

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = \left(\frac{E_r}{E_i}\right)_\perp = \frac{n_2-n_1}{n_2+n_1} \qquad \text{...}(a)$$

$$\left(\frac{E_{02}}{E_{01}}\right)_\perp = \left(\frac{E_t}{E_i}\right)_\perp = \frac{2n_1}{(n_2+n_1)} \qquad \text{...}(b)$$

Discussion

(i) The two components $E_\perp$ and $E_\parallel$ are not transmitted and reflected equally. For case (i) when electric vector is perpendicular to the plane of incidence, there is reflection unless $\theta_i = \theta_r$ *i.e.* unless the two media have the same optical properties.

(ii) The reflected amplitude in case (ii) *i.e.* when E is parallel to the plane of incidence vanishes if $\theta_i + \theta_r = 90°$. The angle of incidence θ_i satisfying this condition is called Brewster's angle θ_p.

(iii) In case (i). the reflected ray suffers a phase change of π at the interface.

(iv) In case (ii) the reflected ray suffers a phase change of π radian only when angle of incidence is greater than Brewster's angle. For $\theta_i < \theta_p$, the reflected and incident waves are in the same phase at the interface.

(v) The refracted ray never suffers a phase change at the interface.

(vi) At grazing incidence $\theta_i = 90°$, the reflection coefficient is equal to unity. The formula then gives a monotonic increase of amplitude as the angle of incidence increases.

9.5. Experimental Verification of Fresnel's Equations.

The angle between plane of incidence and the plane of vibration of **E** in the linearly polarised light is called *azimuth of the wave.* If β_1, β'_1 and β_2 are the azimuths of incident, reflected and refracted electromagnetic waves, then we have

$$\tan\beta_1 = \frac{(E_{01})_\perp}{(E_{0i})_\parallel}\;;\ \tan\beta_1' = \frac{(E_{01})_\perp}{(E_{01}')_\parallel} \text{ and } \tan\beta_2 = \frac{(E_{01})_\perp}{(E_{02})_\parallel} \qquad \text{...(1)}$$

Now by the help of Fresnel's equations, it is possible to express the azimuth's of reflected and refracted waves in terms of incident azimuth as follows :

$$\left(\frac{E'_{01}}{E_{01}}\right)_\perp = -\frac{\sin(\theta_i-\theta_r)}{\sin(\theta_i+\theta_r)} \qquad \text{...(a)}$$

$$\left(\frac{E_{02}}{E_{01}}\right)_\perp = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)} \qquad \text{...(b)}$$

$$\left(\frac{E_{01}}{E_{01}}\right)_\parallel = \frac{\tan(\theta_i-\theta_r)}{\tan(\theta_i+\theta_r)} \qquad \text{...(c)} \qquad \text{...(2)}$$

$$\left(\frac{E_{02}}{E_{01}}\right)_\parallel = \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)\cos(\theta_i+\theta_r)} \qquad \text{...(d)}$$

$$\tan \beta'_1 = \left(\frac{(E'_{01})_\perp}{(E_{01})_\parallel} \right) = \frac{\left(\frac{E'_{01}}{E_{01}} \right)_\perp (E_{01})_\perp}{\left(\frac{E'_{01}}{E_{01}} \right)_\perp ({}^0E_1)_\parallel} = \frac{\left(\frac{E'_{01}}{E_{01}} \right)_\perp}{\left(\frac{E'_{01}}{E_{01}} \right)_\perp} \tan \beta_1$$

Now using (2a) and (2c), we get

$$\tan \beta'_1 = -\frac{[\sin(\theta_i - \theta_r)/\sin(\theta_i + \theta_r)]}{[\tan(\theta_i - \theta_r)/\tan(\theta_i + \theta_r)]} \tan \beta_1 .$$

$$\therefore \qquad \tan \beta_1' = -\frac{\cos(\theta_i - \theta_r)}{\cos(\theta_i + \theta_r)} \tan \beta_1$$

Similarly

$$\tan \beta_2 = \frac{(E_{02})_\perp}{(E_{02})_\parallel} = \frac{(E_{02}/E_{01})_\perp (E_{01})_\perp}{(E_{02}/E_{01})_\parallel (E_{01})_\parallel} .$$

Using equations (1), (2b) and (2d), we get

$$\tan \beta_2 = \frac{[2\cos\theta_i \sin\theta_r/\sin(\theta_i + \theta_r)]}{[2\cos\theta_i \sin\theta_r/\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)]} \tan \beta_1$$

$$\therefore \qquad \tan \beta_2 = \cos(\theta_i + \theta_r) \tan \beta_1 . \qquad \text{...(4)}$$

Obviously equations (2) and (4) are equivalent to Fresnel's equations (2).

For experimental verification of Fresnel's equations the polarising nicol is set with its principal section at 45° to the plane of incidence. Thus equation (3) takes the form

$$\tan \beta_1' = -\frac{\cos(\theta_i - \theta_r)}{\cos(\theta_i + \theta_r)} \qquad \text{...(5)}$$

The angle β_1 between the direction of vibration of reflected light and plane of incidence can be determined experimentally for different values of θ_i.

The theoretical values of β_1' can be obtained by calculating θ_r from Snell's law

$$\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$$

i.e.

$$\sin\theta_r = \frac{\sin\theta_i}{(n_2/n_1)} = \frac{\sin\theta_i}{{}_1n_2}$$

if ${}_1n_2$ is known initially be some other experiment.

The experimental and theoretical values of β_1' are found to be in agreement; thus verifying Fresnel;'s equations.

Experimental Procedure : The refractive index n_2 of the material of the prism is determined for monochromatic light ; say sodium light by means of an adjusted spectrometer. Now polarising angle ϕ_p is calculated from Brewster's law

$${}_1n_2 = \tan\phi_p \quad \text{or} \quad \phi_p = \tan^{-1}({}_1n_2) .$$

Now the telescope is focussed to a position to receive the direct image of the slit and then rotated through $(180° - 2\phi_p)$ and clamped. A cleaned prism face is placed on the prism table such that the cleaned face is along the axis of spectrometer and also parallel to the collimater axis. The prism table rotated to a position to receive the light reflected from this face into the clamped telescope. In this position light is incident on the prism face at the polarising angle ϕ_p. Hence the reflected light is plane polarised with its vibration perpendicular to the plane of incidence (horizontal).

Now a polaroid is fitted on the telescope objective and rotated without disturbing the prism in any way to get a minimum intensity of image of slit. This is known as *tint of passage.* This reading of polaroid pointer is noted.

The prism is now removed, the telescope is again brought into the initial position (*i.e.* in the line with the collimator) and *another polaroid* is mounted on the collimator lens. This polaroid is rotated till the tint of passage reappears. From this setting the collimator polaroid is rotated through 45°. This setting makes vibration of plane of incident light inclined at 45° to the plane of incidence. This condition is necessary to make $\tan \beta_1 = 1$.

The telescope is now rotated from the direct setting through a known small angle α, say about 10°, and clamped. The prism is remounted in its original position and then rotated by rotating the prism table to a position to receive the light reflected from the prism face into position to receive the light reflected from the prism face into the telescope. The angle of incidence is then given by

$$\theta_i = \frac{180 - \alpha}{2}$$

The analysing (or telescope) polaroid is rotated from its initial setting through an angle to restore the tint of passage. This angle of rotation is β_1', then the angle between plane of vibration of reflected light and the plane of incidence is $\beta_1' = 90° - \beta_1$. The angle θ_i is varied in steps of 10° (by turning the telescope) and the angle β_1' is measured for all values of θ_i.

Finally a graph is plotted between $\tan \beta_1'$ and

$$\frac{\cos(\theta_i - \theta_r)}{\cos(\theta_i + \theta_r)}.$$

which comes out to be a straight line passing through the origin and inclined at an angle of 45° with the axes. This verifies the *Fresnel's equations for reflection.*

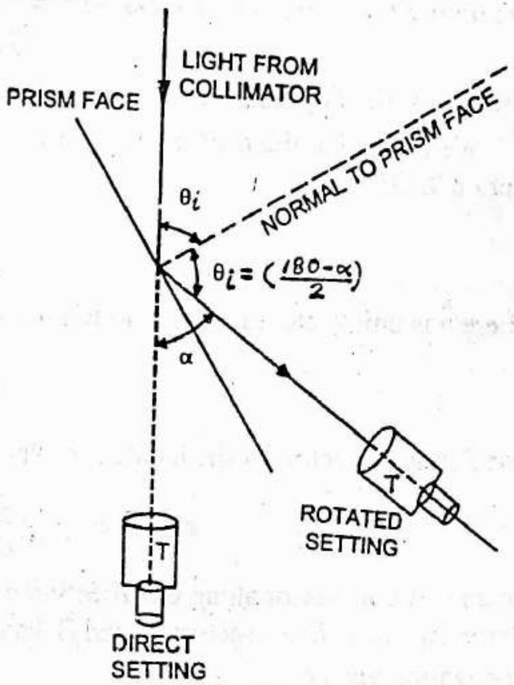


Fig. 9.7

9.6 Reflection and Transmission Coefficients at the Interface Between two Non-conducting Media

Reflection coefficient or reflectivity *at the interface between on two non-conducting media is defined as the fraction of incident energy flux reflected per unit area per second at the interface*

$$i.e. \text{ Reflection coefficient, } R \; \frac{\text{Magnitude of reflected energy flux per unit area per second at interface}}{\text{Magnitude of incident energy flux per unit area per second at interface}}$$

Transmission coefficient or *transmittivity at the interface between two media is defined as the fraction of incident energy flux transmitted (or refracted) per unit area per second at the interface* *

---

* The face of the prism is thoroughly cleaned with chromic acid.

$$i.e.\ \text{Transmission coefficient, } T = \frac{\text{Refracted (or transmitted) energy flux per unit area per second at interface}}{\text{Incident energy flux per unit area per second at interface}} \qquad ...(2)$$

At oblique incidence at the interface the cross sectional areas of incident and reflected beams are different from that of the transmitted beam. Therefore, the transmitted intensity is not complementary to the reflected one; but the total energy of these beams is complementary. Suppose a polarised beam of finite cross-section is incident on unit area of the boundary. Then the cross-section area of the incident. Reflected and transmitted beams are respectively.

$$a' = \cos\theta_i,\ a_1' = \cos\theta_r' \text{ and } a_2 = \cos\theta_r \qquad ...(3)$$

The *amount of energy in the incident beam per second* is

$$\mathbf{J}_1 = \mathbf{S}_1 \cdot \mathbf{a}_1 = \mathbf{S}_1 \cos\theta_i \qquad ...(4)$$

where $\mathbf{S}_1$ is the Poynting vector for incident beam.

We know that the time average of Poynting vector in a non-conducting medium is [refer equation (25) section 9.13]

$$<\mathbf{S}> = \frac{E_0^2}{2Z}\hat{n} \qquad ...(5a)$$

where $\hat{n}$ is unit vector along the direction of energy flow and $\mathbf{Z}$ is *wave impedance* given by $\mathbf{Z} = \sqrt{(\mu/\varepsilon)}$

$$\therefore \qquad <\mathbf{S}> = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\, E_0\,\hat{n} \qquad ...(5b)$$

The Poynting vector for the incident wave at the interface is

$$<\mathbf{S}_1> = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2\,\hat{\mathbf{n}} \text{ or } <\mathbf{S}_1> = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2 \qquad ...(6)$$

where $\mathbf{n}$ is unit vector along the direction of energy flow in the incident beam of along the direction of wave vector $\mathbf{k}_1$ since field vectors $\mathbf{E}$ and $\mathbf{B}$ have been chosen to give the energy flow along the direction of propagation vector.

Therefore the incident energy flux per unit area of the boundary per second is

$$J_1 = <S_1> \cos\theta_i = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2 \cos\theta_i \qquad ...(7)$$

Similarly for reflected and transmitted wave, we have

$$J_1' = <S_1'> \cos\theta_i' = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2 \cos\theta_i \qquad ...(8)$$

(since $\theta_r' = \theta_i$)

and

$$J_2 = <S_2> \cos\theta_r = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{02}^2 \cos\theta_r \qquad ...(9)$$

$\because$ *Reflection coefficient,*

$$R = \frac{J_1'}{J_1} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{02}^2 \cos\theta_i}{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2 \cos\theta_i}$$

$$= \frac{E_{01}^2}{E_{01}^2} = \left(\frac{E_{01}'}{E_{01}}\right)^2 \qquad ...(10)$$

and *transmission coefficient,*

$$T = \frac{J_2}{J_1} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_2}{\mu_0}}\, E_{02}^2 \cos\theta_r}{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_0}}\, E_{01}^2 \cos\theta_i}$$

$$= \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}\frac{E_{02}^2}{E_{01}^2}\cdot\frac{\cos\theta_r}{\cos\theta_i} = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left\{\frac{E_{02}}{E_{01}}\right\}^2 \qquad ...(11)$$

Now from Fresnel's equations

$$\left.\begin{aligned}\left(\frac{E_{01}'}{E_{01}}\right)_{\parallel} &= \frac{\tan(\theta_i-\theta_r)}{\tan(\theta_i+\theta_r)} && ...(a)\\ \left(\frac{E_{02}}{E_{01}}\right)_{\parallel} &= \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)\cos(\theta_i-\theta_r)} && ...(b)\end{aligned}\right] \qquad ...(12$$

$$\left.\begin{aligned}\left(\frac{E_{01}'}{E_{01}}\right)_{\perp} &= -\frac{\sin(\theta_i-\theta_r)}{\sin(\theta_i+\theta_r)}. && ...(a)\\ \left(\frac{E_{02}}{E_{01}}\right)_{\perp} &= \frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)} && ...(b)\end{aligned}\right] \qquad ...(13)$$

When E vector is parallel to the plane of incidence, the coefficients are given by

$$R_{\parallel} = \left(\frac{E_{01}'}{E_{01}}\right)_{\parallel}^2 = \frac{\tan^2(\theta_i-\theta_r)}{\tan^2(\theta_i+\theta_r)} \qquad ...(14)$$

and

$$T_{\parallel} = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left(\frac{E_{02}}{E_{01}}\right)_{\parallel}^2 = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left[\frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)\cos(\theta_i+\theta_r)}\right]^2$$

Using Snell's law,

$$\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$$

$$T_{\parallel} = \frac{\sin\theta_i\cos\theta_r}{\sin\theta_r\cos\theta_i}\cdot\frac{4\cos^2\theta_i\sin^2\theta_r}{\sin^2(\theta_i+\theta_r)\cos^2(\theta_i-\theta_r)}$$

$$= \frac{\sin 2\theta_i\sin 2\theta_r}{\sin^2(\theta_i+\theta_r)\cos^2(\theta_i-\theta_r)} \qquad ...(15)$$

When E vector is perpendicular to the plane of incidence, coefficients are

$$R_{\perp} = \left(\frac{E'_{01}}{E_{01}}\right)_{\perp}^2 = \frac{\sin^2(\theta_i-\theta_r)}{\sin^2(\theta_i+\theta_r)} \qquad ...(16)$$

$$T_{\perp} = \frac{n_2}{n_1}\frac{\cos\theta_r}{\cos\theta_i}\left\{\frac{E_{02}}{E_{01}}\right\}_{\perp}^2 = \frac{\sin\theta_i\cos\theta_r}{\sin\theta_r\cos\theta_i}\left[\frac{2\cos\theta_i\sin\theta_r}{\sin(\theta_i+\theta_r)}\right]^2$$

$$= \frac{\sin 2\theta_i\sin 2\theta_r}{\sin^2(\theta_i+\theta_r)} \qquad ...(17)$$

For normal incidence, we have

$$\left.\begin{aligned} \frac{E_{01}'}{E_{01}} &= \frac{n_2 - n_1}{n_2 + n_1} \qquad \text{...(a)} \\ \text{and} \quad \frac{E_{02}}{E_{01}} &= \frac{2n_1}{n_2 + n_1} \qquad \text{...(b)} \end{aligned}\right\} \qquad \text{...(18)}$$

∴ Reflection and transmission coefficients for normal incidence $(\theta_i + \theta_r = 0)$ are given by

$$R = \left(\frac{E_{01}'}{E_{01}}\right)^2 = \frac{(n_2 - n_1)^2}{(n_2 + n_1)^2} \qquad \text{...(19)}$$

$$T = \frac{n_2}{n_1}\left(\frac{E_{02}}{E_{01}}\right)^2 \frac{\cos\theta_r}{\cos\theta_i}$$

$$= \frac{n_2}{n_1}\left(\frac{2n_1}{n_2 + n_1}\right)^2 = \frac{4n_1 n_2}{(n_2 + n_1)^2} \qquad \text{...(20)}$$

It is obvious that $$R + T = 1, \qquad \text{...(21)}$$

in accordance with the law of conservation of energy.

9.7 Brewster's Law and Degree of Polarisation.

Fresnel equations are

$$\left.\begin{aligned} \left(\frac{E_{01}'}{E_{01}}\right)_{\parallel} &= \frac{\tan(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)} \qquad \text{...(a)} \\ \left(\frac{E_{02}}{E_{01}}\right)_{\parallel} &= \frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)\cos(\theta_i - \theta_r)} \qquad \text{...(b)} \end{aligned}\right] \qquad \text{...(1)}$$

$$\left.\begin{aligned} \left(\frac{E_{01}'}{E_{01}}\right)_{\perp} &= -\frac{\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} \qquad \text{...(a)} \\ \left(\frac{E_{02}}{E_{01}}\right)_{\perp} &= -\frac{2\cos\theta_i \sin\theta_r}{\sin(\theta_i + \theta_r)} \qquad \text{...(b)} \end{aligned}\right] \qquad \text{...(2)}$$

It is obvious from equation [1 (a)] that when $(\theta_i + \theta_r) = \pi/2$, $(E_{01}'/E_{01})_{\parallel} = 0$; so that the reflection coefficient is zero. Thus when the reflected and refracted waves are perpendicular to each other, there is no energy carried by the reflected ray. This angle of incidence $\theta_i = (\pi/2 - \theta_r)$ is known as the *Brewster's angle* or *polarising angle* and is denoted by θ_B or θ_p.

Now from Snell's law

$$\frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_r}$$

or $$\frac{n_2}{n_1} = \frac{\sin\theta_p}{\sin(\pi/2 - \theta_p)} \qquad \left\{\text{because } \theta_i + \theta_r = \frac{\pi}{2}\right\}$$

i.e. $$\tan\theta_p = \frac{n_2}{n_1} \quad \text{or} \quad \theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}({}_1n_2)$$

This is well known *Brewster's law*. Thus if a plane wave of mixed polarisation (or an unpolarised wave) is incident on a plane interface at the *Brewster's angle*, then the only portion of wave with electric vector perpendicular to the plane of incidence will be reflected. Thus, the reflected radiation is completely plane polarised with polarisation vector perpendicular to the plane of incidence. Hence, the *Brewster's* *angle is also known as polarising angle.* For a wave entering glass ($n_2 = 1{\cdot}5$) from air ($n_1 = 1$), $\theta_p = 56^\circ$ and so $\theta_r = (90^\circ - 56^\circ) = 34^\circ$. Thus, if we allow unpolarised light to strike glass at angle of incidence 56°, the reflected light will be plane polarised with polarisation vector perpendicular to the plane of incidence and transmitted light will also be plane polarised with polarisation vector parallel to the plane of incidence as shown in fig. 9.8 (a). Hence again Maxwell's equations predict an effect long known in optics.

For radio waves incidence on water ($n_1 = 9$), refractive index of water radio frequencies from ($n_1 = 1$), $\theta_p = 84^\circ$ and so $\theta_r = 6^\circ$. Therefore, if radio waves are incident on water at angle of incidence 84°, water will not effect radio waves which are polarised with polarisation vector in the plane of incidence and will not transmit radio wave initially polarised with polarisation vector perpendicular to the plane of incidence as shown in fig. 9.8 (b) and 9.8 (c).

Even if the unpolarised wave is reflected at angles other than the Brewster's angle, there is a tendency for the reflected wave to be predominately polarised perpendicular to the plane of incidence. The success of dark glasses which selectively transmits only one direction of polarisation depends on this fact. In such a case the ***degree of polarisation*** is defined as

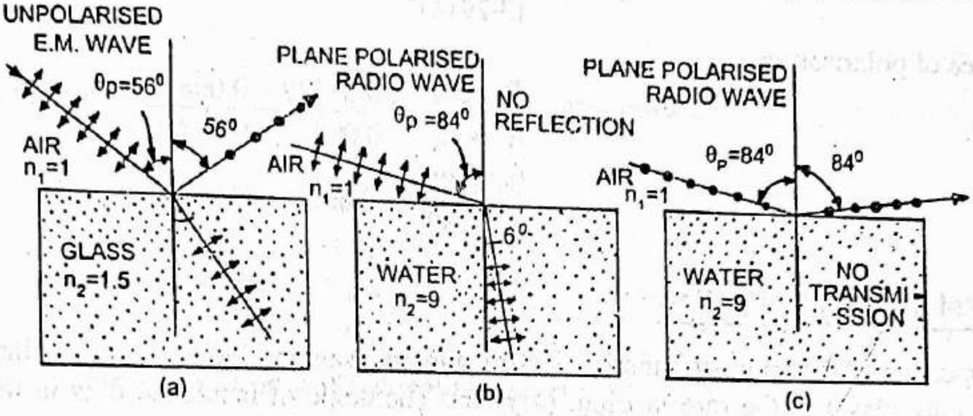


Fig. 9.8

$$\text{degree of polarisation } P = \frac{R_\perp - R_\parallel}{R_\perp + R_\parallel}$$

where $R_\perp$ and $R_\parallel$ are reflection coefficient for perpendicular and parallel components of reflected light are given by

where
$$R_\perp = \frac{\sin^2(\theta_i - \theta_r)}{\sin^2(\theta_i + \theta_r)} \quad \text{and} \quad R_\parallel = \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)}$$

Ex. 1. *Calculate the degree of polarisation for ordinary light reflected from glass (refractive index 1.5) at an angle of incidence 45°.* **(Meerut 1970)**

Solution. From Snell's law

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} = {}_1n_2 = n \text{ (say)}$$

Here $\theta_1 = 45^\circ$ and $n = 1{\cdot}5$.

$$\therefore \qquad \sin\theta_r = \frac{\sin\theta_i}{n} = \frac{\sin 45^\circ}{1{\cdot}5} = \frac{1}{\sqrt{2} \times 1{\cdot}50}$$

or
$$\theta_r = \sin^{-1}\left(\frac{1}{1{\cdot}5\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{2}{3\sqrt{2}}\right) = \sin^{-1}\left(\frac{2\sqrt{2}}{6}\right) = \sin^{-1}(0{\cdot}4713) = 28{\cdot}1^\circ$$

$$\therefore \qquad R_\perp = \frac{\sin^2(\theta_i - \theta_r)}{\sin^2(\theta_i + \theta_r)} = \frac{\sin^2(45° - 28{\cdot}1°)}{\sin^2(45° + 28{\cdot}1°)}$$

$$= \frac{\sin^2 16{\cdot}9°}{\sin^2 73{\cdot}1°} = \frac{(0{\cdot}2294)^2}{(0{\cdot}9568)^2}$$

$$= 0{\cdot}09337.$$

and

$$R_\parallel = \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)} = \frac{\tan^2 16{\cdot}9°}{\tan^2 73{\cdot}1°}$$

$$= \frac{(0{\cdot}3038)^2}{(3{\cdot}2914)^2} = 0{\cdot}00852.$$

∴ Degree of polarisation

$$= \frac{R_\perp - R_\parallel}{R_\perp + R_\parallel} = \frac{0{\cdot}09337 - 0{\cdot}00852}{0{\cdot}09337 + 0{\cdot}00852}$$

$$= \frac{0{\cdot}08485}{0{\cdot}10189} = \mathbf{0{\cdot}833 = 83{\cdot}5\%}.$$

9.8 Total Internal Reflection

Suppose a plane monochromatic electromagnetic wave is incident on the interface from a denser medium (say glass) to the rare medium (say air). The angle of incidence θ_i is in the denser medium and angle of refraction θ is the rarer medium. The word internal implies that the incident and reflected waves are in denser medium (*i.e.* medium of larger refracted index than the refracted wave ($n_1 > n_2$).

Now from Snell's law is

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} \qquad \text{...(1)}$$

Since $\frac{n_2}{n_1} < 1$; this implies $\frac{\sin\theta_i}{\sin\theta_r} < 1$ or $\theta_i < \theta_r$.

Thus for a particular value of angle of incidence θ_i, the angle of refraction θ_r will becomes 90°. This value of angle of incidence for which θ_r becomes 90° is known as *critical angle* and is denoted by θ_c. Substituting $\theta_i = \theta_c$ and $\theta_r = 90$ in (1), we get

$$\frac{\sin\theta_c}{\sin 90°} = \frac{n_2}{n_1}$$

i.e.

$$\sin\theta_c = \frac{n_2}{n_1} \quad \text{or} \quad \theta_c = \left(\sin^{-1}\frac{n_2}{n_1}\right) \qquad \text{...(2)}$$

Thus for wave incident at $\theta_i = \theta_c$, the refracted wave is propagated parallel to the surface (since $\theta_r = 90°$) as shown in fig. 9.9. For glass and air

$$\frac{n_2}{n_1} = \frac{1}{1{\cdot}5} = \frac{2}{3};$$

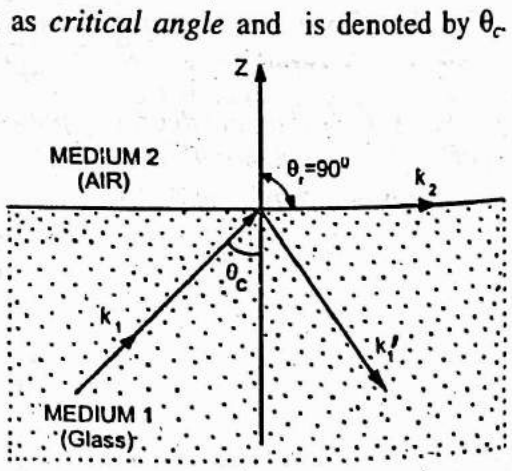


Fig. 9.9

$$\therefore \qquad \theta_c = \sin^{-1}\tfrac{2}{3} = \sin^{-1}(0{\cdot}6667) = 41°55'$$

Now let us see what happens if the angle of incidence θ_i exceeds the critical angle θ_c. From equation (1)

$$\sin\theta_r = \frac{\sin\theta_i}{\left(\dfrac{n_2}{n_1}\right)} = \frac{\sin\theta_i}{\sin\theta_c} > 1 \qquad (\text{since } \theta_i > \theta_c) \qquad \text{...(3)}$$

This equations indicates that angle of refraction θ_r is imaginary; since sine of any real angle can never be greater than unity.

Thus
$$\cos\theta_r = \sqrt{(1-\sin^2\theta_r)} = \sqrt{\left[1-\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2\right]}$$
$$= i\sqrt{\left[\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1\right]} \qquad \text{...(4)}$$

i.e. $\cos\theta_r$ is purely imaginary $\left(\text{since } \dfrac{\sin\theta_i}{\sin\theta_r} > 1\right)$

Now to find the meaning of imaginary θ_r, let us consider the transmitted or refracted electromagnetic wave

$$E_2 = E_{02}\, e^{i\mathbf{k}_2 \cdot \mathbf{r} - i\omega t}$$
$$= \mathbf{E}_{02}\, e^{ik_2 \hat{\mathbf{n}}_2 \cdot \mathbf{r} - i\omega t} \qquad \text{...(5)}$$

($\hat{\mathbf{n}}_2$ being unit vector along $\mathbf{k}_2$)

As transmitted wave lies in xz plane, hence

$$\hat{\mathbf{n}}_2 \cdot \mathbf{r} = (\mathbf{i}\sin\theta_r + \mathbf{k}\cos\theta_r)\cdot(\mathbf{i}x + \mathbf{i}y + \mathbf{k}z)$$
$$= x\sin\theta_r + z\cos\theta_r$$
$$\mathbf{E}_2 = \mathbf{E}_{02}\, e^{ik_2(x\sin\theta_r + z\cos\theta_r) - i\omega t} \qquad \text{...(6)}$$

Substituting value of $\sin\theta_r$ and $\cos\theta_r$ from equation (3) and (4), we get

$$\mathbf{E}_2 = \mathbf{E}_{02}\exp\left[ik_2\left\{x\left(\frac{\sin\theta_i}{\sin\theta_c}\right) + iz\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}\right\}\right]\exp(-i\omega t)$$
$$= \mathbf{E}_{02}\exp\left[-k_2 z\left\{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1\right\}^{1/2}\right]\exp\left[ik_2\left(\frac{\sin\theta_i}{\sin\theta_c}\right)x - i\omega t\right] \qquad \text{...(7)}$$

This equation shows that for $\theta_i > \theta_c$, *the refracted wave is propagated only parallel to the surface and is attenuated exponentially beyond the interface.* Thus we conclude that for $\theta_i > \theta_c$ the wave actually penetrates the second medium where its amplitude is quickly damped according to factor $e^{-k_2 z[(\sin\theta/\sin\theta_r)^2 - 1]^{1/2} z}$. The existence of this damped wave within the second medium has been established in experiments employing electromagnetic waves of a few centimeters wavelength (microwaves).

Energy flow in the second medium

The time-averaged normal component of Poynting vector just inside the surface gives the energy flux carried by transmitted wave per unit area per second and is given by

$$\mathbf{S}_2 \cdot \mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\,[(\mathbf{E}_2 \times \mathbf{H}_2^*)\cdot\mathbf{n}]$$

But
$$\mathbf{H}_2 = \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2\omega_2} \qquad \left\{\text{where } k_2 = \frac{\omega_2}{v_2} = \omega_2\sqrt{\varepsilon/\mu_2} \text{ is real}\right\}$$

$$\therefore \qquad \mathbf{S}_2\cdot\mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\left\{\left[\mathbf{E}_2 \times \frac{\mathbf{k}_2 \times \mathbf{E}_2{}^*}{\mu_2\,\omega_2}\right]\cdot\mathbf{n}\right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{\left[\frac{(\mathbf{E}_2\cdot\mathbf{E}_2{}^*)\,\mathbf{k}_2\,(\mathbf{E}_2\cdot\mathbf{k}_2)\,\mathbf{E}_2{}^*}{\mu_2\,\omega_2}\right]\cdot\mathbf{n}\right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{\left[\frac{(\mathbf{E}_2\cdot\mathbf{E}_2{}^*)\,\mathbf{k}_2\cdot\mathbf{n}}{\mu_2\,\omega_2}\right]\right\}$$

(since $\mathbf{k}_2$ and $\mathbf{E}_2$ are mutually perpendicular)

But $\qquad \mathbf{E}_2.\mathbf{E}_2{}^* = \mathrm{E}_{02}^2\, e^{-2k_2 z\,[(\sin\theta_i/\sin\theta_c)^2-1]^{1/2}}$

and $\qquad \mathbf{k}_2\cdot\mathbf{n} = k_2\cos\theta_r$

$$\therefore \qquad \mathbf{S}_2\cdot\,\mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{k_2\,E_{12}^2}{\mu_2\omega_2}\, e^{-2k_2 z\,\{(\sin\theta_i/\sin\theta_c)^2-1\}^{1/2}}\cos\theta_r\right] = 0 \qquad \text{...(8)}$$

Since $\cos\theta_r$ is purely imaginary, the quantity within bracket is purely imaginary.

Thus we may say that *the net energy flow through the surface into second medium is zero* even though fields exist in the second medium. Hence total internal reflection occurs at $\theta_i \geq \theta_c$

Now, let us re-examine Fresnel's equations by inserting the values of $\sin\theta_r$ and $\cos\theta_r$ from equations (3) and (4) respectively. In the first case when **E** is perpendicular to the plane of incidence, we have(refer eqn. (8a) of section 9.4).

$$\left\{\frac{E'_{01}}{E_{01}}\right\}_{\perp} = \frac{\cos\theta_i - \dfrac{n_2}{n_1}\cos\theta_r}{\cos\theta_i + \dfrac{n_2}{n_1}\cos\theta_r} \qquad \text{...(9)}$$

$$= \frac{\alpha - i\beta}{\alpha + i\beta} \text{ where } \begin{cases} \alpha = \cos\theta_r\; i\beta = \dfrac{n_2}{n_1}\cos\theta_r \\ \text{(since } \cos\theta_r \text{ is purely imaginary)} \end{cases}$$

$$= \frac{(\alpha - i\beta)\,(\alpha - i\beta)}{(\alpha + i\beta)\,(\alpha - i\beta)} = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2} - i\,\frac{2\alpha\beta}{\alpha^2+\beta^2} \qquad \text{...(10)}$$

If we put

$$\frac{\alpha - i\beta}{\alpha + i\beta} = \rho e^{-i\delta_{\perp}} = (\rho\cos\delta_{\perp}) - i\,(\rho\sin\delta_{\perp}) \qquad \text{...(11)}$$

Comparing (10) and (11), we get

$$\rho\cos\delta_{\perp} = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2} \text{ and } \rho\sin\delta_{\perp} = \frac{2\alpha\beta}{\alpha^2+\beta^2} \qquad \text{...(12)}$$

which yields $\rho = 1$ and $\tan\delta_{\perp} = \dfrac{2\alpha\beta}{\alpha^2-\beta^2}$ ...(13)

Thus $\qquad \left\{\dfrac{E_{01}'}{E_{01}}\right\}_{\perp} = \dfrac{\alpha - i\beta}{\alpha + i\beta} = e^{i\delta_{\perp}}$ with $\tan{}^*\dfrac{\delta_{\perp}}{2} = \dfrac{\beta}{\alpha}$ ...(14)

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* If $\tan\delta = \dfrac{2\alpha\beta}{\alpha^2-\beta^2}$ or $\dfrac{2}{\cot\delta/2 - \tan\delta/2} = \dfrac{2}{\alpha/\beta - \beta/\alpha}$

$\therefore \quad \tan\delta/2 = \beta/\alpha.$

Similarly in the second case when **E** is parallel to the plane of incidence (refer equation (16a) of section 9.4).

$$\left\{\frac{E_{01}'}{E_{01}}\right\}_{\parallel} = \frac{\dfrac{n_2}{n_1}\cos\theta_i - \cos\theta_r}{\dfrac{n_2}{n_1}\cos\theta_i + \cos\theta_r}$$

$$= \frac{\cos\theta_i - \dfrac{n_1}{n_2}\cos\theta_r}{\cos\theta_i + \dfrac{n_1}{n_2}\cos\theta_r} = \frac{\alpha - i\beta'}{\alpha + i\beta'}$$

where $\begin{cases} x = \cos\theta_i \\ i\beta' = \dfrac{n_1}{n_2}\cos\theta_r \end{cases}$

$$= e^{i\delta_{\parallel}} \text{ with } \tan\frac{\delta_{\parallel}}{2} = \frac{\beta'}{\alpha'} \qquad \text{...(15)}$$

Thus the components of electric vector parallel and perpendicular the plane of incidence on reflection experience phase retardations of $\delta_{\parallel}$ and $\delta_{\perp}$ respectively; but their amplitudes remain unaltered.

$$|(E_{01}')_{\parallel}| = (E_{01})_{\parallel}| \text{ and } |(E_{01}')_{\perp}| = (E_{01})_{\perp}| \qquad \text{...(16)}$$

Thus for each component the intensity of electromagnetic wave which is totally reflected is equal to the intensity of incident light. Consequently we say that *the incident wave is totally reflected.*

Relative phase difference between parallel and perpendicular components of reflected electromagnetic wave. The relative phase different between parallel and perpendicular components of the reflected wave is given by

$$\delta = \delta_{\parallel} - \delta_{\perp}$$

or

$$\frac{\delta}{2} = \frac{\delta_{\parallel}}{2} - \frac{\delta_{\perp}}{2}$$

$$\therefore \qquad \tan\frac{\delta}{2} = \tan\left(\frac{\delta_{\parallel}}{2} - \frac{\delta_{\perp}}{2}\right) = \frac{\tan\left(\dfrac{\delta_{\parallel}}{2}\right) - \tan\left(\dfrac{\delta_{\perp}}{2}\right)}{1 + \tan\left(\dfrac{\delta_{\parallel}}{2}\right)\tan\left(\dfrac{\delta_{\perp}}{2}\right)}$$

$$= \frac{\dfrac{\beta'}{\alpha} - \dfrac{\beta}{\alpha}}{1 + \dfrac{\beta'}{\alpha}\cdot\dfrac{\beta}{\alpha}} = \frac{\alpha(\beta' - \beta)}{\alpha^2 + \beta'\beta}$$

But $\alpha = \cos\theta_i$, $\beta = \dfrac{n_2}{n_1}\cos\theta_r$ and $\beta' = \dfrac{n_1}{n_2}\cos\theta_r$

$$\therefore \qquad \tan\frac{\delta}{2} = \frac{\cos\theta_i\left\{\dfrac{n_1}{n_2}\cos\theta_r - \dfrac{n_2}{n_1}\cos\theta_r\right\}}{\cos^2\theta_i + \dfrac{n_1}{n_2}\cos\theta_r\cdot\dfrac{n_2}{n_1}\cos\theta_r}$$

$$= \frac{\cos\theta_i \left\{ \frac{n_1}{n_2} - \frac{n_2}{n_1} \right\} \cos\theta_r}{\cos^2\theta_i + \cos^2\theta_r}$$

$$= \frac{\cos\theta_i \left\{ \frac{n_1}{n_2} - \frac{n_2}{n_1} \right\} \left[\left\{ \frac{n_1}{n_2} \sin\theta_i \right\}^2 - 1 \right]^{1/2}}{\cos^2\theta_i + \left\{ \frac{n_1}{n_2} \sin\theta_i \right\}^2 - 1}$$

[Using eqn. (4)]

$$= \frac{\cos\theta_i \frac{n_1}{n_2} \left\{ \frac{n_1^2}{n_2^2} - 1 \right\} \left[\left\{ \frac{n_1}{n_2} \sin\theta_i \right\}^2 - 1 \right]^{1/2}}{\frac{n_1^2}{n_2^2} \sin^2\theta_i - (1 - \cos^2\theta_i)}$$

$$= \frac{\cos\theta_i \frac{n_2}{n_1} \left\{ \frac{n_1^2}{n_2^2} - 1 \right\} \left[\left\{ \frac{n_1}{n_2} \sin^2\theta_i \right\}^2 - 1 \right]^{1/2}}{\left\{ \frac{n_1^2}{n_2^2} - 1 \right\} \sin^2\theta_i}$$

$$\therefore \qquad \tan\frac{\delta}{2} = \frac{\cos\theta_i \sqrt{\left[\sin^2\theta_i - \left\{ \frac{n_2}{n_1} \right\}^2 \right]}}{\sin^2\theta_i} \qquad \text{...(17)}$$

Obviously the phase difference depends upon the angle of incidence θ_i greater than the critical angle. The plane polarised radiation after reflection becomes elliptically polarised. Any desired phase difference δ can be introduced between the two components of reflected light by the proper choice of θ. Therefore any desired type of elliptically polarised radiation can be conveniently produced by the method of total internal reflection. From equation (17) it is easy to see that $n_1/n_2 = \frac{3}{2}$, $\delta = \pi/4$ when $\theta_i = 53°15'$ or $\theta_i = 50°14'$ and so another total internal reflection at same angle the phase difference between reflected components becomes $\pi/2$.The reflected radiation is of course elliptically polarised since in general the components $E_\parallel$ and $E_\perp$ are of unequal amplitudes.

The phenomenon of total internal reflection is applicable in cases where it is required to transmit light without loss in intensity, It is also used to produce elliptically and circularly polarised light.

9.9. Reflection From a Conducting Plane (*i.e.* Metallic Reflection)

Let us consider the reflection and refraction at the interface separating a conducting and a non-conducting media. Let the non-conducting medium '1' be characterised by permittivity ε_1, permeability μ_1, refractive index n_1, while the conducting medium '2' by permittivity ε_2, permeability μ_2, refractive index n_2 and conductivity σ_2. Let a plane monochromatic electromagnetic wave be incident on the interface from medium 1 in medium 2. The reflection and refraction take place but quite differently from the case of interface separating non-conducting media.

The electric and magnetic field intensities in the two media may be expressed as :

In non-conducting medium 'I'

$$\left.\begin{aligned}&\text{For incident wave } \mathbf{E}_1 = \mathbf{E}_{01}\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1 t}\\ &\text{and} \qquad \mathbf{H}_1 = \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\mu_1\omega_1} \text{ where } |\mathbf{k}_1| = \frac{\omega_1}{v_1} = \sqrt{(\mu_1\varepsilon_1)}\,\omega_1\end{aligned}\right] \qquad \ldots(1)$$

$$\left.\begin{aligned}&\text{For reflected wave } \mathbf{E}_1' = \mathbf{E}_{01}'\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1' t}\\ &\text{and} \qquad \mathbf{H}_1' = \frac{\mathbf{k}_1' \times \mathbf{E}_1'}{\mu_1\omega_1'} \text{ where } |\mathbf{k}_1'| = \frac{\omega_1'}{v_1'} = \sqrt{(\mu_1\varepsilon_1)}\,\omega_1'\end{aligned}\right] \qquad \ldots(2)$$

In conducting medium '2'

$$\left.\begin{aligned}&\text{For refracted wave } \mathbf{E}_2 = \mathbf{E}_{02}\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_2 t}\\ &\text{and} \qquad \mathbf{H}_2 = \frac{\mathbf{k}_2 \times \mathbf{E}_2}{\mu_2\omega_2}\end{aligned}\right] \qquad \ldots(3)$$

where

$$|\mathbf{k}_2|^* = \alpha_2 + i\beta_2 \qquad \ldots(4a)$$

$$\left.\begin{aligned}&\text{with } \alpha_2 = \sqrt{(\mu_2\varepsilon_2)}\,\omega_2\left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma_2}{\omega_2\varepsilon_2}\right)^2\right\}} + 1}{2}\right]^{1/2}\\ &\beta_2 = \sqrt{(\mu_2\varepsilon_2)}\,\omega_2\left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2\right\}} - 1}{2}\right]^{1/2}\end{aligned}\right] \qquad \ldots(4b)$$

For good conductors $\sigma_2/\omega\,\varepsilon_2 \gg 1$, so

$$\alpha_2 \approx \beta_2 = \sqrt{\left(\frac{\mu_1\omega_2\sigma_2}{2}\right)}$$

Now using the boundary condition that the tangential component of electric field is continuous at the interface, we obtain

$$E_{1t} + E_{1t}' = E_{2t}$$

---

[*] In section 8.15, we have seen that in a conducting medium both electric and magnetic field vectors decay exponentially and they are no longer in phase. Moreover the propagation vector **k** is complex given by $k = \mu\varepsilon\omega^2\,[1 + (i\sigma/\omega\varepsilon)]$ or equivalently $k = \alpha + i\beta$

$$\text{where } \alpha = \sqrt{(\mu\varepsilon)}\,\omega\left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}} + 1}{2}\right]^{1/2}$$

$$\beta = \sqrt{(\mu\varepsilon)}\,\omega\left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}} - 1}{2}\right]^{1/2}$$

In good conductors $\alpha \approx \beta = \sqrt{\left(\frac{\mu\omega\sigma}{2}\right)}$

i.e.
$$(E_{01})_t\, e^{i\mathbf{k}_1\cdot\mathbf{r} - i\omega_1 t} + (E_{01}')_t\, e^{i\mathbf{k}_1'\cdot\mathbf{r} - i\omega_1' t} = (E_{02})_t\, e^{i\mathbf{k}_2\cdot\mathbf{r} - i\omega_2 t} \qquad ...(5)$$

Since this equality is independent of time, we must have

$$\omega_1 = \omega_1' = \omega_2 = \omega, \text{ (say)} \qquad ...(6)$$

That ***is the incident, reflected and refracted waves all have the same frequency.***

Since equation (5) holds for all points of interface, hence at interface $z = 0$, we must have

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} = (\mathbf{k}_1'\cdot\mathbf{r})_{z=0} = (\mathbf{k}_2\cdot\mathbf{r})_{z=0} \qquad ...(7)$$

This equation shows that $\mathbf{k}_1$, $\mathbf{k}_1'$ and $\mathbf{k}_2$ all lie in the same plane. Equation (7) also implies (refer section 9.3)

$$\theta_i = \theta_r'$$

i.e. angle of incidence is equal to angle of reflection.

and
$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{k_2}{k_1} = \frac{k_2/\sqrt{\mu_0\varepsilon_0}}{k_2/\sqrt{\mu_0\varepsilon_0}} = \frac{k_2/\sqrt{\mu_0\varepsilon_0}}{\omega\sqrt{\mu_1\varepsilon_1}/\sqrt{\mu_0\varepsilon_0}} = \frac{k_2/\omega\sqrt{\mu_0\varepsilon_0}}{\sqrt{(\mu_1\varepsilon_1/\mu_0\varepsilon_0)}} = \frac{n_2}{n_1} \qquad ...(8)$$

which is *well known Snell's law of refraction.*

Hence n_1 and n_2 are refractive indices in medium 1 and medium 2 respectively, given by

$$n_1 = \sqrt{\left(\frac{\mu_1}{\mu_0}\,\frac{\varepsilon_1}{\varepsilon_0}\right)} = \sqrt{(K_{1m}\,K_{1e})}$$

$$n_2 = \frac{k_2}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{c}{\omega}k_2 \text{ where } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$= \text{speed of light in vacuum}$$

Since k_2 is complex, therefore n_2 must be complex and may be expressed as

$$n_2 = (c/\omega)\,k_2 = (c/\omega)\,(\alpha_2 + i\beta) \qquad ...(9a)$$

where α_2 and β_2 are given by equation (4a) with $\omega_2 = \omega$.

Since n_1 is real, it is *customary* to write

$$n_2/n_1 = {}_1n_2 = n\,(1 + ik) \qquad ...(9b)$$

where k is known as *extinction coefficient.*

From equations (9a) and (9b) it is obvious that

$$n = \frac{c\alpha_2}{\omega n_1} \text{ and } nk = \frac{c\beta_2}{\omega n_1} \qquad ...(9c)$$

For extremely good conductors

$$\alpha_2 \approx \beta_2 = \sqrt{\left(\frac{\mu_2\omega\sigma_2}{2}\right)}, \qquad \text{so that } k \to 1. \qquad ...(10)$$

Now from Snell's law

$$\left.\begin{aligned} \frac{\sin\theta_i}{\sin\theta_r} &= \frac{n_2}{n_1} = {}_1n_2 \; i.e. \; \sin\theta_r = \frac{\sin\theta_i}{{}_1n_2} \\ \text{and} \quad \cos\theta_r &= \sqrt{(1-\sin^2\theta_r)} = \sqrt{\left(1 + \frac{\sin^2\theta_i}{({}_1n_2)^2}\right)} \end{aligned}\right] \qquad ...(11)$$

As the refractive index of a conducting medium is very large as compared to that of a non-conducting medium *i.e.* ${}_1n_2 = n_2/n_1$ is very large. This implies that $\cos\theta_r \approx 0$ or $\theta_r = 0$. Thus in a ***good conducting medium, the refracted ray propagates along the normal to the interface irrespective of angle of incidence.***

The refracted or transmitted wave may be expressed as

$$\mathbf{E}_2 = \mathbf{E}_{02}\, e^{i(\alpha_2 + i\beta_2)\,\hat{\mathbf{n}}\cdot\mathbf{r} - o\omega t} \cdot \hat{\mathbf{n}}_2 \qquad \text{being unit vector along } \mathbf{k}_2$$

$$= \mathbf{E}_{02}\, e^{-\beta_2 \hat{\mathbf{n}}_2\cdot\mathbf{r}}\, e^{i\alpha_2 \hat{\mathbf{n}}\cdot\mathbf{r} - i\omega t} \qquad \text{...(12)}$$

This equation indicates that the field vectors in refractd wave exponentially *i.e.* the refracted wave is highly damped wave. The penetration depth or skin depth is given by

$$\delta = \frac{1}{\beta_2} = \frac{1}{\sqrt{(\mu_2\varepsilon_2)}\,\omega\left[\dfrac{\left\{1 + \left(\dfrac{\sigma_2}{\omega\varepsilon_2}\right)^2 + 1\right\}}{2}\right]^{1/2}}$$

$$= \frac{1}{\sqrt{(\mu_2\varepsilon_2)}\,\omega\,\sqrt{(\sigma_2/2\omega\varepsilon)}} = \frac{2}{\mu_2\sigma_2\omega} \qquad \text{...(13)}$$

for good conductors

Obviously the skin depth is independent of the angle of incidence. Now let us find the *Fresnel's equations* in the two usual cases.

Case (i) When E vector is perpendicular to the plane of incidence. Considering fig. 9.4, the continuity of tangential components of electric and magnetic field intensities **E** and **H** at the interface requires that (refer equation (1) and (2) of section 9.4)

$$E_{01} + E_{01}' = E_{02} \qquad \text{...(14)}$$

$$(H_{01} - H_{01}')\cos\theta_i = H_{02}\cos\theta_r \qquad \text{...(15)}$$

We have
$$\mathbf{H}_1 = \frac{\mathbf{k}_1 \times \mathbf{E}_1}{\mu_1\omega_1} = \frac{\mathbf{k}_1\,\hat{\mathbf{n}}_1 \times \mathbf{E}_1}{\mu_1\varepsilon_1} = \sqrt{(\mu_1\varepsilon_1)}\,\frac{\hat{\mathbf{n}}\times\mathbf{E}_1}{\mu_1}$$

$$\left.\begin{aligned} &\text{This given} & H_{01} &= \sqrt{(\varepsilon_1/\mu_1)}\,E_{01} \\ &\text{Similarly} & H_{01}' &= \sqrt{(\varepsilon_1/\mu_1)}\,E_{01}' \\ &\text{and} & H_{02} &= \frac{k_2}{\mu_0\omega}\,E_{02} \end{aligned}\right\} \qquad \text{...(16)}$$

Using (16), equation (15) gives

$$\frac{\varepsilon_1}{\mu_1}(E_{01} - E_{01}')\cos\theta_i = \frac{k_1}{\mu_2\omega}\,E_{02}\cos\theta_r \qquad \text{...(17)}$$

Solving equation (14) and (17), we get

$$\left(\frac{E_{01}'}{E_{01}}\right)_{\perp} = \frac{\cos\theta_i - \dfrac{k_2}{\mu_2\omega}\sqrt{\left(\dfrac{\mu_1}{\varepsilon_1}\right)}\cos\theta_r}{\cos\theta_i + \dfrac{k_2}{\mu_2\omega}\sqrt{\dfrac{\mu_1}{\varepsilon_1}}\cos\theta_r} \qquad \text{...(18)}$$

and
$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \dfrac{k_2}{\mu_2\omega}\sqrt{\left(\dfrac{\mu_1}{\varepsilon_1}\right)}\cos\theta_r} \qquad \text{...(19)}$$

These are the general *Fresnel's equations.*

Now let us consider that *conducting medium is good conductor* for which $\sigma_2/\omega\,\varepsilon_2 \ll 1$, so that $\cos\theta_r = 1$, then equations (18) and (19) take the form

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}=\frac{\cos\theta_i-\frac{k_2}{\mu_2\omega}\sqrt{\frac{\mu_1}{\varepsilon_1}}}{\cos\theta_i+\frac{k_2}{\mu_2\omega}\sqrt{\frac{\mu_1}{\varepsilon_1}}} \qquad ...(20)$$

and
$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp}=\frac{2\cos\theta_i}{\cos\theta_i+\frac{k_2}{\mu_2\omega}\sqrt{\frac{\mu_1}{\varepsilon_1}}} \qquad ...(21)$$

Now noting that

$$k_2=\frac{\omega n_2}{c} \text{ and } n_1=c\sqrt{\mu_1\,\varepsilon_1}, \text{ so that } \frac{k_1}{\mu_2\omega}\sqrt{\frac{\mu_1}{k_1}}\cdot\frac{n_2}{n_1}\cdot\frac{\mu_1}{\mu_2},$$

Equations (20) and (21) taken the form

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\perp}=\frac{\cos\theta_i-\frac{n_2}{n_1}\frac{\mu_1}{\mu_2}}{\cos\theta_i+\frac{n_2}{n_1}\frac{\mu_1}{\mu_2}} \qquad ...(22)$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp}=\frac{2\cos\theta_i}{\cos\theta_i+\frac{n_2}{n_1}\frac{\mu_1}{m_2}} \qquad ..(23)$$

Further for simplicity assuming that the media are non-magnetic *i.e.* $\mu_1=\mu_2=\mu_0$ and using (9b), we get

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp}=\frac{\cos\theta_i-n\,(1-i\kappa)}{\cos\theta_i+n\,(1-i\kappa)}=-\frac{(n-\cos\theta_i)+in\kappa}{(n+\cos\theta_i)-in\kappa} \qquad ...(24)$$

and
$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp}=\frac{2\cos\theta_i}{\cos\theta_i+n\,(1-i\kappa)}=\frac{2\cos\theta_i}{(n+\cos\theta_i)-in\kappa} \qquad ...(25)$$

Now as $[(n-\cos\theta_i)+in\kappa]=ae^{i\delta_1}$

$$(n+\cos\theta_i)-in\kappa=be^{i\delta_2}$$

with
$$a=\sqrt{[(n-\cos\theta_i)^2+n^2\kappa^2]},\ \delta_1=\tan^{-1}\left\{\frac{n\kappa}{n-\cos\theta_i}\right\} \qquad ...(26)$$

$$b=\sqrt{[(n+\cos\theta_i)^2+n^2\kappa^2]},\ \delta_2=\tan^{-1}\left\{\frac{n\kappa}{n+\cos\theta_i}\right\} \qquad ...(27)$$

so equations (24) and (25) reduce to

$$\left(\frac{E'_{01}}{E'_{01}}\right)_{\perp}=\frac{ae^{i\delta_1}}{be^{-i\delta_2}}=\frac{a}{b}e^{i(\delta_1+\delta_2)}$$

$$=\sqrt{\left\{\frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2}\right\}}\,e^{i(\delta_1+\delta_2)} \qquad ...(28)$$

and
$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp}=\frac{2\cos\theta_i}{be^{-\delta_2}}=\frac{2\cos\theta_i}{\sqrt{[(n+\cos\theta_i)^2+n^2\kappa^2]}}\,e^{i\delta_2} \qquad ...(29)$$

Equations (28) and (29) are final equations representing the reflected and transmitted amplitudes relative to incident amplitude. From these equations it is clear that both reflected and transmitted waves exist and they are not in phase with the incident wave. Further the largevalue of n implies that the amplitude of transmitted wave is very small while that of reflected wave is nearly equal to that of incident one.

The reflection coefficient $R_\perp$ is obtained by comparing the normal component of reflected poynting vector with that of incident Poynting vector *i.e.*

$$R_\perp = \frac{S'_1\cdot\mathbf{n}}{S_1\cdot\mathbf{n}} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_2}{\mu_1}}\,|E'_{01}|^2\cos\theta'_r}{\frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}}\,|E_{01}|^2\cos\theta_i} = \frac{|E'_{01}|_\perp^2}{|E_0|_\perp^2}\quad(\text{since } \theta'_r = \theta_i)$$

$$= \left|\frac{E'_{01}}{E_{01}}\right|_\perp^2 = \left\{\frac{E'_{01}}{E_{01}}\right\}_\perp\left\{\frac{E'^{*}_{01}}{E^{*}_{01}}\right\}_\perp$$

$$= \left[\frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2}\right]^{1/2} e^{i(\delta_1+\delta_2)}\left[\frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2}\right] e^{-(\delta_1+\delta_2)}$$

$$= \frac{(n-\cos\theta_i)^2+n^2\kappa^2}{(n+\cos\theta_i)^2+n^2\kappa^2} \qquad \text{...(30)}$$

Case (ii) When E is parallel to plane of incidence.

Considering fig. 9.5, the continuity of tangential components of electric and magnetic field intensities **E** and **H** at the interface requires [refer equations (10) and (11) of section (8.4)].

$$H_{01} + H'_{01} = H_{02} \qquad \text{...(31)}$$

$$(E_{01} - E'_{01})\cos\theta_i = E_{02}\cos\theta_r \qquad \text{....(32)}$$

Using (16), equation (31) gives

$$\sqrt{\frac{\varepsilon_1}{\mu_2}}\,(E_{01} + E'_{01}) = \frac{k_2}{\mu_2\,\omega}E_{02} \qquad \text{...(33)}$$

Solving (32) and (33), we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\sqrt{\frac{\mu_1}{\varepsilon_1}}\,\frac{k_2}{\mu_2\omega}\cos\theta_i - \cos\theta_r}{\sqrt{\frac{\mu_1}{\varepsilon_1}}\,\frac{k_2}{\mu_2\omega}\cos\theta_i + \cos\theta_r} \qquad \text{...(34)}$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\cos\theta_i}{\sqrt{\frac{\mu_1}{\varepsilon_1}}\,\frac{k_2}{\mu_2\omega}\cos\theta_i + \cos\theta_r} \qquad \text{...(35)}$$

If the conducting medium is a good conductor, then $\cos\theta_r = 1$, and writing

$$\sqrt{\frac{\mu_1}{\varepsilon_1}}\cdot\frac{k_2}{\mu_2\omega} = \frac{\mu_1}{\mu_2}\cdot\frac{n_2}{n_1}$$

as before, we get

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_i - 1}{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_i + 1} \text{ and } \left\{\frac{E_{02}}{E_{01}}\right\}_{\parallel} = \frac{2\cos\theta_i}{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_i + 1}$$

Again considering $\mu_1 = \mu_2 = \mu_0$ and writting $n_2/n_1 = {}_1n_2 = n\,(1 + i\kappa)$ above equation reduce to

$$\left(\frac{E'_{01}}{E_{01}}\right)_{\parallel} = \frac{n\,(1+i\kappa)\cos\theta_i - 1}{n\,(1-i\kappa)\cos\theta_i + 1} = \frac{\left\{n - \frac{1}{\cos\theta_i}\right\} + in\kappa}{\left\{n + \frac{1}{\cos\theta_i}\right\} + in\kappa} \qquad \text{...(36)}$$

and

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\cos\theta_i}{n\,(1+i\kappa)\cos\theta_i + 1} = \frac{2}{\left\{n + \frac{1}{\cos\theta_i}\right\} + in\kappa} \qquad \text{...(37)}$$

Again writting

$$\left(n - \frac{1}{\cos\theta_i}\right) + in\kappa = a'e^{i\delta'_1}$$

and

$$\left(n + \frac{1}{\cos\theta_i}\right) + in\kappa = b'e^{i\delta'_2}$$

with

$$a' = \left[\left\{n - \frac{1}{\cos\theta_i}\right\} + n^2\kappa^2\right]^{1/2}; \delta'_1 = \tan^{-1}\frac{n\kappa}{\left(n - \frac{1}{\cos\theta_i}\right)} \qquad \text{...(38)}$$

$$b' = \left[\left(n + \frac{1}{\cos\theta_i}\right)^1 + n^2\kappa^2\right]^{1/2}; \delta'_2 = \tan^{-1}\frac{n\kappa}{\left[n + \frac{1}{\cos\theta_i}\right]} \qquad \text{...(39)}$$

Equations (36) and (37) take the form

$$\left(\frac{E'_{01}}{E_{01}}\right) = \frac{a'e^{i\delta'_1}}{b'e^{i\delta'_2}}\left[\frac{\left(n - \frac{1}{\cos\theta_i}\right)^2 + n^2\kappa^2}{\left(n + \frac{1}{\cos\theta_2}\right)^2 + n^2\kappa^2}\right]^{1/2} e^{i(\delta'_1 - \delta'_2)} \qquad \text{...(40)}$$

and

$$\left[\frac{E_{02}}{E_{01}}\right]_{\parallel} = \frac{2}{b'e^{i\delta'_2}} = \frac{2}{\left[\left(n + \frac{1}{\cos\theta_i}\right)^2 + n^2\kappa^2\right]^{1/2}} e^{-i\delta'_2}. \qquad \text{...(41)}$$

These equations indicate again that both reflected and transmitted waves exist and they are not in phase with the incident wave. Moreover the large value of n implies that amplitude of transmitted wave is extremely small,

$$R_{\parallel} = \left|\frac{E'_{02}}{E_{01}}\right|^2_{\parallel} = \frac{\left[n - \frac{1}{\cos\theta_i}\right]^2 + n^2\kappa^2}{\left[n + \frac{1}{\cos\theta_i}\right]^2 + n^2\kappa^2} \qquad \text{...(42)}$$

At normal incidence $\theta_i = 0$; $\cos\theta_i = 1$; therefore equations (30) and (42) take identical form

$$\left.\begin{aligned} R = R_{\perp} = R_{\parallel} &= \frac{(n-1)^2 + n^2\kappa^2}{(n+1)^2 + n^2\kappa^2} \qquad \text{...(a)} \\ &= 1 - \frac{4n}{(n+1)^2 + n^2\kappa^2} \qquad \text{...(b)} \end{aligned}\right\} \qquad \text{...(43)}$$

For extremely good conductors $\kappa \to 1$ and

$$n \to \frac{c\alpha_2}{\omega n_1} \to \frac{c}{\omega n_1}\sqrt{\frac{\mu_2 \omega \sigma_2}{2}} \to \frac{c}{\omega c\sqrt{\mu_1 \varepsilon_1}}\sqrt{\frac{\mu\omega\sigma_2}{2}} \to \sqrt{\frac{\sigma_2}{2\omega\varepsilon_1}} \qquad (\text{since } \mu_1 = \mu_2)$$

Also

$$n = \sqrt{\frac{\sigma_2}{2\omega\varepsilon_1}} \gg 1.$$

Making these substitutions, equation (43b) gives

$$R = 1 - \frac{2}{n} = 1 - 2\sqrt{\frac{2\omega\varepsilon_1}{\sigma_2}} \qquad \text{...(44)}$$

This equation shows that at low frequencies for substance of high conductivity, the reflection coefficient R is close to unity and therefore all the energy will be reflected. This explains why metals are opaque to light. The little energy that flows into the metal is rapidly dissipated by the heat loss associated with the induced currents. For example taking $\sigma = 5{\cdot}6 \times 10^7$ mho/m, the value for copper and assuming $\mu_1 = \mu_2,\ \varepsilon_1 = \varepsilon_2$, we find that for frequency 10^{10} sec^{-1} (3 cm wavelength), $R = 0{\cdot}9997$; for lower frequencies R is even closer to unity *i.e.* perfect reflection takes place. It is only for radiation of very short wavelength that the deviation of R from unity becomes significant for extremly good conductors like copper, silver and aluminium. The skin depth for these materials is very small *i.e.* these materials absorb waves strongly. Thus all waves which are strongly absorbed are also strongly reflected by metals *i.e. all good conductors are good absorbers and good reflectors.*

Experimental verification of the law metallic reflection. At optical *frequencies* and normal incidence at metals, the reflection coefficient, $R = 1 - \frac{2}{n} = 1 - 2\sqrt{\frac{2\omega\varepsilon_1}{\sigma_2}}$. The value of this reflection coefficient is usually determined by measuring the *emissivity* e of the surface, then by Kirchoff's law

$$e = 1 - R = \frac{2}{n} = 2\sqrt{\frac{2\omega\varepsilon_1}{\sigma_2}} \qquad \text{...(45)}$$

This implies $e\sqrt{\sigma_2} = 2\sqrt{2\omega\varepsilon_1} =$ *constant at a given frequency i.e.* for all *metals* $e\sqrt{\sigma_2}$ should be constant at a given frequency of incident radiation. Extensive measurement by Hagen and Rubens confirmed this prediction for a wide range of metals within frequency range $10^{23} \sim 10^{14}$ Hertz. The average value of $e\sqrt{\sigma}$ obtained from observation on 12 metals at $\lambda = 2{\cdot}55 \times 10^{-5}$ m. was $6{\cdot}96 \times 10^6$, where as the value calculated from conductivity and frequency was $6{\cdot}68 \times 10^6$. The close agreement between experimental and theoretical result verifies the law of metallic reflection. Hagen and Rubens also showed that the relation (44) is not valid below $\lambda = 0{\cdot}5 \times 10^{-5}$ m. Performing experiment below this specific wavelength, they showed that the reflecting power of most metals increases rapidly with wavelength and approaches the value given by (44).

9.10. Group Velocity.

If a region consists of two or more wave trains, then the *physical velocity of propagation of waves is called the* **group velocity.** The velocity of individual wave is known as phase velocity and is given by $v_p = \omega/k$ where $k = |\mathbf{k}| =$ magnitude of propagation vector and ω is angular frequency. The phase velocity is not the velocity of propagation of any physical quantity; this simply represents the velocity of a point of constant phase on the wave. Signals are transmitted with the group velocity and not with the phase velocity. The group velocity $\mathbf{v}_G$ is given by the derivative

$$\mathbf{v}_G = \frac{\partial\omega}{\partial\mathbf{k}} \qquad ...(1)$$

the ratio ω/k is called the phase velocity.

Let us derive relation (1). Let the two wave trains propagating along z-axis have the same amplitude but slightly different frequencies ω_1 and ω_2 and phase velocities v_1 and v_2, so that they can be expressed by the equations

$$E_1 = E_0 \cos\omega_1\left(t - \frac{z}{v_1}\right) \qquad ...(2)$$

$$E_2 = E_0 \cos\omega_2\left(t - \frac{z}{v_2}\right) \qquad ...(3)$$

The resultant amplitude of wave may be obtained by the principle of superposition as

$$\begin{aligned} E = E_1 + E_2 &= E_0\cos\omega_1\left(t - \frac{z}{v_1}\right) + E_0\cos\omega_2\left(t - \frac{z}{v_2}\right) \\ &= 2\,E_0\cos\left\{\frac{\omega_1}{2}\left(t - \frac{z}{v_1}\right) + \frac{\omega_2}{2}\left(t - \frac{z}{v_2}\right)\right\} \\ &\quad \cos\left\{\frac{\omega_1}{2}\left(t - \frac{z}{v_1}\right) - \frac{\omega_2}{2}\left(t - \frac{z}{v_2}\right)\right\} \end{aligned} \qquad ...(4)$$

Let us write $\omega_1 = \omega;\ \omega_2 = \omega + d\omega$

Then $\omega_1 - \omega_2 = -\,d\omega$ and $\omega_1 + \omega_2 = 2\omega$ as $d\omega \ll \omega$

Similarly $\frac{\omega_1}{v_1} + \frac{\omega_2}{v_2} = 2\frac{\omega}{v}$ where $v = \frac{v + v_2}{2}$

Then equation (4) becomes

$$E = 2E_0\cos\omega\left(t - \frac{z}{v}\right)\cos\left\{-\frac{d\omega}{v_2}t + \frac{1}{2}\left(\frac{\omega_2}{v_2} - \frac{\omega_1}{v_1}\right)z\right\}$$

Now
$$\frac{\omega_2}{v_2} - \frac{\omega_1}{v_1} = d\left(\frac{\omega}{v}\right)$$

so that
$$E = 2E_0\cos\frac{d\omega}{2}\left[t - \frac{d\,(\omega/v)}{d\omega}z\right]\cos\omega\left(1 - \frac{z}{v}\right) \qquad ...(5)$$

The equation represents a wave, travelling with phase velocity v whose amplitude is

$$2E_0\cos\frac{d\omega}{2}\left(t - \frac{z}{v_G}\right) \qquad ...(6)$$

where
$$v_G = \frac{d\omega}{d\,(\omega/v)} = \frac{d\omega}{dk} \qquad ...(7)$$

where $k = \omega/v$.

Equation (6) represents that amplitude itself is being propagated, along Z axis with velocity v_G and since amplitude velocity is signal velocity, this is *group velocity* and is given by equation (7) or (1) in vector form.

9.11. Propagation of Electromagnetic Waves Between Parallel Conducting Planes.

We know that in a good conductor the field penetrates only to a distance comparable with the skin depth, which reduces to zero in the limits of perfect conductivity. Thus a perfect conductor behaves as a perfect reflector and hence allows no field to penetrate within it. Consequently the electric and magnetic fields are zero within the conducting plane and our boundary conditions reduce to the following two statements.

(i) *The tangential component of* **E** *at the surface of the conducting plane is zero,* since it must be zero within the plane and we know that the tangential component is continuous.

(ii) *The normal componet of* **B** *at the surface of the conducting plane is zero,* since it must be zero with the plane and the normal component of **B** is continuous.

The other two conditions on the normal component of **D** and the tangential component of **H** lead to information about the surface charge and surface current respectively. Since the surface is conducting and can carry charge and current ; therefore the normal component of **D** outside the conducting plane measures the surface charge density and the tangential component of **H** measures the surface current density, which appear on the surface. Because *of no penetration of electromagnetic waves, we need to consider only incident waves striking the conducting plane and the corresponding reflected waves.*

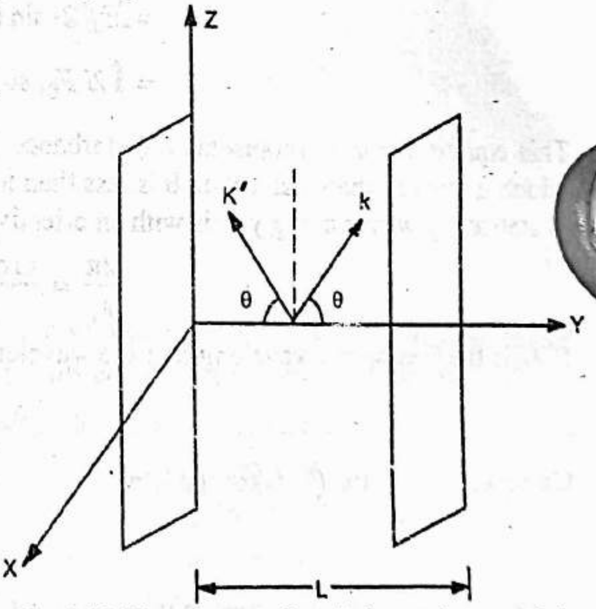


Fig. 9.10. Wave propagations between two parallel conducting planes.

Now let us consider the propagation of electromagnetic waves in the region between two parallel perfectly conducting planes separated by a distance L. Let the planes be in the $X-Z$ plane situated at $y=0$ and $y=L$ as shown in fig. 9.10. If an electromagnetic wave is incident on any plane, it will be reflected from the wall whenever it strikes it and hence by a process of multiple reflections the wave will propagate in a direction parallel to planes. Since in fig 9.10 the x and z directions are physically indistinguishable no generality is lost if we consider only waves with wave vectors in y–z plane in particular, those making an angle θ with y-axis. Such waves will strike on perfectly conducting surface at $y=L$ and will be reflected as waves whose propagation vector k_1' makes an angle θ with the minus y-axis. When these waves are reflected again by the surface at $y=0$, they become waves of initial type again and have propagation vector **k**. Thus the propagation between two conducting planes can be described in terms of exponential factors

$$\left.\begin{aligned} &\text{for initial wave } e^{i\mathbf{k}\cdot\mathbf{r}} \rightarrow e^{ik(y\cos\theta + z\sin\theta) - i\omega t} \\ &\text{For } \textit{reflected wave (once)}\ e^{i\mathbf{k}'\cdot\mathbf{r}} \rightarrow e^{ik(-y\cos\theta + z\sin\theta) - i\omega t} \end{aligned}\right\} \qquad ...(1)$$

where $|\mathbf{k}| = |\mathbf{k}'| = \omega/c$, since both waves are in same medium.

Since in electromagnetic waves **k**, **H** and **E** are mutually perpendicular, therefore for such waves there are only two possible polarisation which may be described by saying that for one **E** is parallel to X-axis or at right angles to z for both the direct and reflected waves (*i.e.* $E_z = 0$) and for the other **H** is parallel to X-axis or at right angles to z for both the direct and reflected waves (*i.e.* $H_z = 0$). These are known as the *transverse electric* (abbreviated *TE*). and *transverse- magnetic* (abbreviated *TM*) modes of propagation. Here we shall consider only *TE* waves.

The electric fields for incident and reflected waves in *TE* case may be expressed as

$$\left.\begin{aligned}\mathbf{E}_I &= \mathbf{i}E_{01}\, e^{ik(y\cos\theta + z\sin\theta) - i\omega t} && \text{(incident wave)}\\ \mathbf{E}_I' &= \mathbf{i}E_{01}'\, e^{ik(-y\cos\theta + z\sin\theta) - i\omega t} && \text{(reflected wave)}\end{aligned}\right\} \quad ...(1)$$

where **i** is unit vector along X-axis.

By the principle of superposition the electric field in the region between two conducting planes in *TE* case will be given by

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_I + \mathbf{E}_I'\\ &= \mathbf{i}\,[E_{01}\, e^{ik(y\cos\theta + z\sin\theta) - i\omega t}] + [E_{01}'\, e^{ik(-y\cos\theta + z\sin\theta) - i\omega t}]\end{aligned} \quad ...(2)$$

According to boundary condition that the tangential component of electric field must vanish at the surface of conducting plane, we must have

$$0 = E_{01}\, e^{ikz\sin\theta\, i\omega t} + E_{01}'\, e^{ikz\sin\theta - i\omega t}$$

This equation is satisfied for all z and all t if

$$E_{01} = E_{01}' \quad ...(3)$$

This condition simply indicates that the reflection at the conducting plane involves a phase change of π and no change in amplitude.

Then **E** will be given by

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{i}}E_{01}\,(e^{iky\cos\theta} - e^{-iky\cos\theta})\, e^{i(kz\sin\theta - i\omega t)}\\ &= \hat{\mathbf{i}}E_{01}\, 2i\sin(k\cos\theta \bullet y)\, e^{i(kz\sin\theta - \omega t)}\\ &= \hat{\mathbf{i}}\, 2i\, E_{01}\sin\left(\frac{\omega\cos\theta}{c}y\right) e^{i[\{(\omega\sin\theta)/c\}z - \omega t]}\end{aligned}$$

This equation simply represents a disturbance propagating along z-axis with an apparent veloity* $c/\sin\theta$ which is greater than c since $\sin\theta$ is less than unity and varying sinusoidally along y-axis, or corresponding to stationary waves along y-axis with an effective wavelength λ_c (along y-axis) given by

$$\frac{2\pi}{\lambda_c} = \frac{\omega\cos\theta}{c} \text{ or } \lambda_c = \frac{2\pi c}{\omega\cos\theta} \quad ...(5)$$

If λ_0 is the free space wavelength or the wavelength in the absence of reflection, then

$$\lambda_0 = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \quad ...(6)$$

Using this, equation (5) takes the form

$$\lambda_c = \frac{\lambda_0}{\cos\theta} \quad ...(7)$$

The standing wave phenomenon was observed by Lippmann not in free space but in a sensitive emulsion of a photographic plate. The emulsion was blackened along surfaces half a wave-length apart in the standing wave pattern the so called *Lippmann fringes*, The phenomenon can also be demonstrated with the longer waves of the microwave region.

So far we have considered the boundary condition at $y = 0$: the same boundary condition at $x = L$ requires that (refer eqn. 4)

$$\sin\left(\frac{\omega\cos\theta}{c}L\right) = 0 \qquad \text{for all } z \text{ and all } t.$$

This gives

---

*The phase velocity $(c/\sin\theta) > c$, the speed of light. This apparently contradicts relativity *i.e.* no signal can be propagated with a velocity greater than the velocity of light. The resolution of this apparent difficulty is that energy is propagated with a smaller velocity than c, namely, the so called group velocity. Signals are transmitted with group velocity and not with the phase velocity.

$$\frac{\omega \cos \theta L}{c} = n\pi$$

or
$$L = n\frac{\pi c}{\omega \cos \theta} = \frac{n\lambda_c}{2} \qquad \text{using (5)} \qquad \text{...(8a)}$$

i.e.
$$\lambda_c = \frac{2L}{n} \qquad \text{...(8b)}$$

This equation indicates that the dependence of disturbance on y can not have any arbitrary wavelength ; but only an infinite set of discrete wavelengths. For a given n or a given mode, the effective wavelength λ_c is determined by equation (8b). This is independent of the free space wavelength λ_c or the corresponding frequency ω.

Now it is convenient to introduce phase velocity v_c and the *effective wavelength* λ_c along z-axis.

The apparent phase velocity v_g along the z-direction is given by

$$v_g = \frac{c}{\sin \theta} \text{ (refer equation (4))} \qquad \text{...(9)}$$

$$\lambda_g = \frac{v_g}{\nu} = \frac{v_g}{c/\lambda_0} = \frac{v_g \lambda_0}{c} = \frac{\left(\frac{c}{\sin \theta}\right)\lambda_0}{c} = \frac{\lambda_0}{\sin \theta} \qquad \text{...(10)}$$

For many purposes it is convenient to eliminate θ. From equations (7) and (10) we have

$$\cos \theta = \frac{\lambda_0}{\lambda_c} \text{ and } \sin \theta = \frac{\lambda_0}{\lambda_g}$$

Squaring and adding, we get

$$\left(\frac{\lambda_0}{\lambda_c}\right)^2 + \left(\frac{\lambda_0}{\lambda_g}\right)^2 = 1$$

or
$$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} \qquad \text{...(11)}$$

This equation determines the wavelength λ_g along the z-axis, in terms of free space wavelength λ_0 and the wavelength λ_c along y-direction or the direction normal to z. This equation gives an intersting fact. If λ_c is fixed and λ_0 is increased or the frequency is decreased, $1/\lambda_q^2$ decreases or λ_g increases with consequent increase of phase velocity v_g, until finally when λ_0 equals λ_c, then λ_g, and v_g both become infinite. If λ_0 is further increased beyond λ_c, then the wavelength λ_g and phase velocity v_g become imaginary, so that the exponential in (5) containing z viz. $e^{i\omega \sin \theta z/c} = \text{'}e^{i\omega z/v_g}\text{'}$ instead of representing wave propagated along z-axis, represents a decreasing exponential and hence the attenuated wave. *The free space wavelength at which the disturbance changes over from being propagated to being attenuated is called the* **cut off wavelength** and the corresponding frequency is called the **cut off frequency.** The reason for the subscript c is now clear : it means *'cut off.* The cut off wavelength is the longest wavelength of a wave in free space that can be propagated for a given mode '*n*'.

The problem of propagation of electromagnetic waves between two parallel planes is very simple and yet instructive case of *wave-guide propagation. A wave guide is simply a hollow pipe of infinite length* and has recently come into much prominence in connection with the applications of microwaves or electromagnetic waves whose lengths are of the order of magnitude of laboratory dimensions. We have seen that the waves travelling in the direction along the pipe can be propagated and the free space wavelength of these propagating waves is less than or equal to λ_0 ; otherwise the waves will be attenuated.

Group Velocity. We have seen that the phase velocity $v_0 = c/\sin\theta$ exceeds velocity of light. This apparently contracts the theory of relativity where we have postulated that no signal can be propagated with a velocity greater than the velocity of light. The resolution of the apparent difficulty is that the energy is propagated along the guide with a smaller velocity than the velocity of light, namely with the so called *group velocity. Signals are transmitted with the group velocity and not with the phase velocity. The phase velocity is not the velocity of propagation of any physical quantity : this simply represents the velocity of a point of constant phase on the wave.*

To determine the group velocity *i.e.* the velocity of energy propagation we shall calculate the time average energy density and time averaged-Poynting vector. Then dividing the Poynting vector (power flow) by the energy density, the velocity of energy propagation (group velocity) may be obtained.

In terms of λ_g and λ_c, the electric field (4) may be written as

$$\mathbf{E} = \hat{\mathbf{i}} E' \sin\left(\frac{2\pi\lambda}{\lambda_c}\right) e^{i\,[(2\pi z/\lambda_g) - \omega t]} \qquad \text{...(12)}$$

(where $E' = 2iE_{01}$)

The magnetic field induction in the guide is readily obtained from

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and assuming that } \mathbf{B}\,(\mathbf{r}, t) = B\,(\mathbf{r})\, e^{-i\omega'}$$

we quickly find

$$B\,(\mathbf{r}, t) = \hat{\mathbf{j}} E' \frac{2\pi}{\omega\lambda_g} \sin\frac{2\pi y}{\lambda_c}\, e^{i}\,\{(2\pi z/\lambda_{g)} - \omega t\,\} + \hat{\mathbf{k}} iE' \frac{2\pi}{\omega\lambda_g} \cos\frac{2\pi y}{\lambda_c}\, e^{i}\,\{(2\pi z/\lambda_g) - \omega t\} \qquad \text{...(13)}$$

The time average Poynting vector is

$$\begin{aligned} \langle S_z \rangle &= \tfrac{1}{2} R_e\,(\mathbf{E}^* \times \mathbf{H}) = \tfrac{1}{2} R_e\,(E_x{}^* H_y) \\ &= \tfrac{1}{2} R_e \left[E'^* \sin\left(\frac{2\pi y}{\lambda_c}\right) \frac{1}{\mu_0} E' \left(\frac{2\pi}{\omega\lambda_g}\right) \sin\left(\frac{2\pi y}{\lambda_c}\right) \right] \\ &= 2\,E'^*\,E' \left(\frac{2\pi}{\mu_0 \omega \lambda_g}\right) \sin^2\left(\frac{2\pi y}{\lambda_c}\right) \end{aligned}$$

Integrating this expression from $y = 0$ to $y = L$ across the guide yields the total average power travelling along the guide

$$\langle S_x \rangle = \int_0^L \langle S_z \rangle\, dy = \tfrac{1}{4} E'^*\, E' \left(\frac{2\pi}{\mu_0 \omega \lambda_g}\right) L \qquad \text{...(14)}$$

$$\left(\text{since } \int_0^L \sin^2 \frac{2\pi y}{\lambda_c}\, dy = \frac{L}{2}\right)$$

The . e averaged energy density is

$$\langle u \rangle = \tfrac{1}{4} \text{Re}[\mathbf{E}^* \bullet \mathbf{D} + \mathbf{B}^* \bullet \mathbf{H}]$$

$$= \tfrac{1}{4} \text{Re} \left[\varepsilon_0\, E'^* E' \sin^2\left(\frac{2\pi y}{\lambda_c}\right) + \frac{1}{\mu_0} E'^*\, E' \left(\frac{2\pi}{\omega\lambda_g}\right)^2 \sin^2\left(\frac{2\pi y}{\lambda_c}\right) + \frac{1}{\mu_0} E'^*\, E' \left(\frac{2\pi}{\omega\lambda_g}\right)^2 \cos^2\left(\frac{2\pi y}{\lambda_c}\right) \right] \qquad \text{...(15)}$$

Integrating in y-direction from $y = 0$ to $y = L$ across the guide, effectively replaces $\sin^2\left(\frac{2\pi y}{\lambda_c}\right)$ and $\cos^2\left(\frac{2\pi y}{\lambda_c}\right)$ by $\frac{L}{2}$. Thus

$$\begin{aligned}<S> &= \int_0^L <u>\,dy = \tfrac{1}{4}E'^{*}E'\frac{L}{2}\left[\varepsilon_0+\frac{4\pi}{\mu_0\omega^2}\left(\frac{1}{\lambda_g^2}+\frac{1}{\lambda_c^2}\right)\right]\\ &= \tfrac{1}{4}E'^{*}E'\frac{L}{2}\left[\varepsilon_0+\frac{4\pi^2}{\mu_0\omega^2\lambda_0^2}\right]\\ &= \tfrac{1}{4}E'^{*}E'\frac{L}{2}\left[\varepsilon_0+\frac{1}{\mu_0c^2}\right] = \tfrac{1}{4}E'^{*}E'\frac{L}{2}[\varepsilon_0+\varepsilon_0]\\ &= \tfrac{1}{4}E'^{*}E'\varepsilon_0L \end{aligned} \qquad ...(16)$$

The velocity of *energy propagation* or *group velocity*

$$\begin{aligned} v_G = \frac{<S>}{<U>} &= \frac{\tfrac{1}{4}E'^{*}E'\left(\frac{2\pi}{\mu_0\,\omega\lambda_g}\right)L}{\tfrac{1}{4}E'^{*}E\varepsilon_0L} = \frac{2\pi}{\mu_0\varepsilon_0\omega\lambda_g}\\ &= \frac{2\pi c^2}{\omega\lambda_g} = c\frac{\lambda_0}{\lambda_g} = c\sin\theta \end{aligned} \qquad ...(17)$$

Obviously group velocity v_G is smaller than the velocity of light.

The group velocity v_G may be obtained by usual relation $v_G = \frac{\partial\omega}{\partial k}$, the ratio $\frac{\omega}{k}$ is called the phase velocity which in this case becomes

$$v_c = \frac{\partial\omega}{\partial k_g}. \qquad ...(18)$$

Equation (11) may be expressed as

$$k^2 = k_c^2 + k_g^2 \qquad (\text{as } k = 2\pi/\lambda).$$

In free space $k = \omega/c$, therefore

$$\omega^2/c^2 = k_c^2 + k_g^2$$

or

$$\{\omega = c\sqrt{(k_c^2 + k_g^2)}\}. \qquad ....(19)$$

Differentiating this equation with respect to k_g, we get

$$\begin{aligned} v_G = \frac{\partial\omega}{\partial k_g} &= c\,.\tfrac{1}{2}(k_c^2+k_g^2)^{-1/2}\times 2K_g = \frac{ck_g}{\omega/c} \text{ using(19)}\\ &= c^2.\left(\frac{k_g}{\omega}\right) = \frac{c^2}{c/\sin\theta} = c\sin\theta. \end{aligned} \qquad ...(20)$$

e have $v_g = c/\sin\theta$ and $v_G = c\sin\theta$.

Thus it is readily apparent.

$$v_g v_G = c^2. \qquad ...(21)$$

9.12. Wave Guides

A wave guide is a hollow pipe of infinite extent. We shall now consider the propagation of electromagnetic waves along a hollow conducting pipe (wave guide) of arbitrary (simply connected) cross section uniform along its length. We shall assume that the walls of the wave guide are perfectly conducting and take the z-axis along the wave guide. For simplicity we further assume that the interior of wave guide is vacuum. As shown in chapter 7, each component of **E** and **B** must satisfy wave equation which in vacuum has the form

$$\left.\begin{aligned}\nabla^2\mathbf{E}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}&=0 \text{ or } \nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \quad ...(a)\\ \nabla^2\mathbf{B}-\mu_0\varepsilon_0\frac{\partial^1\mathbf{B}}{\partial t^2}&=0 \text{ or } \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0 \quad ...(b)\end{aligned}\right\} \quad ...(1)$$

In addition to these wave equations Maxwell's equations and boundary conditions must be satisfied. As the wave is propagated along Z-axis, we assume solutions of the form

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t)&=\mathbf{E}(x,y)\,e^{ik_gz-i\omega t} \quad ...(a)\\ \mathbf{B}(\mathbf{r},t)&=\mathbf{B}(x,y)\,e^{ik_gz-i\omega t} \quad ...(b)\end{aligned}\right\} \quad ...(2)$$

Appropriate linear combinations can be formed to give travelling or standing waves in the Z-direction. The wave number k_g is known from preceding section but however we assume that it is unknown parameter which may be real or complex. With this assumed Z dependence of fields, the wave equation reduces to two dimensional form

$$\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}-k_g^2\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}=0$$

$$\left.\begin{aligned}\text{or}\quad \frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]\mathbf{E}&=0 \quad ...(a)\\ \text{and}\quad \frac{\partial^2\mathbf{B}}{\partial x^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]\mathbf{B}&=0 \quad ...(b)\end{aligned}\right\} \quad ...(3)$$

Now if $\nabla_\perp^2$ is transverse part of Laplacing operator ∇^2, then

$$\nabla_\perp^2=\nabla^2-\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad ...(4)$$

In these notations equation (3) takes the form

$$\left[\nabla_\perp^2+\left(\frac{\omega^2}{c^2}-k_g^2\right)\right]\left\{\frac{\mathbf{E}}{\mathbf{B}}\right\}=0 \quad ...(5)$$

Now Maxwell's curl equations for free space are

$$\left.\begin{aligned}\nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t} \quad (a)\\ \text{and}\quad \nabla\times\mathbf{B}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \text{ or } \nabla\times\mathbf{B}&=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad ...(b)\end{aligned}\right\} \quad ...(6)$$

which in terms of components can be written as

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=-\frac{\partial B_x}{\partial t}\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=-\frac{\partial B_y}{\partial t}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=-\frac{\partial B_z}{\partial t}\end{aligned}\right] \quad ...(7a)$$

and

$$\left.\begin{aligned}\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}&=\frac{1}{c^2}\frac{\partial E_x}{\partial t}\\ \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}&=\frac{1}{c^2}\frac{\partial E_y}{\partial t}\\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}&=\frac{1}{c^2}\frac{\partial E_z}{\partial t}\end{aligned}\right]\qquad\text{...(7b)}$$

From the form of exponential $e^{ik_g z-i\omega t}$ it is apparent that

$$\frac{\partial}{\partial z}\rightarrow ik_g \text{ and } \frac{\partial}{\partial t}\rightarrow -i\omega$$

Therefore equations (7) lead to

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-ik_gE_y&=i\omega B_x &&\text{....}(a)\\ ik_gE_x-\frac{\partial E_z}{\partial y}&=i\omega B_z &&\text{....}(b)\\ \frac{\partial \mathbf{E_y}}{\partial x}-\frac{\partial \mathbf{E_x}}{\partial y}&=i\omega B_z &&\text{...}(c)\end{aligned}\right]\qquad\text{...(8)}$$

and

$$\left.\begin{aligned}\frac{\partial B_z}{\partial y}-ik_gB_y&=-\frac{i\omega}{c^2}\mathbf{E_x} &&\text{....}(a)\\ ik_gB_x-\frac{\partial B_z}{\partial x}&=-\frac{i\omega}{c^2}\mathbf{E_y} &&\text{....}(b)\\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}&=-\frac{i\omega}{c^2}\mathbf{E_z} &&\text{...}(c)\end{aligned}\right]\qquad\text{...(9)}$$

Substituting values of B_y from (8b) in (9a), we get

$$\frac{\partial B_z}{\partial y}-ik_g\left(\frac{ik_g\mathbf{E_x}-\frac{\partial \mathbf{E_z}}{\partial x}}{i\omega}\right)=-\frac{i\omega}{c^2}\mathbf{E_x}$$

i.e.

$$\frac{\partial B_z}{\partial y}+\frac{k_g}{\omega}\frac{\partial \mathbf{E_z}}{\partial x}=\left(\frac{ik_g^2}{\omega}-\frac{i\omega}{c^2}\right)\mathbf{E_x}$$

i.e.

$$\mathbf{E_x}=\frac{i}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\left[k_g\frac{\partial \mathbf{E_z}}{\partial x}-\omega\frac{\partial B_y}{\partial y}\right]\qquad\text{...(10)}$$

Now eliminating B_x from (8a) and (9b), we get

$$\mathbf{E_y}=\frac{i}{\left[\frac{\omega^2}{c^2}-k_g^2\right]}\left[k_g\frac{\partial \mathbf{E_z}}{\partial q}-\omega\frac{\partial B_y}{\partial x}\right]\qquad\text{...(11)}$$

Similarly eliminating $\mathbf{E}_y$ and $\mathbf{E}_x$ in turn from (8) and (9), we get

$$B_x = \frac{i}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\left(k_g\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial \mathbf{E}_z}{\partial y}\right) \qquad \text{....(12)}$$

$$B_y = \frac{i}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\left(k_g\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial \mathbf{E}_z}{\partial x}\right) \qquad \text{...(13)}$$

Equation (10), (11), (12) and (13) show that it is sufficient to determine $\mathbf{E}_z$ and B_z as the appropriate solutions of the two dimensional wave equation (5). The other components can then be calculated from (10), (11), (12) and (13). In general cylindrical guide with perfectly conducting walls, such as that shown in fig. 9.11, is under consideration, then the appropriate boundary conditions are that the tangential component of **E** and the normal component of **B** should vanish on the surface of the conductor. The boundary condition for two dimensional wave equation for $\mathbf{E}_z$ viz.

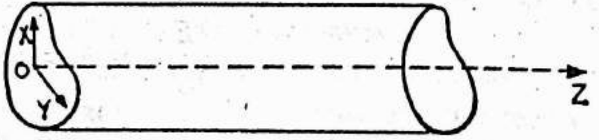


Fig. 9.11 Hollow cylindrical wave guide of arbitrary cross sectional shape

$$\nabla_\perp^2 \mathbf{E}_z + \left(\frac{\omega^2}{c^2}-k_g^2\right)\mathbf{E}_z = 0 \qquad \text{...(14)}$$

that the tangential component of **E** should vanish on the surface of the conductor requires

$$\mathbf{B}_z\big|_S = 0 \qquad \text{...(15)}$$

The normal component of **B** at the surface requires

$$|\mathbf{B}\cdot\mathbf{n}|_S = 0$$

where **n** is unit vector normal the surface

i.e.
$$\left| B_z(x,y)\, e^{ik_g z - i\omega t}\cdot \mathbf{n}\right|_S = 0$$

this condition implies
$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \qquad \text{...(16)}$$

where $\partial/\partial x$ is the normal derivative at the point on the surface. The two dimensional wave equations (5) for $\mathbf{E}_z$ and B_z together with the boundary conditions on $\mathbf{E}_z$ and B_z at the surface of the cylinder allow only certain values of axial wave number k_g for a given frequency ω. Because of different boundary conditions on $\mathbf{E}_z$ and B_z they can not be generally satisfied simultaneously. Consequently the fields may be divided into two distinct classes.

Transverse Magnetic (TM) Mode. In this case $B_z = 0$ always *i.e.* magnetic fields is always perpendicular to the direction of propagation; hence the name transverse magnetic. These type of wave are also sometimes named as *electric type waves* or **E** waves.

Substituting $B_z = 0$, equations (10) to (13) take to form

$$\mathbf{E}_x = \frac{ik_g}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\frac{\partial \mathbf{E}_z}{\partial x} = \frac{ik_g}{(k_o^2-k_g^2)}\frac{\partial \mathbf{E}_z}{\partial x} \qquad \text{...(17)}$$

$$\mathbf{E_y} = \frac{ik_g}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\frac{\partial \mathbf{E_z}}{\partial y} = \frac{ik_g}{(k_o^2-k_g^2)}\frac{\partial \mathbf{E_z}}{\partial y} \qquad \text{...(18)}$$

$$\mathbf{B_x} = -\frac{i\omega}{\left(\frac{\omega^2}{c^2}-k_g^2\right)c^2}\frac{\partial \mathbf{E_z}}{\partial y} = -\frac{i\omega}{(k_o^2-k_g^2)\,c^2}\frac{\partial \mathbf{E_z}}{\partial y} \qquad \text{...(19)}$$

$$\mathbf{B_y} = \frac{i\omega}{\left(\frac{\omega^2}{c^2}-k_g^2\right)c^2}\frac{\partial \mathbf{E_z}}{\partial y} = \frac{i\omega}{(k_o^2-k_g^2)\,c^2}\frac{\partial \mathbf{E_z}}{\partial x} \qquad \text{...(20)}$$

where $k_0 = \frac{\omega}{c}$ = free space wave number

Thus in *TM* mode, all the transverse components of **E** and **B** can be expressed in terms of longitudinal component of the electric field. This component may be obtained by solving the two dimensional wave equation (5) for $\mathbf{E_z}$ *viz.*

$$\nabla_\perp^2 \mathbf{E_z} + \left(\frac{\omega^2}{c^2}-k_g^2\right)\mathbf{E_z} = 0 \qquad \text{...(21)}$$

or

$$\nabla_\perp^2 \mathbf{E_z} + (k_0^2-k_g^2)\,\mathbf{E_z} = 0$$

The boundary condition for this equation is $\mathbf{E_z}|_S = 0$ ...(i)

(ii) Transverse electric (TE) Mode. In this case $\mathbf{E_z} = 0$ always *i.e.* electric field is always perpendicular to the direction of propagation of the wave, hence the name *transverse electric mode.* These types of waves are also called magnetic type waves or *H*-waves.

Substituting $\mathbf{E_z} = 0$, equations (10) to (13) take the form

$$\mathbf{E_x} = \frac{i\omega}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\frac{\partial \mathbf{B_z}}{\partial y} = \frac{i\omega}{k_0^2-k_g^2}\frac{\partial \mathbf{B}_z}{\partial y} \qquad \text{...(22)}$$

$$\mathbf{E_y} = -\frac{i\omega}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\frac{\partial \mathbf{B_z}}{\partial x} = -\frac{i\omega}{k_0^2-k_g^2}\frac{\partial \mathbf{B}_z}{\partial y} \qquad \text{...(23)}$$

$$B_z = \frac{ik_g}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\frac{\partial B_z}{\partial x} = -\frac{ik_g}{k_0^2-k_g^2}\frac{\partial B_z}{\partial x} \qquad \text{...(24)}$$

$$B_y = \frac{ik_g}{\left(\frac{\omega^2}{c^2}-k_g^2\right)}\frac{\partial B_z}{\partial y} = -\frac{ik_g}{k_0^2-k_g^2}\frac{\partial B_z}{\partial y} \qquad \text{...(25)}$$

Thus in *T*E mode all transverse components of **E** and **B** can be expressed in terms of longitudinal component of magnetic field. This component may be obtained by solving two dimensional wave equation (5) for B_z *viz.*

$$\nabla_\perp^2 B_z + \left(\frac{\omega^2}{c^2} - k_g^2\right) B_z = 0$$

or

$$\nabla_\perp^2 B_z + (k_0^2 - k_g^2)\, B_z = 0 \qquad \text{...(26)}$$

with the boundary condition

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \qquad \text{...(i)}$$

According to equations (22)–(25), this condition ensures that normal component of **B** is zero.

Thus the problem of determining the electromagnetic field in a wave-guide reduces to that of finding solutions of two dimensional wave equation

$$[\nabla_\perp^2 + (k_o^2 - k_g^2)]\,\psi = 0 \text{ or } [\nabla_\perp^2 + k_c^2]\,\psi = 0 \qquad \text{...(27)}$$

subject to boundary conditions $\psi|_S = 0$ and $\partial\psi/\partial n\,|_S = 0$.

Here $$k_c^2 = k_0^2 - k_g^2 \text{ or } k_0^2 = k_c^2 + k_g^2 \qquad \text{...(28a)}$$

Equation (28 a) is equivalent to $$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} \qquad \text{.....(28 b)}$$

which is same as equation (11) of preceding section 9.11 Thus the conclusions drawn in the preceding section are quite general and are not limited to the case of propagation between parallel conducting planes.

For a given cross section the solution ψ exists only for a certain definite eigen values of the parameter k_c^2. It is easy to see that the constant k_c^2 must be non-negative, since roughly speaking ψ must be oscillatory in order to satisfy required boundary condition on opposite sides of the cylinder. These will from a spectrum of eigen values $(k_c^2)_\lambda$ and corresponding solutions ψ_λ $(\lambda = 1, 2, 3)$ form an orthogonal set. These different solutions are called the *modes of the guide*. For a given frequency ω the wave number k_g is determined for each value of λ

$$(k_g^2)_\lambda = k_0^2 - (k_c^2)_\lambda = \omega^2/c^2 - (k_c^2)_\lambda \qquad \text{...(29)}$$

Since $k_g^2 = k_0^2 - k_c^2$, therefore we note that for $k_0 > k_c$ or $\omega > \omega_c$, the wave number k_g is real ; hence waves of such modes can propagate in the guide But if $k_c < k_c$ or $\omega < \omega_c$, k_g is imaginary which is turn implies the attenuation of **E** and **B** given by (2) : hence such modes can not propagate and are called *cut off modes*. The frequency given by

$$(\omega_c)\lambda = [c]\,(k_c) \qquad \text{...(30)}$$

is called cut off frequency. Then the *wave number* can be written

as $$(k_g)\lambda = \frac{1}{c}\sqrt{[\omega^2 - (\omega_c^2)\,\lambda]} \qquad \text{...(31)}$$

From this it follows that a guide acts a sort of high pass filter in the sense that only frequencies greater that cut off frequency can be propagated in the guide. Moreover at any given frequency only a finite mumber of modes can propagate. It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.

The velocity of propagation *of the wave (i.e. group velocity) along the wave guide* is given by the derivative

$$v_z = \frac{\partial\omega}{\partial k_g} = \frac{\partial}{\partial k_g}\, c\,(k_g^2 + k_c^2)^{1/2}$$

$$= \frac{ck_g}{\sqrt{(k_0^2 + k_c^2)}} = \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \qquad ...(32)$$

For a given k_c : this varies from o to c when k_g varies from o to ∞.

The **phase velocity** v_g in the guide is given by

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(\omega^2 - \omega_c^2)/c}} = \frac{c}{\sqrt{\left[1 - \left(\frac{\omega_c^2}{\omega^2}\right)\right]}}$$

Obviously $v_g > c$ and becomes infinite exactly at cut off frequency. The energy in the guide is propagated with the group velocity.

TEM **Waves.** In addition to *TE* and *TM* modes, there is a degenerate mode, called the transverse electromagnetic (*TEM*) mode, in which both E_z and B_z vanish. If we substitute $E_z = B_z = 0$ in equations (10)–(13), we note that there is no non-zero component of **E** or **B**. This implies that *TEM* **wave can not be propagated along the wave-guide.**

9.13. Rectangular Wave Guide

The most commonly used wave guide is that of rectangular cross-section having inner dimension a and b as shown in fig. 9.21.

The solution of two dimensional wave equation

$$(\nabla_\perp^2 \neq k_c^2)\,\psi = 0 \qquad ...(1)$$

can be carried out in rectangular coordinates as follows :

TE Mode. For *TE* mode $E_z = 0$: hence equation (1) is to be written for B_z ; which takes the form

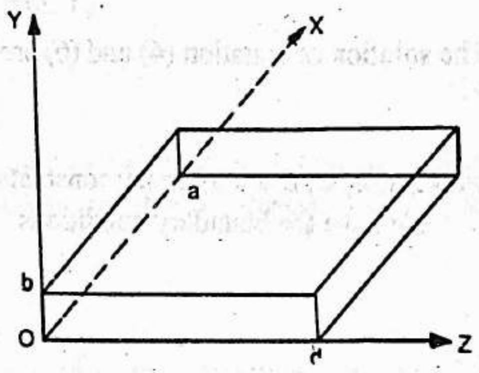


Fig. 9.12

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z = 0 \qquad ...(2)$$

The boundary conditions are

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0$$

i.e.

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

We shall solve equation (2) by the method of separation of variables. Therefore writing

$$B_z(x, y) = X(x)\,Y(y) = XY \qquad ...(3)$$

where X is a function of x only and Y is a function of y only.

Susbtituting equation (3) in (2) and dividing by XY, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

or
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

In above equation L.H.S. is a function of x only, while R.H.S. is a function of y only. Hence this equation will be satisfied if both sides are equal to a constant say p^2 *i.e.*

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = +p^2$$

or
$$\frac{\partial^2 X}{\partial x^2} + (k_c^2 - p^2) X = 0$$

or
$$\frac{\partial^2 X}{\partial x^2} + q^2 X = 0 \qquad \text{...(4)}$$

where
$$q^2 = k_c^2 - p^2 \qquad \text{...(5)}$$

and
$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = p^2$$

or
$$\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0 \qquad \text{...(6)}$$

The solution of equation (4) and (6) are

$$X(x) = A\cos qx + B\sin qx \qquad \text{...(7)}$$
$$Y(y) = C\cos py + D\sin py \qquad \text{...(8)}$$

where A, B, C, D are arbitrary constants.

We have the boundary conditions

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$
$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

These conditions are equivalent to

$$\frac{\partial X}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and
$$\frac{\partial Y}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b.$$

Differentiating equations (7) and (8), we get

$$\frac{\partial X}{\partial x} = -Aq\sin qx + Bq\cos qx$$
$$\frac{\partial Y}{\partial y} = -Cp\sin qy + Dp\cos py$$

Applying boundary condition $\left.\frac{\partial Z}{\partial x}\right|_{x=a} = 0$, we get

$Bq = 0$ ∴ This gives $B = 0$

Now applying boundary condition $\left.\frac{\partial X}{\partial x}\right|_{x=a} = 0$, we get

$$-Aq\sin qa = 0$$

We must take $A \neq 0$ since otherwise $X = 0$ and $B_z = 0$. Hence

$$\sin qa = 0 \text{ or } qa = m\pi, \text{ that is,}$$

$$q = \frac{m\pi}{a} \quad (m \text{ integer}) \qquad \text{...(11)}$$

In precisely the same manner we conclude that $D = 0$ and p must be restricted to values $p = n\pi/b$ where n is an integer. In this way be obtain the solutions

$$X(x) = A \cos\left(\frac{m\pi}{a}\right)x\,; \quad Y(y) = C\cos\left(\frac{n\pi}{b}\right)y \qquad \text{...(12a)}$$

where

$$m = 1, 2, 3, \ldots\ldots \quad n = 1, 2, 3, \ldots.$$

and

$$k_c^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The solution for $B_z\,(x, y)$ is consequently

$$B_z\,(x, y) = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad \text{...(13a)}$$

when

$$(k_c^2)_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \qquad \text{...(13b)}$$

Here the indices mn specify the mode. The *cut off frequency* ω_{mn} is given by

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad \text{...(14)}$$

The modes corresponding to m and n are represented as TE_{mn}. The case $m = n = 0$ gives a static field which does not represent a wave propagation ; hence the mode TE_{00} represents non-trivial solution. If $a > b$, the lowest *cut off frequency* results for $m = 1$ and $n = 0$,

i.e.

$$\omega_{10} = \frac{\pi c}{a} \quad \text{or} \quad (k_c)_{10} = \frac{\pi}{a} \qquad \text{...(15)}$$

The mode (TE_{10}) represents the *dominant TE mode* and is the one used in most practical situations. The values E_x, E_y, B_x and B_y for TE mode may be obtained from equations (22)–(25) of preceding section by substituting the solution for B_z, which is

$$B_z\,(r, t) = B_z\,(x, y)\; e^{ik_z z - i\omega t}$$

$$= B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\; e^{ik_z z - i\omega t} \qquad \text{...(16)}$$

Thus we have

$$\left.\begin{aligned} E_x &= \frac{in\pi\omega}{k_c^2 b} B_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ E_y &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_x &= -\frac{im\pi k_g}{k_c^2 a} B_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_y &= -\frac{in\pi k_g}{k_c^2 a} B_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{ik_g z - i\omega t} \end{aligned}\right\} \quad \text{...(17)}$$

For *TE* mode, these equations yield

$$\left(\text{put } m = 1,\ n = 0, k_c^2 = \frac{\pi^2}{a^2}\right)$$

$$\left.\begin{aligned} E_x = B_y = E_z = 0\,;\ B_z &= B_0 \cos\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ \text{and} \qquad B_x &= -\frac{ik_g a}{a} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ E_y &= \frac{i\omega a}{\pi c} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t}\,. \end{aligned}\right\} \quad \text{...(18)}$$

The presence of a factor i in B_x (and E_y) means that there is a spatial (or temporal) phase difference of $\pi/2$ between B_x (and E_y) and B_z in the propagation.

***TM* Mode :** For *TM* mode $B_z = 0$; hence equation (1) is to be written for E_z which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \quad \text{...(19)}$$

The boundary conditions are $E_z = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$.

Solving equation (19) as for *TE* case, we note that the solution of equation (19) is of the form

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad \text{...(20)}$$

where k_c^2 and hence ω_{mn} are still given by equations (12b) and (14). This implies that *TE and TM modes of a rectangular guide have the same set of cut off frequencies*. However in this case $m = 1$ and $n = 0$ represents non-trival solution since this gives $E_z = 0$ and hence all components of **E** and **B** will be zero.

It is obvious that in this case the lowest mode has $m = n = 1$ and may be represented by TM_{11}. The **cut off frequency of lowest mode** is given by

$$\omega_{11} = \pi c\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = \frac{\pi c}{a}\left[1 + \frac{a^2}{b^2}\right]^{1/2} \quad \text{...(21)}$$

Since $a < b$, therefore the *cut off frequency of lowest TM mode is greater than that of the lowest TE mode by the factor* $\left[1 + \left(\frac{a^2}{b^2}\right)\right]^{1/2}$ The fields E_x, E_y, B_x, B_y for *TM* mode may be obtained from equations (17)–(22) of preceding section if we substitute

$$E_z(\mathbf{r}, t) = E_0(x, y)\, e^{ik_g z - i\omega t}$$

$$E_z(\mathbf{r}, t) = E_0 \sin\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{-ik_g z - i\omega t} \qquad ...(22)$$

or

Thus we have

$$\left.\begin{aligned} E_x &= \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b.}\right] e^{ik_g z - i\omega t} \\ E_y &= \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_x &= -\frac{i\omega n\pi}{k_c^2 bc} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_y &= \frac{i\omega n\pi}{k_c^2 ac^2} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \end{aligned}\right] \qquad ...(22)$$

Ex. 2. *What must be the width of a rectangular guide such that the energy of electromagnetic radiation, whose free space wavelength is 3·0 cm, travels down the guide at 95% of the speed of light ?* **(Meerut 1970)**

Solution. We know that the energy of electromagnetic radiation is propagated down the guide with *group velocity* given by

$$v_z = \frac{ek_g}{k_0} = \frac{c\sqrt{(k_0^2 - k_c^2)}}{k_0} = c\left[1 - \frac{k_c^2}{k_0^2}\right]^{1/2} \qquad ...(1)$$

For rectangular wave guide

$$(k_c^2)_{mn} = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{a^2}\right]$$

For *TE mode* and $a > b$; the maximum cut off frequency is given by $m = 1$ and $m = 0$,

$$(k_c^2)_{10} = \pi^2\left[\frac{1}{a^2} + \frac{0}{a^2}\right] = \frac{\pi^2}{a^2} \qquad ...(2)$$

If λ_0 is *free space wavelength,* then $\qquad k_c = 2\pi/\lambda_0 \qquad ...(3)$

Using (2) and (3), equation (1) gives

$$v_z = c\left[1 - \frac{(\pi^2/a^2)}{(2\pi/\lambda c)^2}\right]^{1/2} = c\left[1 - \frac{\lambda_0^2}{4\pi^2}\right]^{1/2}$$

Squaring, we get

$$v_z^2 = c^2\left[1 - \frac{\lambda_0^2}{4a^2}\right]$$

or

$$\frac{v_z^2}{c^2} = 1 - \frac{\lambda_0^2}{4a^2}$$

This gives

$$a = \frac{\lambda_0}{2\left[1 - \frac{v_z^2}{c^2}\right]^{1/2}}$$

Here $v_z = \frac{95}{100}\, c = 0{\cdot}95\, c$ and $\lambda_0 = 3$ cm $= 0{\cdot}03$ m

$$\therefore \qquad a = \frac{0{\cdot}03}{2\left[1-\left(\frac{0{\cdot}95}{c}\right)^2\right]^{1/2}} = \frac{0{\cdot}03}{2\,[1-(0{\cdot}95)^2]^{1/2}}$$

$$= \frac{0{\cdot}03}{2\times 0{\cdot}12} = 0{\cdot}048 \text{ m} = 4{\cdot}8 \text{ cm.}$$

9.14. Circular Wave Guide

The next most important example of wave guides after the rectangular guide is that in which the bounding surfaces are circular cylinders. This includes both the circular guide and the coaxial line, which has two concentric cylinder, with the wave propagated in the angular space between them. In either case the two dimensional wave equation *viz.*

$$(\nabla_\perp^2 + k_c^2)\,\psi = 0 \qquad \text{...(1)}$$

is to be solved in cylinderical coordinates, but here we shall restrict ourselves for cases where cross-section is circular. Let a be the radius of circular cross-section

We have

$$\nabla_\perp^2 = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$$

Therefore two dimensional wave equation (1) in this case becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\psi + k_c^2\,\psi = 0 \qquad \text{...(2)}$$

If a is the radius of circular cross-section, then the boundary conditions are

$$\psi\Big|_{r=a} = 0 \text{ and } \frac{\partial\psi}{\partial n}\Big|_{r=a} = 0 \qquad \text{for all values of } \theta \qquad \text{...(3)}$$

Now let us write Maxwell's equations in cylinderical coordinates (r, θ, z). Maxwell's curl equations in free space are

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

and

$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \qquad \text{...(4)}$$

In cylindrical coordinates equations take the form

$$\left(\frac{1}{r}\frac{\partial E_r}{\partial\theta} - \frac{\partial E_\theta}{\partial z}\right)\hat{n}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]\hat{n}$$

$$= -\left[\frac{\partial B_r}{\partial t}\hat{n}_r + \frac{\partial B_\theta}{\partial t}\hat{n}_\theta + \frac{\partial B_z}{\partial t}\hat{n}_z\right]$$

and

$$\left[\frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial r}\right]\hat{n}_r + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{n}_z$$

$$= \frac{1}{c^2}\left[\frac{\partial E_r}{\partial t}\hat{n}_r + \frac{\partial E_\theta}{\partial t}\hat{n}_\theta + \frac{\partial E_z}{\partial t}\hat{n}_z\right]$$

Comparing coefficients of $\hat{n}_r$, $\hat{n}_\theta$ and $\hat{n}_z$ on either sides, we get

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial E_z}{\partial r}-\frac{\partial E_\theta}{\partial r}&=-\frac{\partial B_r}{\partial t}\\ \frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}&=-\frac{\partial B_\theta}{\partial t}\\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,E_\theta)-\frac{\partial E_r}{\partial \theta}\right]&=-\frac{\partial B_z}{\partial t}\end{aligned}\right]\text{ and}$$

and

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial B_z}{\partial \theta}-\frac{\partial B_\theta}{\partial z}&=\frac{1}{c^2}\frac{\partial E_r}{\partial t}\\ \frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}&=\frac{1}{c^2}\frac{\partial E_\theta}{\partial t}\\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,B_\theta)-\frac{\partial B_r}{\partial \theta}\right]&=\frac{1}{c^2}\frac{\partial E_z}{\partial t}\end{aligned}\right] \qquad ...(5)$$

But

$$\mathbf{E}(r,\theta,z,t)=\mathbf{E}(r,\theta)\,e^{ik_gz-i\omega t}$$
$$\mathbf{B}(r,\theta,z,t)=\mathbf{B}(r,\theta)\,e^{ik_gz-i\omega t}$$

From the form of exponential $e^{ik_gz-i\omega t}$, it is apparent that

$$\partial/\partial z\rightarrow ik_g \quad \text{and} \quad \partial/\partial t\rightarrow -i\omega t$$

Substituting these values in (5), we get

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial E_z}{\partial \theta}-ik_g\,E_\theta&=i\omega\,B_r &&...(i)\\ ik_g\,E_r-\frac{\partial E_r}{\partial r}&=i\omega B_\theta &&...(ii)\\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,E_\theta)-\frac{\partial E_r}{\partial \theta}\right]&=i\omega\,B_z &&...(iii)\\ \frac{1}{r}\frac{\partial B_z}{\partial \theta}-ik_g\,B_\theta&=\frac{i\omega}{c^2}E_r &&...(vi)\\ ik_g\,B_r-\frac{\partial E_z}{\partial r}&=-\frac{i\omega}{c^2}\,E_\theta &&...(v)\\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,B_\theta)-\frac{\partial B_r}{\partial \theta}\right]&=-\frac{i\omega}{c^2}\,E_z &&...(vi)\end{aligned}\right] \qquad ...(6)$$

Eliminating B_θ from (ii) and (iv), we get

$$E_r=\frac{i\omega}{\left[\frac{\omega^2}{c^2}-k_g^2\right]}\left[\frac{k_g}{\omega}\frac{\partial E_z}{\partial r}+\frac{1}{r}\frac{\partial B_z}{\partial \theta}\right] \qquad ...(7)$$

Eliminating B_r from (i) and (v), we get

$$E_0=-\frac{i\omega}{\left[\frac{\omega^2}{c^2}-k_g^2\right]}\left[\frac{\partial B_z}{\partial r}-\frac{1}{r}\frac{k_g}{\omega}\frac{\partial E_z}{\partial \theta}\right] \qquad ...(8)$$

Eliminating E_θ from (i) and (v), we get

$$B_r = + \frac{i}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \left[k_g \frac{\partial B_z}{\partial r} - \frac{\omega}{c^2 r} \frac{\partial E_z}{\partial \theta} \right] \qquad ...(9)$$

and finally eliminating E_r from (ii) and (iv), we get

$$B_\theta = \frac{i}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \left[\frac{\omega}{c^2} \frac{\partial E_z}{\partial r} + \frac{k_g}{r} \frac{\partial B_z}{\partial \theta} \right] \qquad ...(10)$$

It is obvious that all the transverse components of field viz. E_r , E_θ , B_r , B_θ may be obtained from B_z and E_z. Thus we have to solve two dimensional wave equation (2) for E_z and B_z. Let us now consider two usual cases :

Case (i). TM Mode. For *TM* mode $B_z = 0$; hence equations (7), (8), (9) and (10) take the form

$$\left.\begin{aligned} E_r &= \frac{ik_g}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \frac{\partial E_z}{\partial r} & ...(a) \\ E_\theta &= \frac{ik_g}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \frac{\partial E_z}{\partial \theta} & ...(b) \\ B_r &= - \frac{i\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right] c^2 r} \frac{\partial E_z}{\partial \theta} & ...(c) \\ B_\theta &= \frac{i\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right] c^2} \cdot \frac{\partial E_z}{\partial r} & ...(d) \end{aligned}\right] \qquad ...(11)$$

Obviously in *TM* mode all transverse components are expressible in terms of E_z ; hence we have to solve two dimensional wave equation for E_z ; which is

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] E_z + k_g^2 E_z = 0 \qquad ...(12a)$$

with boundary condition $E_z = 0$ at $r = a$ for all values of θ. ....(12b)

Let us solve above equation by the method separation of variables. Therefore writing

$$E_z(r, \theta) = R(r)\, \Theta(\theta) \qquad ...(13)$$

where $R(r)$ is a function of r only, while $\Theta(\theta)$ is a function of θ only. Substituting $E_z = \Theta R$ in (12a) and dividing throughout by $R\,\Theta$, we get

$$\frac{1}{Rr} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} + k_c^2 = 0 \qquad ...(14)$$

Muliplying throughout by r^2, we get

$$\frac{r}{R} \frac{\partial}{\partial r} \left(\frac{\partial R}{\partial r} \right) + k_c^2 r^2 = - \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} \qquad ...(15)$$

In above equation L.H.S. is a function of r only ; while R.H.S. is a function of θ only. This is only possible if each side is equal to the same constant say n^2 *i..e*

$$-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = n^2 \qquad \text{...(16a)}$$

and

$$\frac{r}{R}\frac{r}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k_c^2 r^2 = n^2 \qquad \text{...(17)}$$

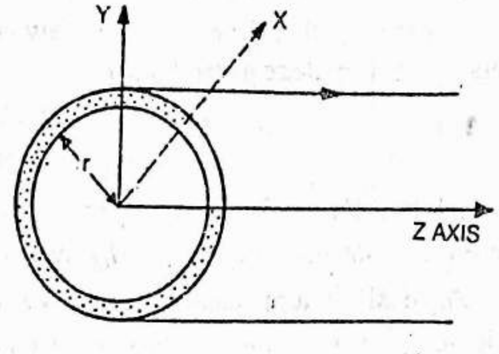


Fig. 9.13. Circular wave guide.

Equation (16a) may be expressed as

$$\frac{\partial^2\Theta}{\partial\theta^2} + n^2\Theta = 0$$

the solution of equations (16b) is

$$\Theta = A_n \cos(n\theta) + B_n \sin(n\theta) \qquad \text{...(18)}$$

Further the relative amplitudes of A_n and B_n determine the orientation of the field in the guide and for a circular guide and for any particular values of n, the θ-axes can always be oriented to make either A_n or B_n equal to zero. Let us choose B_n equal to zero, so that

$$\Theta = A_n \cos n\theta$$

Now equation (17a) may be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \left(k_c^2 - \frac{n^2}{r^2}\right)R = 0 \qquad \text{...(1}$$

or

$$\frac{1}{r}\frac{\partial}{\partial(k_c r)}\left[r\frac{\partial R}{\partial(k_c r)}\right] + \left[1 - \frac{n^2}{(k_c r)^2}\right] = 0 \qquad \text{...(20)}$$

Equation (20) is a Bessel's equation in terms of $(k_c r)$ which has two independent solutions namely Bessel's function $J_n(k_c r)$ and Neumann's function $N_n(k_c r)$.

The Bessel's and Neumann's function [$J_n(x)$ and $N_n(x)$] have the following properties :

(i) At $x = 0$, the Bessel's function $J(x)$ is proportional to x^n while Neumann's function $N_n(x)$ becomes infinite.

(ii) For large values of x, $J_n(x)$ and $N_n(x)$ approach the values

$$\left.\begin{aligned} J_n(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{(2n+1)}{4}\pi\right) \\ N_n(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{2n+1}{4}\pi\right) \end{aligned}\right] \qquad \text{...(21)}$$

From this it is clear that in a circular guide, where the field must be finite at $r = 0$, we must use only Bessel's function $J_n(k_c r)$. Thus the accepted solution of equation (10) is

$$R = J_n(k_c c) \qquad \text{...(22)}$$

Hence

$$E_z(k_c r) = R(r)\,.\,\Theta(\theta)$$

$$= J_n(k_c r)\,A_n \cos n\theta = A_n J_n(k_{cr}) \cos n\theta \qquad \text{...(23a)}$$

$\therefore$

$$E_z(r, \theta, z, t) = A_n J_n(k_c r) \cos n\theta\, e^{ik_g z - i\omega t} \qquad \text{...(23b)}$$

The boundary condition for *TM* mode is

$$E_z\Big|_{r=a} = 0. \text{ This implies } J_n(k_c a) = 0 \qquad \text{...(24)}$$

Equation (24) has infinite number of roots but since for the propagation to be possible $k_g = \sqrt{(k_0^2 - k_c^2)}$ should be real *i.e.* k_c should be small $k_c^2 \ll k_0^2$ or $k_c^2 \ll \omega^2/c^2$ or otherwise extremely high frequencies will be required therefore only first few roots of equation (24) will be of practical interest. The first few roots $J_n(x_{nm}) = 0$ are given below

$$\left.\begin{aligned} x_{01} &= 2{\cdot}405\,;\ x_{11} = 3{\cdot}832\,;\ x_{21} = 5{\cdot}135 \\ x_{01} &= 5{\cdot}520\,;\ x_{12} = 7{\cdot}016\,;\ x_{22} = 8{\cdot}417 \end{aligned}\right\} \qquad ...(25)$$

The various TM modes are represented by TM_{01}, TM_{02}, etc. *i.e.* in general TM_{nm}. It is obvious that since there is no root x_{00} or $(k_c a)_{00}$. *TM_{00} wave can not exist.*

Physically n represented the number of cycles of variation of E_z found as θ varies around the complete cylinder. The subscript m indicates the number of zeros of electric field along a radial path from the centre to the inside surface of the guide wall. Thus if we let x_{nm} be the mth value of x for which $J_n(x) = 0$; then the cut off value of k_c is given by

$$k_c a = x_{nm} \text{ or } k_c = \frac{x_{nm}}{a} \qquad ...(26)$$

Hence *cut off wavelength* is given by

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi a}{x_{nm}} \qquad ...(27)$$

the phase velocity is

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(k_0^2 - k_c^2)}} \qquad ...(28a)$$

$$= \frac{\omega}{k_0 \sqrt{\left[1 - \left(\frac{k_c^2}{k_0^2}\right)\right]}}$$

$$= \frac{c}{\sqrt{\left[1 - \frac{k_c^2}{k_0^2}\right]}}$$

$$= \frac{c}{\sqrt{\left[1 - \frac{\omega_c^2}{\omega_0^2}\right]}} \qquad ...(28b)$$

It is obvious that phase velocity is greater than the speed of light, and *group velocity* or *velocity of propagation of wave is*

$$v_z = \frac{\partial \omega}{\partial k_g} = \frac{\partial}{\partial k_g}(ck_o) = \frac{\partial}{\partial k_g}\left[c\,(k_g^2 + k_c^2)^{1/2}\right]$$

$$= \frac{ck_g}{\sqrt{(k_g^2 + k_c^2)}}$$

$$= \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \qquad ...(29)$$

For a given k_c, this values varies from *o* to *c* when k_c varies from *o* to ∞.

Substituting solution for E_z from (23b) and using

$$k_c^2 = k_0^2 - k_g^2 = (\omega^2/c^2) - k_g^2$$

the *transverse field components* may be expressed from equation (11) *viz.*

$$\left.\begin{aligned}
E_r &= \frac{ik_g}{k_c^2} A_n J_n'(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} && \text{...(a)}\\
E_\theta &= -\frac{ik_g n}{k_c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} && \text{...(b)}\\
B_r &= +\frac{i\omega n}{k_c^2 c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} && \text{...(c)}\\
B_\theta &= \frac{i\omega}{k_c^2 c^2} A_n J_n'(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} && \text{...(d)}
\end{aligned}\right] \quad \text{...(30)}$$

where J_n' represents derivative of J_n with respect to r.

Case (ii). ***TE* Mode.** For *TE* mode $E_2 = 0$; hence equations (7), (8), (9) and (10) take the form

$$\left.\begin{aligned}
E_r &= \frac{i\omega}{[(\omega^2/c^2) - k_g^2]r} \frac{\partial B_z}{\partial \theta} && \text{...(a)}\\
E_\theta &= \frac{i\omega}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial r} && \text{...(b)}\\
B_r &= \frac{ik_g}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial r} && \text{...(c)}\\
B_\theta &= \frac{ik_g}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial \theta} && \text{...(d)}
\end{aligned}\right] \quad \text{...(31)}$$

Obviously in *TE* mode all transverse components E_r, E_θ, B_r, B_θ are expressible in terms of B_z, hence we have to obtain the solution for two dimensional wave equation (1) for B_z. The solution is obtained exactly as in *TM* mode we get

$$B_z(r, \theta) = A_n J_n(k_c r) \cos n\theta \quad \text{...(32)}$$

The boundary condition for *TE* mode is $\partial B_z/\partial r = 0$ at $r = a$ for all values of θ.

This implies

$$\left.\frac{\partial J_n(k_c r)}{\partial r}\right|_{r=a} = 0 \quad \text{or} \quad J'_n(k_c a) = 0 \quad \text{...(33)}$$

The derivative $J_n'(k_c r)$ with respect to r of the Bessel function may be obtained from

$$J'_n(k_c r) = \frac{\partial J(k_c r)}{\partial r} = \frac{n}{k_c r} J_n(k_c r) - J_{n+1}(k_c r) \quad \text{...(34)}$$

The first few roots of $J'_n(x_{nm}) = 0$ are

$$\left.\begin{aligned}
x_{01} &= 3{\cdot}832, & x_{11} &= 1{\cdot}842, & x_{21} &= 3{\cdot}05\\
x_{02} &= 7{\cdot}016, & x_{12} &= 5{\cdot}330, & x_{22} &= 6{\cdot}71
\end{aligned}\right\} \quad \text{...(35)}$$

If we let x_{nm} by the m^{th} value of x for which $J_n'(x) = 0$, then cut off value of k_c is given by

$$k_c a = x_{nm} \quad \text{or} \quad k_c = \frac{x_{nm}}{a}. \quad \text{...(36a)}$$

Hence *cut off wavelength*

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi a}{x_{nm}} \quad \text{...(36b)}$$

where x_{nm} are given by equation (35). Thus we can calculate the cut off wavelength of various modes. The equations for phase velocity and group velocity are the same as those obtained for *TM* mode except for the use of equation (35) for x_{nm} in place of equation (25).

The field components for *TE* mode may by expressed as

$$\left.\begin{aligned} E_z &= 0 \text{ and } B_z = A_n J_n(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} \\ E_r &= -\frac{i\omega n}{k_c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} \\ E_\theta &= -\frac{i\omega}{k_c^2} A_n J'_n(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} \\ B_r &= \frac{ik_g}{k_c^2} A_n J'_n(k_c r) \cos n\theta \, e^{k_g z - i\omega t} \\ B_\theta &= -\frac{ik_g n}{k_c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} \end{aligned}\right\} \quad \text{...(37)}$$

9.15. Resonant Cavities

If the cylindrical wave guide of finite length has its end faces then it forms what is known as *resonant cavity* (or cavity resonator). Such cylindrical cavity resonator may be rectangular or circular cavity resonator depending upon whether the cross-section of the cavity resonator is rectangular of circular. We shall restrict ourselves to case where cross-section is rectangular. We shall assume that :

(i) the end faces are plane and perpendicular to the axis of cavity.

(ii) the walls are perfectly conducting.

(iii) the interior of cavity is vacuum.

An electromagnetic wave propagating along the axis of a cavity will be reflected back and forth from the end faces, thus giving rise to *standing wave pattern.* If r is the amplitude of reflection coefficient, then the amplitude of transverse electric field is r times of the amplitude of incident wave ; consequently the amplitude of resulting disturbance, obtained by superposition of incident and reflected waves, will vary from something proportional to $(1 + r)$; where the two waves add, to something proportional to $(1 - r)$; where they oppose each other. The ratio of maximum to minimum E or $\frac{1+r}{1-r}$ is called the *standing wave ratio* in voltage. We see that it can go to unity $r = 0$ when there is no reflection to $r = 1$ when there is perfect reflection. For metallic reflection which we shall assume $r = 1$, so that the *standing wave ratio is infinite.* The points along the guide where the resultant transverse electric field is maximum are called *standing wave maxima* and those where it is minimum are called *standing wave minima.* For in infinite standing wave ratio the standing wave minima become the nodes and in any case they are separated by a half wave length.

It is observed that certain standing wave field distributions be set up in the cavity which correspond to different cavity modes. The possible frequencies are known as resonant frequencies of the cavity. Consider a rectangular cavity consisting of the region bounded by the six planes $x = 0, x = a, y = 0, y = b, z = 0, z = d.$

Case (i). *TE* Mode. In this mode $E_x = 0$; let us put

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_z \quad \text{...(1)}$$

where

$$\mathbf{E}_\perp = (\mathbf{i}\,\mathbf{E}_x + \mathbf{j}\,E_y)$$

represents the transverse or perpendicular component of electric field $\mathbf{E}_z$ and $\mathbf{E}_r$ is component of electric field along which wave is propagating *i.e.* along z-axis.

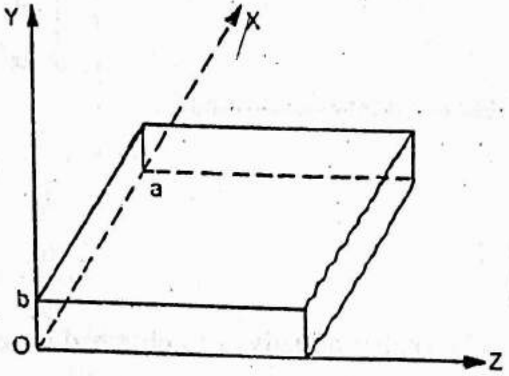


Fig. 9.14. Rectangular resonant cavity.

The transverse component $\mathbf{E}_\perp^+$ of electric field propagating along positive Z-direction may be expressed as

$$\mathbf{E}_\perp^+(\mathbf{r},t) = \mathbf{E}_\perp^+(x, y,)\, e^{ik_gz - i\omega t} = \mathbf{E}_\perp^+(x, y)\, e^{ik_gz - i\omega t} \qquad ...(2)$$

The transverse component of electric field of reflected wave propagating along negative Z-axis is expressed as

$$\mathbf{E}_\perp^-(\mathbf{r},t) = \mathbf{E}_\perp^-(x, y)\, e^{ik_gz - i\omega t} \qquad ...(3)$$

The resultant *transverse electric field*, which results due to superposition of two *TE* waves, one incident propagating along positive Z-axis and the other reflected propagating along negative Z-axis, is given by

$$\mathbf{E}_\perp(\mathbf{r}\,.\,t) = \mathbf{E}_\perp^+(\mathbf{r},t) + \mathbf{E}_\perp^-(\mathbf{r},t)$$

$$= [E_\perp^+(x, y)\, e^{ik_gz} + E_\perp^-(x, y)\, e^{-ik_gz}]\, e^{-i\omega t} \qquad ...(4)$$

The boundary condition is that the tangential component of $\mathbf{E}(\mathbf{r}, t)$ must vanish at $z = 0$ and $z = d$. The boundary condition $\mathbf{E}_\perp(\mathbf{r}, t) = 0$ at $z = 0$ for all values of x, y and t implies

$$\mathbf{E}_\perp^+(x, y) + \mathbf{E}_\perp^-(x, y) = 0$$

i.e.

$$\mathbf{E}_\perp^{(-)}(x, y) = -\,\mathbf{E}_\perp^+(x, y) \qquad ...(5)$$

this equation (5) gives

$$\mathbf{E}_\perp(\mathbf{r}, t) = \mathbf{E}_\perp^+(x, y)\,(e^{ik_gz} - e^{-ik_gz})\, e^{-i\omega t}$$

$$= 2i\,\mathbf{E}_\perp^+(x, y)\left(\frac{e^{ik_gz} - e^{-ik_gz}}{2i}\right) e^{-i\omega t}$$

$$= 2i\,\mathbf{E}_\perp^+(x, y)\sin k_gz\; e^{-i\omega t} \qquad ...(6)$$

The boundary condition $\mathbf{E}_\perp(\mathbf{r}, t) = 0$ at $z = d$ implies that

$$\sin(k_gd) = 0 \text{ or } k_gd = p\pi,\ p = 0, 1, 2, 3, ...$$

This gives

$$k_g = \frac{\pi p}{d} \qquad ...(7)$$

Substituting this value of k_g in equation (6), we get

$$E_\perp(\mathbf{r}, t) = 2i\,\mathbf{E}_\perp^+(x, y)\sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t} \qquad ...(8)$$

This equation may be expressed as

$$iE_z(\mathbf{r}, t) + jE_y(\mathbf{r}, t) = 2i\,\{\mathrm{i}E_x^+(x, y) + \mathbf{j}\mathbf{E}_y^+(x, y)\}\ \sin\frac{p\pi z}{d}\ e^{-i\omega t}$$

This gives

$$\left.\begin{aligned} E_x(\mathbf{r}, t) &= 2i\,E_x^+\ (x, y)\ \sin\frac{p\pi z}{d}\ e^{-i\omega t} \qquad ...(a)\\ E_y(\mathbf{r}, t) &= 2i\,\mathbf{E}_y^+\ (x, y))\ \sin\frac{p\pi z}{d}\ e^{-i\omega t} \qquad ...(b)\end{aligned}\right\} \qquad ...(9)$$

In *TE* mode, the value of B^+ may be obtained by solving the equation

$$(\nabla_\perp^2 + k_c^2)\, B_z^+ = 0$$

or

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mathbf{k}_c^2\right] B_z^+ = 0$$

with the boundary conditions

$$\left.\begin{aligned} \frac{\partial B_z^+}{\partial x} &= 0 \text{ at } x=0 \text{ and } x=a \\ \frac{\partial B_z^+}{\partial y} &= 0 \text{ at } y=0 \text{ and } y=b \end{aligned}\right\}$$

This solution has already been obtained in section 8.13 in the case of rectangular wave guide and is given by

$$B_z^+(\mathbf{r}, t) = B_0 \cos\frac{m\pi x}{h} \cos\frac{n\pi y}{b}\, e^{-ik_z z - i\omega t} \qquad \text{...(10a)}$$

i.e.

$$B_z^+(x, y) = B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad \text{...(10b)}$$

with

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \text{...(11)}$$

This in turn provides

$$\left.\begin{aligned} E_x^+(x, y) &= -\frac{in\pi\omega}{k_c^2 b} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi x}{b} && \text{...(a)} \\ E_y^+(x, y) &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} && \text{...(b)} \end{aligned}\right\} \qquad \text{...(12)}$$

Substituting these values in (9), we get

$$\mathbf{E}_x(\mathbf{r}, t) = \frac{2\omega}{k_c^2}\left\{\frac{n\pi}{b}\right\} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\left\{\frac{p\pi z}{d}\right\} e^{-i\omega t} \qquad \text{...(13a)}$$

$$\mathbf{E}_y(\mathbf{r}, t) = \frac{2\omega}{k_c^2}\left\{\frac{m\pi}{d}\right\} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi z}{b} \sin\left\{\frac{p\pi y}{d}\right\} e^{-i\omega t} \qquad \text{...(13b)}$$

In *TE* mode we already have $E_z(\mathbf{r}, t) = 0.$ ...(13c)

The accompanying magnetic field components for *TE* mode may be obtained by using Maxwell's curl equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

which in terms of components can be written as

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} && \text{...(a)} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} && \text{...(b)} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} && \text{...(c)} \end{aligned}\right\} \qquad \text{...(14)}$$

From the form of exponential $e^{-i\omega t}$, it is apparent that

$$\frac{\partial}{\partial t} \Rightarrow -i\omega$$

therefore equations (14) may be expressed as

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=i\omega B_x &&\text{...(a)}\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=i\omega B_y &&\text{...(b)}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_z}{\partial y}&=i\omega B_z &&\text{...(c)}\end{aligned}\right\}\qquad\text{...(15)}$$

In *TE* mode $E_z = 0$, therefore above equtions yield

$$\left.\begin{aligned}-\frac{\partial E_y}{\partial z}&=i\omega B_x &&\text{...(a)}\\ \frac{\partial E_x}{\partial z}&=i\omega B_y &&\text{...(b)}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=i\omega B_z &&\text{...(c)}\end{aligned}\right\}\qquad\text{...(16)}$$

From equation (16a), we have

$$\begin{aligned}B_x &= -\frac{1}{i\omega}\frac{\partial E_y}{\partial z}\\ &= -\frac{1}{i\omega}\frac{\partial}{\partial z}\left\{-\frac{2m\pi\omega}{k_c^2 a}B_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi y}{d}e^{-i\omega t}\right\}\\ &= -\frac{2i}{k_c^2}\left[\frac{m\pi}{b}\right]\left[\frac{p\pi}{d}\right]B_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}e^{-i\omega t}\end{aligned}\qquad\text{...(17a)}$$

From equation (16 b) we have

$$\begin{aligned}B_y &= \frac{1}{i\omega}\frac{\partial E_x}{\partial z}\\ &= \frac{1}{i\omega}\frac{\partial}{\partial z}\left\{\frac{2\pi n\omega}{k_c^2 b}B_0\cos\frac{m\pi y}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}e^{-i\omega t}\right\}\\ &= -\frac{2i}{k_c^2}\left[\frac{n\pi}{b}\right]\left[\frac{b\pi}{d}\right]B_0\cos\frac{m\pi y}{a}\sin\frac{n\pi y}{h}\cos\frac{p\pi z}{d}e^{-i\omega t}\end{aligned}\qquad\text{...(17d)}$$

Similarly from equation (16c), we have

$$\begin{aligned}B_z &= \frac{1}{i\omega}\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right]\\ &= \frac{1}{i\omega}\left[\frac{\partial}{\partial x}\left\{-\frac{2m\pi\omega}{k_c^2 a}B_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}e^{-i\omega t}\right\}\right.\\ &\qquad\left.-\frac{\partial}{\partial y}\left\{\frac{2n\pi\omega}{k_c^2 b}B_0\cos\frac{m\pi y}{b}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}e^{-i\omega t}\right\}\right]\\ &= \frac{1}{i\omega}\left[-\frac{2m\pi\omega}{k_c^2 a}\cdot\frac{m\pi}{a}\cdot B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}e^{-i\omega t}\right.\\ &\qquad\left.-\frac{2n\pi\omega}{k_c^2 b}\cdot\frac{n\pi}{b}B_0\cos\frac{n\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}e^{-i\omega t}\right]\\ &= \frac{2i}{k_c^2}\left[\left\{\frac{m\pi}{a}\right\}^2+\left\{\frac{n\pi}{b}\right\}^2\right]B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}e^{-i\omega t}\end{aligned}$$

$$= 2i\ B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \qquad \text{...(17c)}$$

Equation (13) and (17) express components of field in resonant cavity for *TE* mode.

The *resonant frequency* of the cavity may be obtained by the use of *guide equations*

$$k_g^2 = k_0^2 - k_c^2\,. \qquad \text{...(18a)}$$

The equation implies that for $k_g < k_c$; k_g is imaginary. This leads to attenuation rather than propagation. Equation (18a) may be expressed as

$$k_0 = k_c^2 + k_g^2\,. \qquad \text{...(18b)}$$

Substituting values of k_g and k_c from equations (7) and (11), and using $k_0 = \omega/c$, we get

$$\frac{\omega^2}{c^2} = \left[\frac{\pi m}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 + \left[\frac{\pi p}{d}\right]^2$$

or

$$\omega^2 = c^2\pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right] \qquad \text{...(19)}$$

i.e. resonant angular frequency

$$\omega_r = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right]^{1/2} \qquad \text{...(20a)}$$

and *resonant linear frequency*

$$f_r = \frac{\omega_r}{2\pi} = \frac{c}{2}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right]^{1/2} \qquad \text{...(20b)}$$

This equation represents that the resonant frequencies form a discrete set depending on the choice of m, n and p. Because propagation in cavity resonators may take place in more than one direction and in various modes, cavity resonators in general have a large number of possible modes of resonance. A cavity resonator for a specific application may be designed and excited in such a manner however, that only one mode of resonance is obtained over a limited frequency range.

From equations (13) and (17) it is obvious that if any of the two integers m, n and p are zero, all the field components would become zero. This implies that TE_{000} , TE_{001} , TE_{010} ,TE_{100} , modes do not exist in the cavity. The physically possible lowest modes are of the type TE_{101} , TE_{011} , (*i.e.* where only one integer is zero). The smallest allowed frequency of these modes is called the *fundamental frequency* or *first harmonic.* The higher allowed values which are integral multiples of the fundamental are called *over ones or higher harnomics*

Case (ii) TM Mode. In *TM* mode $B_z = 0$: and the value of $\mathbf{E}_z^+$ may be obtained by solving the equation

$$(\nabla_\perp^2 + k_c^2)\,E_z^+ = 0 \quad \text{or} \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right]E_z^+ = 0 \qquad \text{...(21)}$$

with the boundary condition $E_z^+ = 0$ at $x = 0, x = a$; $y = 0$ and $y = b$.

This solution has already been obtained in section 9.13 for rectangular wave guide and is given by

$$E_z^+(\mathbf{r}, t) = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,e^{ik_g z - i\omega t} \qquad \text{...(22a)}$$

or

$$E_z^+(x, y) = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{...(22b)}$$

with the condition

$$k_0^2 = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 \qquad \text{...(23)}$$

The transverse components E_x^+ and E_y^+ may be obtained from equations [refer equations (17) and (18) of section 9.12]

$$\left.\begin{aligned} E_x^+ &= \frac{ik_g}{k_c^2}\frac{\partial E_z^+}{\partial x} && \text{...(a)}\\ E_y^+ &= \frac{ik_g}{k_c^2}\frac{\partial E_z^+}{\partial y} && \text{...(b)}\\ \text{where}\quad k_c^2 &= k_g^2 - k_g^2 && \text{...(c)} \end{aligned}\right\} \quad \text{...(24)}$$

Substituting value of E_z^+ from (22a) in above equations, we get

$$\left.\begin{aligned} E_x^+(\mathbf{r},t) &= \frac{ik_g}{k_c^2}\left[\frac{m\pi}{a}\right] E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, e^{ik_g z - i\omega t} && \text{...(a)}\\ E_y^+(\mathbf{r},t) &= \frac{ik_g}{k_c^2}\left[\frac{n\pi}{b}\right] E_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, e^{ik_g z - i\omega t} && \text{...(b)} \end{aligned}\right\} \quad \text{...(25)}$$

If $\mathbf{E}_\perp^+$ is the transverse component of electric field propagating along positive z-direction, then

$$\mathbf{E}_\perp^+(\mathbf{r},t) = i\,\mathbf{E}_z(\mathbf{r},t) + \mathbf{j}\mathbf{E}_y^+(\mathbf{r},t)$$

and it may be expressed as

$$\mathbf{E}_\perp^+(\mathbf{r},t) = \mathbf{E}_\perp^+(x,y)\; e^{ik_g z - i\omega t}$$

If $\mathbf{E}_\perp^-$ is the transverse component of electric field in *TM* wave propagating along negative z-axis, then

$$\mathbf{E}_\perp^-(\mathbf{r},t) = \mathbf{E}_\perp^-(x,y)\; e^{-ik_g z - i\omega t}$$

∴ Resultant *transverse electric field* is given by

$$\begin{aligned} \mathbf{E}_\perp(\mathbf{r},t) &= \mathbf{E}_\perp^+(\mathbf{r},t) + \mathbf{E}_\perp^-(\mathbf{r},t)\\ &= \mathbf{E}_\perp^+(x,y)\, e^{ik_g z - i\omega t} + \mathbf{E}_\perp^-(x,y)\, e^{-ik_g z - i\omega t} \end{aligned} \qquad \text{...(26)}$$

The boundary condition that the tangential component of electric field vanishes at $z = 0$ implies that $\mathbf{E}_\perp(\mathbf{r},t) = 0$ at $z = 0$ or $\mathbf{E}_\perp^+(x,y) = -\mathbf{E}_\perp^-(x,y)$ for all values of z and t.

Using this substitution, equation (26) becomes

$$\mathbf{E}_\perp(\mathbf{r},t) = 2i\,\mathbf{E}_\perp^+(x,y)\sin k_g z\, e^{-i\omega t} \qquad \text{...(27)}$$

The boundary condition that the tangential component of electric field vanishes at $z = d$ implies that

$$\sin k_g d = 0 \quad \text{or} \quad k_g d = p\pi \quad \text{or} \quad k_g = \frac{p\pi}{d} \qquad \text{...(28)}$$

where p is an integer. With this subtitution equation (27) takes the form

$$\mathbf{E}_\perp(\mathbf{r},t) = 2i\,\mathbf{E}_\perp^+(x,y)\sin\frac{p\pi z}{d}\, e^{-i\omega t} \qquad \text{...(29)}$$

In terms of components equation (29) may be expressed as

$$\left.\begin{aligned} E_x(\mathbf{r},t) &= 2i\,E_x^+(x,y)\sin\frac{p\pi z}{d}\, e^{-i\omega t} && \text{...(a)}\\ E_y(\mathbf{r},t) &= 2i\,E_y^+(x,y)\sin\frac{p\pi z}{d}\, e^{-i\omega t} && \text{...(b)} \end{aligned}\right\} \quad \text{...(29)}$$

Substituting values from (25) we get

$$\left.\begin{aligned} E_x(\mathbf{r},t) &= -\frac{2k_z}{k_c^2}\left(\frac{m\pi}{a}\right)E_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(a)\\ E_y(\mathbf{r},t) &= -\frac{2k_z}{k_c^2}\left(\frac{n\pi}{a}\right)E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(b) \end{aligned}\right\} \quad \ldots(30)$$

We already have

$$B_z(\mathbf{r},t) = 0. \quad \ldots(31)$$

The components B_x, B_y and B_z may be obtained by using Maxwell curl equation

$$\nabla \times \mathbf{B} = -\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

which in terms of components can be written as

$$\left.\begin{aligned} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial x} &= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} \quad \ldots(a)\\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= -\frac{1}{c^2}\frac{\partial E_y}{\partial t} \quad \ldots(b)\\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -\frac{1}{c^2}\frac{\partial E_z}{\partial t} \quad \ldots(c) \end{aligned}\right\} \quad \ldots(32)$$

From the form of exponential $e^{-i\omega t}$, it is apparent that $\partial/\partial t \rightarrow -i\omega$ and keeping in mind that $B_z = 0$ for *TM* mode; the equations (32) takes the form

$$\left.\begin{aligned} \frac{\partial B_y}{\partial z} &= -\frac{i\omega}{c^2}E_x \quad \ldots(a)\\ \frac{\partial B_y}{\partial z} &= +\frac{i\omega}{c^2}E_y \quad \ldots(b)\\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \frac{i\omega}{c^2}E_z \quad \ldots(c) \end{aligned}\right\} \quad \ldots(33)$$

Substituting value of E_y from (30b), in (33b), we get

$$\frac{\partial B_x}{\partial z} = +\frac{i\omega}{c^2}\frac{2k_g}{k_c^2}\left\{\frac{n\pi}{b}\right\}E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t}$$

Integrating with respect to z and using equation (28) viz. $k_g = p\pi/d$, we get

$$B_x = +\frac{2i\omega E_0}{k_c^2c^2}\left\{\frac{n\pi}{b}\right\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(34a)$$

Similarly substituting value of E_x from (30a) in (33a), integrating with respect to z and using equation (28), we get

$$B_y = -\frac{2i\omega E_0}{k_c^2c^2}\left\{\frac{m\pi}{a}\right\}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} \quad \ldots(34b)$$

Now using (34a) and (34b), equations (33c) gives

$$E_y = \frac{c^2}{i\omega}\left[\frac{\partial B_y}{i\omega} - \frac{\partial B_x}{\partial y}\right]$$

$$= \frac{c^2}{i\omega}\left[\frac{\partial}{\partial x}\left\{-\frac{2i\omega E_0}{k_c^2 c^2}\left\{\frac{m\pi}{a}\right\}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t}\right\}\right.$$

$$\left. - \frac{\partial}{\partial y}\left\{\frac{2i\omega E_0}{k_c^2 c^2}\left\{\frac{n\pi}{b}\right\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t}\right\}\right]$$

$$E_z = \frac{2E_0}{k_c^2}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t}$$

Using equation (23), we have

$$E_z = 2E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} \qquad \text{...(34c)}$$

Writing $k_g = p\pi/d$ (equation 28), the *field components for TM mode in resonant cavity* may be expressed as

$$\left.\begin{aligned}
B_x(\mathbf{r}, t) &= \frac{2i\omega}{k_c^2 c^2}\left(\frac{n\pi}{b}\right)E_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(a)}\\
B_y(\mathbf{r}, t) &= -\frac{2i\omega}{k_c^2 c^2}\left(\frac{m\pi}{a}\right)E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(b)}\\
B_z(\mathbf{r}, t) &= 0 && \text{...(c)}\\
E_x(\mathbf{r}, t) &= -\frac{2}{k_c^2}\left(\frac{mx}{a}\right)\left(\frac{p\pi}{d}\right)E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(d)}\\
E_y(\mathbf{r}, t) &= -\frac{2}{k_c^2}\left(\frac{nx}{b}\right)\left(\frac{p\pi}{d}\right)E_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(e)}\\
E_z(\mathbf{r}, t) &= 2E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\,e^{-i\omega t} && \text{...(f)}
\end{aligned}\right\}$$

The resonant frequency is again calculated by guide equation

$$k_0^2 = k_c^2 + k_g^2$$

i.e.

$$\frac{\omega^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \qquad \text{...(35)}$$

or *resonant angular frequency*

$$(\omega_r)_{mnp} = \pi c\left\{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\right\}^{1/2} \qquad \text{...(36)}$$

which is same as in case of *TE* mode. From field equations (35) it is apparent that the following mode TM_{mnp} do not exist

$$TM_{000},\ TM_{001},\ TM_{010},\ TM_{101},\ TM_{011}$$

The physically possible lowest mode is of type TM_{110}

If two or more cavity modes have the same resonant frequency, then the system is said to be degenerate and the resonant frequency is called degenerate frequency. The order of the degeneracy is equal to the number of modes beloging to the degenerate frequency.

The cavity resonators display the properties typical of *resonant circuits* in that they can store energy in oscillating electric and magnetic fields. Furthermore practical cavity resonators dissipate a fraction of the stored energy in each cycle of oscillation. The fraction of energy dissipated in a cavity resonator is about $\frac{1}{20}$ th of the fraction dissipated per cycle in an *L—C* circuit : that is why the cavity resonators are usually superior to conventional LC circuits by a factor of about twenty. The additional advantage of cavity resonators is that they have resonant frequencies with range upward from a few hundred mega cycles-just the region where it is impossible to construct a L—C circuit.

QUESTIONS AND PROBLEMS

1. (a) Obtain the boundary conditions satisfied by the electromagnetic field vectors **E**, **D**, **B** and **H** on the plane interface between two media. **(Meerut 1994, 76)**

   (b) A plane wave travelling in a homogeneous isotropic dielectric is incident obliquely at the plane surface of another dielectric, Determine the reflection and transmission coefficient if the incident wave is polarised with its electric vector.

   (i) perpendicular to the plane of incidence (ii) in the plane of incidence. **(Meerut 1971)**

2. A plane electromagnetic wave is incident on a plane boundary between two non-conducting media. Specify the boundary conditions to be satisfied by the wave and hence derive the Fresnel formula for the reflected and transmitted intensities. Discuss important consequences of these formulae.

3. (a) Derive Fresnel's equaltons for reflection and refraction of electro-magnetic waves at a plane boundary separating two non-conducting isotropic linear media. **(Meerut 1998, Rohilkhand 1997, 96, 91; 87)**

   (b) Discuss metallic reflection and refraction. **(Rohilkhand 1987, 84)**

   (c) Find out an expression for the reflection power of a metallic surface. **(Rohilkhand 1997, 82; Agra 1977, 72)**

4. Derive Fresnel's equations for reflection of light from a dielectric surface. How are the relations verified experimentally ? **(Agra 1970)**

5. What are Fresnel's equations for reflection of plane polarized light from transparent media ? Indicate how they are derived and how they can be experimentally verified. **(Agra 1936)**

6. Derive Fresnel's equations for reflection and refraction of electromagnetic wave at a plane boundary separating two media when the incident wave is polarised with its E vector parallel to the plane of incidence. Hence show that there is an angle of incidence for which there is no reflected wave. Name this angle. **(Garhwal 1993)**

7. Plane electromagnetic waves are incident obliquely at the plane interface between two dielectric media with electric vector lying in the plane of incidence. Obtain the amplitude of electric vectors in the reflected and transmitted waves as fractions of that in the incident wave. What is the reflection coefficient ? Show that there is an angle of incidence for which there is no reflected wave. Find this angle ? What is this angle called ? **(Rohilkhand 1988, 86, 80)**

8. Discuss the reflection and refraction of electromagentic waves at the interface of two non-conducting media and derive the Fresnel equations. **(Meerut 1992)**

9. Discuss the reflection and refraction of electromagnetic waves at a plane interface between dielectrics. Under what condition will the incident wave be totally internally reflected ? **(Meerut 1997)**

10. Show on the basis of electromagnetic theory, that all good codnuctors are good absorbers and good reflectors. **(Garhwal 1993)**

11. Obtain the necessary relation between the incident and reflected electromagnetic field amplitudes for reflection at a plane metal surface and show how to determine the complex refractive index optically in terms of suitable parameters **(Meerut 1987)**

12. (a) $A_{\parallel}$ and $A_{\perp}$ are parallel and perpendicular components of plane electromagnetic wave incident at an angle θ, obtain expression for the corresponding components in the reflected beam if the second medium is denser (refractive index n).

    (b) Show that if first medium is denser in (a), there is a difference of phase between the reflected parallel and perpendicular components given by

$$\tan\frac{\delta}{2} = \frac{\cot\theta\sqrt{[\sin^2\theta - (1/n^2)]}}{\sin^2\theta}$$

    (c) The degree of polarisation is defined as

$$P = \frac{I_{\perp} - I_{\parallel}}{I_{\perp} + I_{\parallel}}$$

where $I_{\perp}$ and $I_{\parallel}$ are perpendicular and parallel components of intensity. Calculate the value of ρ for ordinary light reflected from glass (index 1·5) at an angle of incidence of 45°. **(Meerut 1970)**

13. Deduce Brewster's law on the basis of electromagnetic theory and explain how it can be verified experimently. What should be the altitude (angle above horizon) of the sun so that sunlight reflected from a still lake is completely plane polarised. (UPCS 1997)

[Hint. Brewster's law $n = \tan\theta_p$; $\theta_p = \tan^{-1}(n) = \tan^{-1}(4/3 = 53^\circ)$ and altitude $\theta = 90^\circ - \theta_p = 37^\circ$].

14. Derive Fresnel's equations for reflection and refraction of electromagnetic waves at a plane boundary separating two dielectric media. Discuss the phenomenon of total internal reflection.

15. Discuss the phenomenon of total internal reflection on the basis of electromagnetic theory of light. (Rohilkhand 1997, 96)

16. Establish the Maxwell's equations for the electromagnetic field. Discuss the phenomenon of total internal reflection on the basis of these equations. (Meerut 1980)

17. What happens when a plane wave is incident on a plane surface with boundaries having different electric and magnetic properties ? Discuss the condition of the incident angle when the ray suffers maximum degree of polarisation after reflection. (Agra 1997, 89)

18. (a) Establish Maxwell's equations for the electromagnetic field. (Agra 1974)

(b) Discuss the phenomenon of reflection from a conducting medium on the basis of these equations.

19. Electromagnetic waves are incident normally on a metal surface of refractive incing n and extenction coefficient k. Show that the ratio of the reflected intensity to incident intensity is, $R = \frac{n^2 + k^2 - 2n + 1}{n^2 + k^2 + 2n + 1}$ (Rohilkhand 1986; Agra 1995)

[Hint. put $\frac{n_2}{n_1} = n_2 = n + ik$ in section 9.9]

20. What are peculiarities of metallic reflection ? Give its theory in brief and describe method for its verification. (Rohilkhand 1993)

21. What is a wave guide ? Describe the propagation of electromagnetic wave along a hollow wave guide of uniform cross-section. Explain how the cut off modes arise. Show that the phase velocity of waves becomes infinite exactly at the cutoff. (Agra 1938)

22. What is a wave guide ? For transverse electric waves perfectly propagating in a rectangular wave guide with perfectly conducting walls; find (Meerut 1989, 87)

(i) the cut off wavelength.

(ii) the magnetic field induction.

(iii) The velocity with which energy is transmitted along the guide. (Meerut 1989, 74)

23. What is a resonant cavity ? Discuss its superiority over reasonant circuit. For *TE* wave in the cavity with perfectly conducting walls, derive expression for (i) cut off wavelength, (ii) resonant frequency, (iii) electric and magnetic fields.

24. What is an electromagnetic cavity resonator ? Describe the excitation of T.E.M. waves in the class of cavity resonators produced by placing plane end faces on a length of a cylindrical wave guide. (Meerut 1983)

25. What are cavity resonators ? How do they resonate ? How they resemble with electronic resonant circuits ? Discuss in detail the theory of a rectangular cavity resonator, showing what lowest modes are possible in TE mode. (Meerut 1991, 81)

26. How is the polarisation of an electromagnetic wave affected when it crosses the plane interface between two dielectrices ? Explain the significance of Brewster's angle. What is the total internal reflection and why do we get it ? (Kanpur 1983; Rohilkhand 1985, 81)

27. Write short notes on

(i) Total internal reflection (Meerut 98)

(ii) Brewster's law (Meerut 1997)

(iii) Fresnel equations. (Rohilkhand 1999)

7

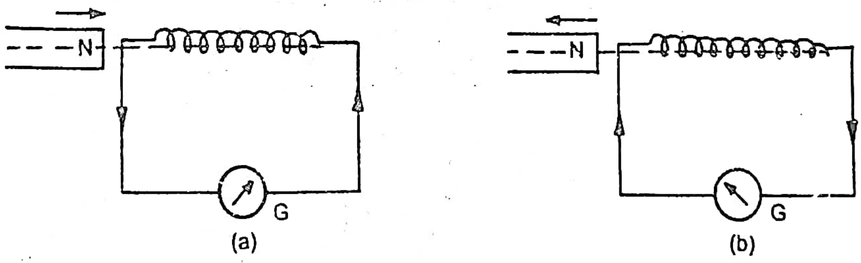
Electromagnetic Induction

7.1. Introduction

In this chapter we shall discuss the behaviour of time-dependent electric and magnetic fields. This forms the basis of Maxwell's equations of electro-magnetic field.

Oersted's discovery of the magnetic field associated with an electric current suggested to several investigators that the magnetism in some way may give rise to electricity. The method by which this could be done was discovered by Faraday and Henry independently in the early nineteenth century. Faraday's priority was admitted even by Henry, Let us now consider the following experiments in which magnetism gives rise to electricity :

1. Let there be a magnet near a coil connected to a galvanometer as shown in fig. 7.1. When the magnet is quickly brought towards the coil [fig. 7.1 (a)] with its north pole facing the coil, the galvanometer shows the deflection while magnet is in motion,



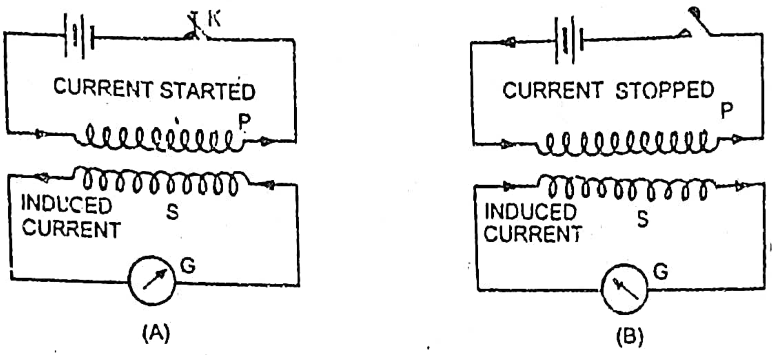
(a) (b)

Fig. 7.1

The indicates a momentary current in the coil. When the magnet is moved away from the coil [7.1 (b)], the galvanometer again shows a deflection, but in opposite direction, which means that now the current in the galvanometer is in opposite direction. If the experiment is repeated with south pole of the magnet facing the coil, the deflections in the galvanometer are reversed. It is further observed that the deflection increases with increase in the velocity of the magnet relative to the coil.

The same results are obtained if the magnet is kept fixed and coil moved.

2. Consider a primary coil P connected to a battery and a secondary coil S connected to a galvanometer as shown in fig. 7.2. When a current is started in the primary coil by closing key K [fig. 7.2 (a)], the galvanometer gives a momentary deflection, showing an induced current in the secondary coil in a direction opposite to that in a primary. When the current in the primary (or coil) is flowing steadily, then galvanometer gives no deflection. Also when the primary circuit is switched off by opening key K [fig. 7.2 (b)], there is a again a momentary deflection in the galvanometer, but in opposite direction ; this indicates an induced current in the secondary in the same direction as in the primary. Thus the current in the secondary



(A) (B)

Fig. 7.2

coil is induced when the current in the primary coil is changed. Similar effects are also observed when the relative positions of the coils are changed. The strength of the induced current is much more increased when the coils are wound on a piece of iron.

The above experiments definitely indicate that whenever there is a change in the magnetic flux linked with a closed circuit, an emf and current are induced in the circuit and last only so long as the change in the flux lasts. This phenomenon is called *electromagnetic induction.*

The magnetic flux through a circuit may be changed in the following ways :

(i) By relative motion between a magnet and the circuit,

(ii) By changing current in the same circuit,

(iii) By changing current in the neighbouring circuit,

(iv) By rotating a coil in a magnetic field.

7.2. Faraday's Laws of Electromagnetic Induction

The experimental observations of electromagnetic induction have been summed up in the form of two laws inunciated below which are known as Faraday's laws of electromagnetic induction.

1. *Whenever the magnetic flux linked with a closed circuit changes, an induced e.m.f. is set up in the circuit whose magnitude at any instant is proportional to the rate of change of magnetic flux linked with the circuit.*

If e is the magnetic flux linked with the circuit at any instant t and e is the induced e.m.f., then

$$e \propto \frac{d\phi}{dt}$$

2. *The direction of induced e.m.f. is such that it opposes the change in flux that produces it.* This law is known as **Lenz's law,** because althrough the direction of induced e.m.f. was determined by Faraday, but it was expressed in a law by Lenz.

These laws can be mathematically expressed as

$$e = -\frac{d\phi}{dt} \qquad ...(1)$$

where e is the e.m.f. induced in the circuit in volts and ϕ is the instantaneous flux linked with the circuit in webers.

Explanation of Faraday's Laws :

To explain Faraday's laws of electromagnetic induction we consider the coil and magnet experiment.

When the north pole of the magnet is placed in near the coil, a definite number of lines of magnetic induction originating from this pole pass through the coil (fig. 7.3) *i.e.* a definite magnetic flux is linked with the coil. If the magnet is moved towards the coil, the flux through the coil increases, while if the magnet is moved away from the coil, the flux through the coil decreases. In either case an e.m.f. is induced in the coil during the motion of the magnet. Greater is the speed of the magnet, greater is the rate of change of flux linked with the coil and greater is the induced e.m.f. Thus induced e.m.f. in a coil at any instant is proportional to the rate of change of magnetic flux linked with the coil.

The direction of induced e.m.f. of current is in accordance to Lenz's law. When the north pole of magnet is moved towards the coil, the induced current flows in a direction so as to oppose the motion of the magnet towards the coil. *This is only possible when nearer face of the coil acts as a magnetic north pole which necessitates an anticlockwise induced current in the coil* (fig. 7.4a). Then the *repulsion* between two similar poles opposes the motion of the magnet towards the coil. Similarly, when the magnet is moved away from the coil, the direction of induced current is such as to make the *nearer face of the coil as a south pole which necessiates a clockwise induced current in the coil.* Then the *attraction* between two opposite poles opposes the motion of the magnet away from the coil. In either case, therefore work has to be done in moving the magnet. It is the mechanical work which appears as electrical energy in the coil. Thus ***the production of induced e.m.f. or induced current in the coil is in accordance with the law of conservation of energy.***

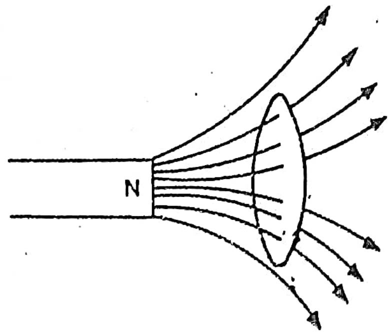
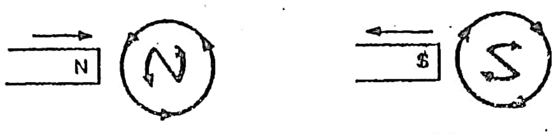


Fig. 7.3



(a) (b)

Fig. 7.4

7.3. Faraday's Laws From Lorentz Force.

Consider a circuit shown in fig. 7.5. *ab* and *cd* are parallel horizontal conducting rails and *pq* is a conducting rod of length *l*, which can slide without friction on the rails. A uniform magnetic field of induction *B* exists at the locality of rails. The field *B* is perpendicular to plane of paper and downward.

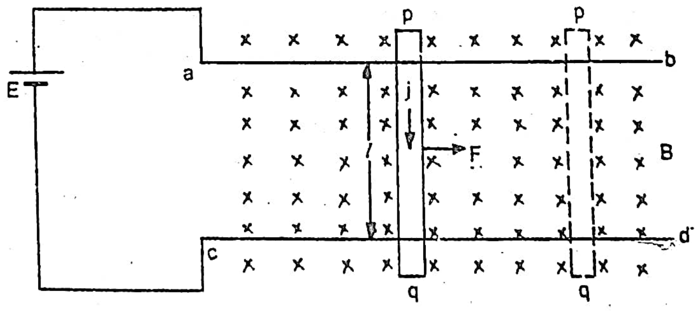


Fig. 7.5

A current *i* amp is allowed to flow throught the rod *pq*.

The force acting on the rod, $\mathbf{F} = i\,\overrightarrow{l} \times \mathbf{B}$.

The magnitude of force is *Bil* and its direction is as shown in fig. 7.5.

Due to this force let dx be the displacement of rod in time dt. The work done for this displacement.

$$\begin{aligned} dW &= F\,dx \\ &= Bil\,dx. \end{aligned}$$

If E is the e.m.f. of battery, the energy given by battery in time dt is $E\,i\,dt$.

This energy is used up in two parts :

(i) A part is used overcoming the resistance R of circuit *i.e.*

$$i^2 R\,dt.$$

(ii) The remaining part is used as work for displacement of rod.

∴ Therefore conservation theorem for energy gives

$$E\,i\,dt = i^2 R\,dt + Bil\,dx$$

i.e.

$$i^2 R\,dt = E\,i\,dt - Bil\,dx.$$

Concelling the common factor i (since $i \neq 0$)

$$i R\,dt = E\,dt - Bl\,dx$$

This gives

$$i = \frac{E - Bl\,\frac{dx}{dt}}{R}. \qquad ...(1)$$

In this equtiort $Bl\,\frac{dx}{dt}$ must have dimensions of e.m.f.

But $l\,dx\ (= dA)$ is the area swept by rod pq is time dt. Therefore

$$Bl\,\frac{dx}{dt} = B.\frac{dA}{dt}.$$

As B is constant, we have

$$Bl\,\frac{dx}{dt} = \frac{d}{dt}(BA) = \frac{d\phi}{dt}.$$

Where $d\phi = d(BA)$ is the change in magnetic flux in time dt. Therefore

$$\frac{d\phi}{dt} = Bl\,\frac{dx}{dt}$$

gives the rate of change of flux.

From equation (1), the e.m.f. induced in the circuit due to rate of change of flux,

$$\begin{aligned} e = -\frac{d\phi}{dt} &= -Bl\,\frac{dx}{dt} \\ &= -Blv. \end{aligned}$$

The negative sign shows that the e.m.f. induced in the circuit opposes the applied e.m.f. E. The property of induced e.m.f. is same as the force of friction in mechanical motion.

7.4. A Conducting Rod Moving Through a Uniform Magnetic Field :

Let ab be a thin conducting rod of length l moving with velocity $\mathbf{v}$ parallel to the direction of Y-axis in a uniform field of magnetic induction $\mathbf{B}$. The length of the rod is along X-axis and the magentic field is along Z-axis (fig. 7.6). Since the rod is conducting, it contains charged particles. The force acting on any charge q of the conducting rod is given by

$$\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B}) \qquad ...(1)$$

The direction of this force **F** is perpendicular to both the directions of velocity **v** and magnetic induction **B**. If the charge q is positive, the direction of the force is along positive X-axis and if the charge q is negative, the direction of the force is along negative X-axis.

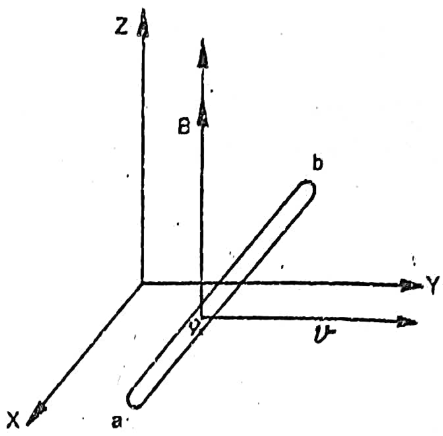


Fig. 7.6

In the *steady state* when the conductor moves with a constant velocity **v**, there must be developed an equal and opposite force at every point inside the rod to balance the magnetic force given by equation (1). The internal force to balance the magnetic force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ can only arise from an electric field developed inside the rod. The magnetic force **F** pushes the negatively charged electrons towards one end of the rod and the other end of the rod becomes positively charged. Thus process of separation of positive and negative charges continues until the accumulation of excess charges at the ends of the conducting rod established an electric field E such that the resultant force on every charge carrier is zero *i.e.*

$$q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) = 0$$

i.e.
$$q\mathbf{E} = -q(\mathbf{v} \times \mathbf{B})$$

i.e.
$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}. \qquad ...(2)$$

Notes. 1. The e.m.f. induced between the two ends of the rod is

$$e = \int_0^l \mathbf{E} \cdot d\mathbf{l} = \left| -\int_0^l (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \right| = vBl. \qquad ...(3)$$

2. If the observer is the frame of the conducting rod, then there will be no electric field inside the rod and there will be only uniform magnetic field ; because the charges are at rest in the frame of conducting rod, thus no magnetic force will act due to the magnetic field.

7.5 Induction Law of Moving Circuits :

Consider a circuit of any arbitrary shape moving with instantaneous velocity **v** in uniform magnetic field **B**. Consider an element PQ of length δl In time δt, it takes the position $P'Q'$. The displacement of element in time δt, $= \mathbf{v}\,\delta t$.

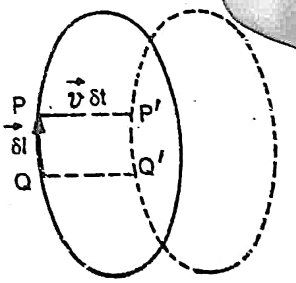


Fig. 7.7

If the velocity of the conducting electrons of element, relative to wire along the length of wire is **u**, then their relative velocity is $\mathbf{v} + \mathbf{u}$.

The magnetic force on each electron of charge $(-e)$

$$\mathbf{F}_m = -e(\mathbf{v} + \mathbf{u}) \times \mathbf{B}$$

The component of this force along the element δl is $\mathbf{F}_{11} = [-e(\mathbf{v} + \mathbf{u}) \times \mathbf{B}] \cdot \hat{n}_l$ where $\hat{n}_l$ is unit vector along $\overrightarrow{PQ}$. As **u** and n_l are along same direction

$$\because \quad \mathbf{u} \times \mathbf{B} \cdot \hat{n}_l = \mathbf{B} \cdot \hat{n}_l \times \mathbf{u} = 0$$

$$\therefore \quad \mathbf{F}_{11} = -e\mathbf{v} \times \mathbf{B} \cdot \hat{n}_l \qquad ...(1)$$

This equation shows that there is an electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ induced in the wire. Its component along the wire is $\mathbf{v} \times \mathbf{B} \cdot \hat{n}_l$.

The induced e.m.f. is the line integral of this field along the boundary of the circuit *i.e.* induced e.m.f.

$$e = \oint (\mathbf{v} \times \mathbf{B}) \cdot \hat{n}_l \, dl \qquad ...(2)$$

But $$\mathbf{v}\,dt \times \hat{n}_l\,dl = \text{area } PP'Q'Q$$

swept by element dl in time dt and hence $(\mathbf{v}\,dt \times n_l\,dl) \cdot \mathbf{B}$ represents the flux change across the element in time dt.

$\therefore$ The flux change across entire circuit

$$d\phi = \oint (\mathbf{v}\,dt \times \hat{n}_l\,dl) \cdot \mathbf{B}$$

$$\therefore \quad \frac{d\phi}{dt} = \oint (\mathbf{v} \times \hat{n}_l\,dl) \cdot \mathbf{B} = \oint \mathbf{B} \cdot (\mathbf{v} \times \hat{n}_l)\,dl$$

$$= -\oint (\mathbf{v} \times \mathbf{B}) \cdot \hat{n}_l\,dl \qquad \text{...(3)}$$

Comparing (2) and (3), we get

$$e = -\frac{d\phi}{dt}. \qquad \text{...(4)}$$

This is Faraday's law. Hence the Feraday's laws holds for circuits moving in uniform magnetic field **B**.

Solved Examples

Ex. 1. *A vertical copper disc of diameter* 20 cm, *makes* 10 *revolutions per second about a horizontal axis passing through its centre. A uniform magnetic field of* 100 *gauss acts perpendicular to the plane of the disc. Calculate the potential difference between its centre and rim in volts.* **(Rohil. 1996; Agra 98)**

Sol. The magnetic flux linked with the disc is given by

$$\phi = BA.$$

The induced e.m.f. *i.e.* potential difference between the rim and centre is given by

$$e = -\frac{d\phi}{dt} = -\frac{d}{dt}(BA) = B\frac{dA}{dt} \text{ (numerically).}$$

Given $B = 100 \text{ gauss} = 100 \times 10^{-4} \text{ weber/m}^2 = 10^{-2} \text{ weber/m}^2$,

$$r = 10 \text{ cm.} = 0{\cdot}1 \text{ meter}$$

$$\frac{dA}{dt} = \text{Area swept out by disc in unit time.}$$

$$= \pi r^2 \times \text{number of revolutions per second}$$

$$= 3{\cdot}14 \times ({\cdot}1)^2 \times 10 = 0{\cdot}314 \text{ m}^2/\text{s}$$

$$\therefore \quad e = B\frac{dA}{dt} = (10^{-2} \text{ weber/m}^2) \times (0{\cdot}314 \text{ m}^2/\text{sec})$$

$$= 3{\cdot}14 \times 10^{-3} = \mathbf{3{\cdot}14 \text{ milli volts.}}$$

Ex. 2. *The rails of a railway track are* 1·5 *meter apart and assumed to be insulated from one another. Calculate the e.m.f. in volts that will exist between the rails if a train is passing at* 100 *km/hour. Assume that horizontal component of earth's magnetic field is* 0·36 *oersted and* tan θ = 1·036 *where* θ *is the angle of dip.*

Sol. If V and H are the vertical and horizontal components of earth's magnetic field respectively, then

$$V = H \tan\theta = (0{\cdot}36 \times 10^{-4} \text{ weber/m}^2) \times (1{\cdot}036)$$

Velocity of the train

$$v = \frac{\text{distance}}{\text{time}} = \frac{100 \times 10^3 \text{ m}}{60 \times 10 \text{ sec.}} = \frac{1000}{36} \text{ m/sec.}$$

Distance between the rails, $d = 1{\cdot}5$ m.

The area swept out by the train in unit time $= \frac{dA}{dt} = d\,.\,v$

$$= (1{\cdot}5\text{ m}) \times \left(\frac{1000}{36}\text{ m/sec.}\right) = \frac{1000 \times 1{\cdot}5}{36}\text{ m}^2\text{/sec.}$$

∴ The induced e.m.f. $e = -\dfrac{d\phi}{dt} = V\dfrac{dA}{dt}$ (numerically)

$$= (0{\cdot}36 \times 10^{-4} \times 1{\cdot}036\text{ weber/m}^2) \left(\frac{1000 \times 1{\cdot}5}{36}\text{ m}^2\text{/sec.}\right)$$

$$= 1{\cdot}55 \times 10^{-3}\text{ volts} = \mathbf{1{\cdot}554\ milli\text{-}volts.}$$

Ex. 3. *A long solenoid diameter 5 cm. has 100 closely and evenly wound turns per cm. of its length. At the centre of the solenoid a small coil of 50 turns and 2 cm. diameter is placed such that the magnetic field produced by the solenoid is parallel to the axis of the coil. The current in the solenoid is changed from 2 amp. to zero and then raised to 2 amp in the opposite direction at a steady rate over a period of 0·1 second, calculate the e.m.f. induced in the coil during this change.* **(Kanpur 1972)**

Sol. The magnetic induction B at the centre of the solenoid,

$$B = \mu_0 nI\text{ weber/m}^2$$

Here $\mu_0 = 4\pi \times 10^{-7}$ weber/amp–m,

$$n = 100\text{ turns/cm} = 10^4\text{ turns/m and } I = 2\text{ amp.}$$

$$B = (4\pi \times 10^{-7}\text{ weber/amp–m}) \times (10^4\text{ turns/m}) \times (2\text{ amp})$$

$$= 2{\cdot}512 \times 10^{-2}\text{ weber/m}^2.$$

Area of the central coil = number of turns × area of each turn

$$= (n\,.\,\pi r^2)$$

$$= 50 \times 3{\cdot}14 \times \left(\frac{1}{100}\text{ m}\right)^2 = 1{\cdot}57 \times 10^{-2}\text{ m}^2$$

The initial flux through the coil,

$$\phi = \mathbf{B} \cdot \mathbf{A} = 2{\cdot}512 \times 10^{-2} \times 1{\cdot}57 \times 10^{-2}\text{ webers}$$

$$= 3{\cdot}94 \times 10^{-4}\text{ webers.}$$

As the current is changed from +2 amp, to −2 amp, the flux will change from $+3{\cdot}94 \times 10^{-4}$ webers to $-3{\cdot}94 \times 10^{-4}$ webers, therefore the change in flux linked with the coil in time interval of 0·1 sec is

$$\delta\phi = 3{\cdot}94 \times 10^{-4} - (-3{\cdot}94 \times 10^{-4}) = 7{\cdot}88 \times 10^{-4}\text{ webers}$$

The magnitude of e.m.f. induced

$$e = \frac{d\phi}{dt} = \frac{7{\cdot}88 \times 10^{-4}}{0{\cdot}1} = 7{\cdot}88 \times 10^{-3}\text{ volts} = \mathbf{7{\cdot}88\ milli\ volts.}$$

Ex. 4. *An all metal aeroplane drives down vertically at 30 km/sec. at a place where the horizontal component of the earth-field is 0·4 oersted. If the wing span is 30 metre, what will be the resulting potential difference between the tips ?*

Sol. The vertical distance covered in one sec

$$= 30\text{ km} = 30 \times 10^3\text{ m} = 3 \times 10^4\text{ m}$$

The distance between the wing tips = 30 m

The area swept by the wing span in one sec.

$$\frac{dA}{dt} = 3 \times 10^4 \times 30\text{ m}^2\text{/sec.}$$

The potential difference induced between the wing tips

$$e = -\frac{d\phi}{dt} = -\frac{d}{dt}(\mathbf{B}\cdot\mathbf{A}) = -\frac{d}{dt}(HA) = H\frac{dA}{dt} \text{ (numerically).}$$

$$= (0{\cdot}4 \times 10^{-4} \text{ weber/m}^2) \times (3 \times 10^4 \times 30 \text{ m}^2/\text{sec})$$

$$= \mathbf{360 \text{ volts.}}$$

Ex. 5. *A conducting rod of length L is rotating with a constant angular velcity ω in a uniform magnetic field of induction* **B**. *What is the e.m.f. developed between the ends of the rod ?*

Sol. Let *OA* be the rod of length *L* rotating with a constant angular velocity ω in a uniform magnetic field of induction **B**. Consider a small element of length *dl* of the rod at a distance *l* from the end *O*. The area swept out by the rod per second by the element of length *dl* of the rod is *v dl* where *v* is the linear velocity of the element of length *dl*.

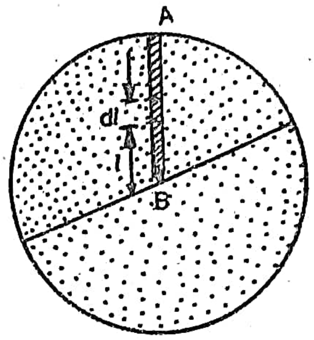


Fig. 7.8

The e.m.f. developed between the ends of the element *dl* is,

$$de = B\frac{dA}{dl} = B\,v\,dl$$

$$= B\omega l\,dl \text{ (since } v = \omega l)$$

Therefore the e.m.f. developed between the ends of the rod *OA* of length *L* is

$$e = \int_0^L B\omega l\,dl = \omega B\int_0^L l dl$$

$$= \frac{1}{2}\omega B L^2 .$$

Ex. 6. *A wire of length* 2 *metre perpendicular to X-Y plane is moved with velocity* $\mathbf{v} = (2\mathbf{i}+3\mathbf{j}+\mathbf{k})$ m/s *through a region of uniform induction* $\mathbf{B} = (\mathbf{i}+2\mathbf{j})$ *weber/m*^{2}. *Calculate the potential difference between the ends of the wire.*

Sol. The electric field strength inside the wire is given by

$$\mathbf{E} = -(\mathbf{v}\times\mathbf{B}) = -(2\mathbf{i}+3\mathbf{j}+\mathbf{k})\times(\mathbf{i}+2\mathbf{j})$$

$$= -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{vmatrix} = (2\mathbf{i}-\mathbf{j}-\mathbf{k}) \text{ volt/m.}$$

$$e = -(\mathbf{v}\times\mathbf{B})\cdot\mathbf{l}$$

$$= (2\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot(2\mathbf{k}) = -2 \text{ volt.} = 2 \text{ volt (numerically)}$$

Ex. 7. *A wire of length* 1 *m moves at right angles to its length at a speed of* 100 m/s *in uniform magnetic field* 1 *wb/m*^{2} *which is also acting at right angles to the length of the wire. Calculate the e.m.f. induced in the wire when the direction of motion is*

(i) at right angles to the field.

(ii) inclined at 30° *to the direction of field.* **(Rohilkhand 1986)**

Sol. E.M.F. induced $e = (\mathbf{v}\times\mathbf{B})\cdot\mathbf{l}$

(i) $|\mathbf{v}\times\mathbf{B}| = vB\sin 90 = vB$

and direction of $\mathbf{v}\times\mathbf{B}$ is normal to **v** and **B** *i.e.* collinear with $\mathbf{l}$

$$\therefore\ e = vBl = 100\times 1\times 1 = \mathbf{100 \text{ volt.}}$$

(ii) $|\mathbf{v}\times\mathbf{B}| = vB\sin 30 = \dfrac{vB}{2}$

and direction of $\mathbf{v}\times\mathbf{B}$ is normal to **v** and **B** *i.e.* collinear with $\mathbf{l}$

$$\therefore\quad e = \frac{vB}{2}\cdot l = \frac{100\times 1\times 1}{2} = \mathbf{50 \text{ volts.}}$$

Ex. 8. *A conducting rod AB (fig. 7.9) makes contact with the metal rails CA and DB. If a uniform magnetic field of induction 0·5 weber/m^2 acts perpendicular to the plane of fig. 7.9 pointing downward; then*

(a) Calculate the magnitude and direction of the e.m.f. induced in the rod when it is moving towards the right with the velocity of 4 m/s.

(b) If the resistance of the circuit ABCD is 0·2 ohm (assumed constant), calculate the force required to maintain the motion neglecting friction.

(c) Compare the rate at which mechanical work is done by the force with the rate of heat developed in the circuit.

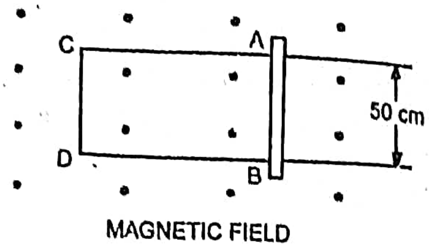


Fig. 7.9

Sol. (a) If a conductor of length l moves with velocity $\mathbf{v}$ in a region of uniform magnetic induction $\mathbf{B}$, the induced e.m.f.

$$e = \mathbf{l} \cdot (\mathbf{v} \times \mathbf{B}).$$

Taking positive X-axis along right, positive Y-axis along upward (*i.e.* from B to A) and positive Z-axis vertically upward, we have

$$\mathbf{l} = 50\,\mathbf{j} \text{ cm.} = 0{\cdot}5\,\mathbf{j} \text{ m}, \quad \mathbf{v} = 4\mathbf{i} \text{ m/s}$$

and $$\mathbf{B} = -\ 0{\cdot}5\,\mathbf{k} \text{ weber/m}^2$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vector along X, Y and Z axes.

$$\therefore \quad e = 0{\cdot}5\,\mathbf{j} \cdot [4\mathbf{i} \times (-0{\cdot}5\,\mathbf{k})] = \mathbf{1 \text{ volt.}}$$

The direction of this induced e.m.f. is from B to A.

(b) Resistance of the circuit = 0·2 ohm

$$\text{Induced current, } I = \frac{V}{R} = \frac{1}{0{\cdot}2} = 5 \text{ amp.}$$

The force required to maintain the motion

$$= \text{side thrust on the moving rod due to current } I$$
$$= IlB = (5 \text{ amp.}) \times (0{\cdot}5 \text{ m}) \times (0{\cdot}5 \text{ weber/m}^2)$$
$$= 1{\cdot}25 \text{ weber amp./m} = \mathbf{1{\cdot}25 \text{ Newton.}}$$

(c) Mechanical work done by force per sec

$$P = Fv = 1{\cdot}25 \times 4 = 5 \text{ watt}$$

Rate of heat development

$$= I^2R = 5^2 \times {\cdot}2 = 5 \text{ watt}.$$

That is whole mechanical work done appears as heat.

Ex. 9. *Two long parallel horizontal rails, a distance d apart and each having a resistance λ per unit length, are joined at one end by a resistance R. A perfectly conducting rod MN of mass m is free to slide along the rails without friction (Fig. 7.10). There is a uniform magnetic field of induction B, normal to plane of paper and directed into the paper. A variable force F is applied to rod MN, such that as the rod moves, a constant current i flows through R.*

(i) Find the velocity of the rod and the applied force F as a function of distance x of the rod from R.

(ii) What fraction of work done per second by F is converted into heat ?

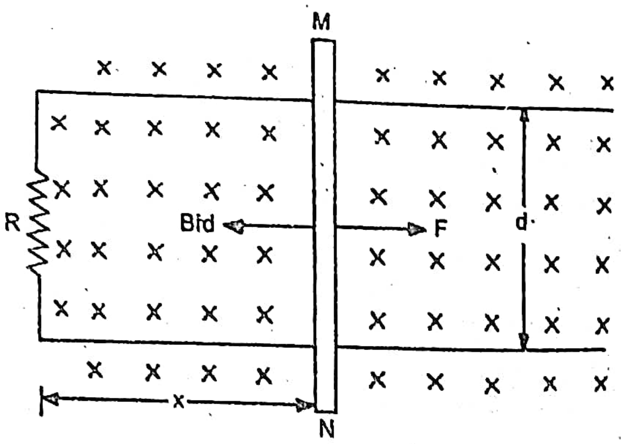


Fig. 7.10

Solution. (i) If rod has instantaneous velocity v at a distance x from R, the induced e.m.f. $= Bvd$.
Instantaneous resistance of current $= R + 2\lambda x$.

$$\therefore \quad \text{Induced current} \quad i = \frac{e}{R+2\lambda x} = \frac{Bvd}{R+2\lambda x} = \text{constant}$$

$$\therefore \quad \text{velocity } v = \frac{(R+2\lambda x)\, i}{Bd}$$

If F is the force acting on rod–

$$F - Bid = m\frac{dv}{dt}$$

or

$$F - Bid = m\frac{dv}{dt}\cdot\frac{dx}{dt}$$

$$= mv\frac{d}{dx}\left\{\frac{(R+2\lambda x)\, i}{Bd}\right\}$$

$$= mv\frac{2\lambda i}{Bd}$$

$$\therefore \quad F = Bid + m\frac{(R+2\lambda x)\, i}{Bd}\cdot\frac{2\lambda i}{Bd}$$

$$= Bid + \frac{2m\lambda (R+2\lambda x)}{B^2 d^2}\, i^2$$

(ii) Work done per second $= Fv$.

Heat produced per second $= i^2 (R + 2\lambda x)$

$$\text{Required ratio} = \frac{i^2 (R+2\lambda x)}{F.v}$$

$$= \frac{i^2 (R+2\lambda x)\, Bd}{F\,(R+2\lambda x)\, i} = \frac{iBd}{F}$$

$$= \frac{Bid}{Bid + \dfrac{2m\lambda (R+2\lambda x)\, i^2}{B^2 d^2}}$$

$$= \frac{1}{1 + \dfrac{2m\lambda (R+2\lambda x)\, i^2}{B^2 d^2\, Bid}}$$

$$= \frac{1}{1 + \dfrac{2m\lambda (R+2\lambda x)\, i}{B^3 d^3}}$$

Ex. 10. *Space is divided by the line AD into two regions. Region I is field free and region II has a uniform magnetic field B directed into the plane of the paper. ACD is a semicircular conducting loop of radius r with centre at O, the plane of the loop being in the plane of the paper. The loop in now made to rotate with a constant angular velocity ω about an axis passing through O and perpendicular to the plane of paper. The effective resistance of the loop is R.*

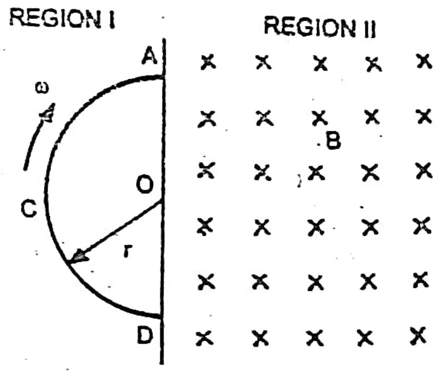


Fig. 7.11

(i) Obtain an expression for the magnitude of induced current in the loop.

(ii) Show the direction of current when the loop is entering into region II.

(iii) Plot a graph between the induced e.m.f. and the time of rotation for two periods of rotation.

Solution. (i) When the loop is rotated about an axis passing through centre O and perpendicular to the plane of the paper, the angle between magnetic field vector $\vec{B}$ and area $\vec{A}$ is always 0°. When the loop is in region *I*, the magnetic flux linked with loop $= BA \cos 0 = 0$.

(since $B = 0$ in region *I*)

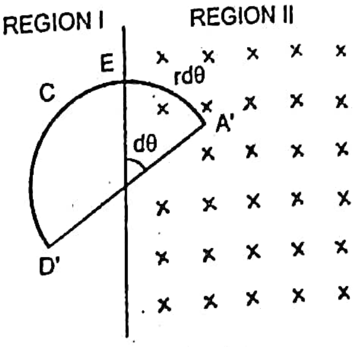


Fig. 7.12 (a)

When the loop enters the magnetic field in region II, the magnetic flux is linked with it, given by $\phi = BA$.

$$\therefore \qquad \text{e.m.f. induced, } e = -\frac{d\phi}{dt} = -\frac{d}{dt}(BA) = -B\frac{dA}{dt}$$

$$= B\frac{dA}{dt} \text{ (numerically).}$$

If $d\theta$ is the angle rotated by the loop in time dt, then

$$dA = \text{Area of triangle } OEA'$$

$$= \frac{1}{2} r (r d\theta) = \frac{1}{2} r^2 d\theta$$

$$\therefore \qquad e = B \cdot \frac{1}{2} r^2 \frac{d\theta}{dt} = B \cdot \frac{1}{2} r^2 \omega = \frac{1}{2} B r^2 \omega$$

$\therefore$ As resistance of loop is R,

$$\therefore \qquad \text{Current induced, } i = \frac{e}{R} = \frac{1}{2}\frac{Br^2 \omega}{R}.$$

This is required expression for current induced in the loop.

(ii) According to Lenz's law the direction of current induced is to oppose the change in magnetic flux. So when entering in II region the field produced by the current induced must be upward. For this the current in the loop must be *anticlockwise*.

(iii) When the loop enters the magnetic field, the magnetic flux linked increases and e.m.f. $e = \frac{1}{2} B r^2 \omega$ is induced in one direction when it comes out of field, the flux decreases and e.m.f. is induced in opposite sense. The graph for representing e.m.f. induced versus time for t two periods ($T = 2\pi/\omega$) is shown in fig. 7.12 (b).

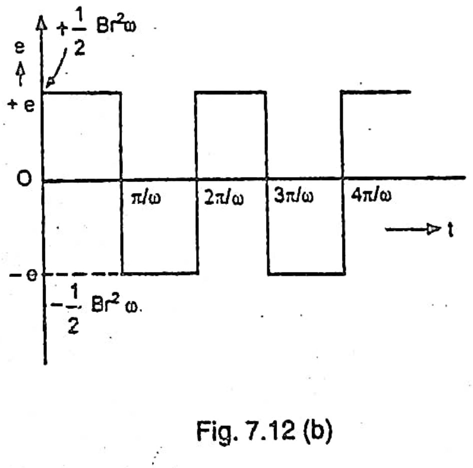


Fig. 7.12 (b)

Ex. 11. *A square metal wire loop of side 10 cm and resistance 1 Ω is moved with a constant velocity v_0, in a uniform magnetic field of induction $B = 2$ weber/m^2 as shown in fig. 7.13. The magnetic field lines are perpendicular to the plane of loop directed into the paper. The loop is connected to a network of resistors each of value 3Ω. The resistance of lead wires OS and PQ are negligible. What should be the speed of the loop*

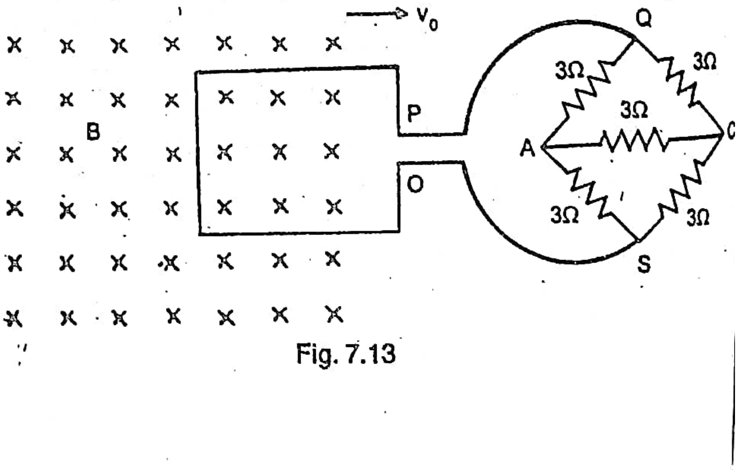


Fig. 7.13

v_0 so as to have a steady current of 1 mA in the loop ? Indicate also the direction of current in the loop.

Solution. The network of resistors is a balanced Wheatstone's bridge. Therefore the resistance AC is ineffective. The equivalent resistance R' of the net work is

$$\frac{1}{R'} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \text{ or } R' = 3\Omega.$$

The resistance of loop is 1 Ω, therefore effective resistance of circuit,

$$R = 3 + 1 = 4\Omega.$$

∴ The e.m.f. induced in loop,

$$e = Bv_0 l.$$

∴ Current in loop,

$$i = \frac{e}{R} = \frac{Bv_0\,l}{R}.$$

∴ Speed of loop,

$$v_0 = \frac{i\,R}{Bl}.$$

Here $i = 1\ mA = 10^{-3}$, $l = 0{\cdot}1$ m, $B = 2$ weber/m^2.

∴ Speed of loop,

$$v_0 = \frac{10^{-3} \times 4}{2 \times 0{\cdot}1} = 2 \times 10^{-2} \text{ m/s} = 2 \text{ cm/sec.}$$

According to Fleming right hand rule, the direction of *induced current is clockwise* from O to P.

Ex. 12. *A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction B, perpendicular to (x, y) plane. At the moment $t = 0$, a connecting bar MN starts sliding translation wise from the apex of parabola with constant acceleration f. Find the e.m.f. induced in the loop thus formed as a function of y.*

Solution. Let co-ordinates of N at any time t be (x, y). Then

$$MN = 2x$$

When bar moves along Y-axis, the emf induced

$$e = -\frac{d\phi}{dt}$$

$$= \left(B\,2x\,\frac{dy}{dt}\right) = B\,2x\,v \qquad ...(1)$$

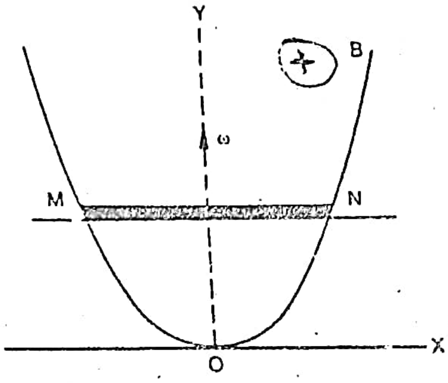


Fig. 7.14

From relation $v^2 = u^2 + 2as$, we have

$$v^2 = 0 + 2fy = 2fy$$

$$v = \sqrt{(2fy)}$$

and from equation of parabola

$$y = ax^2, \qquad x = \sqrt{(y/a)}$$

∴ Equation (1) gives

Induced emf.,

$$e = B.2\sqrt{\frac{y}{a}}\ \sqrt{2fy}$$

$$e = By\sqrt{\frac{8f}{a}}$$

Ex. 13. *Two long parallel wires of zero resistance are connected to each other by a battery of 1·0 V. The separation between the wires is 0·5 m. A metallic bar, which is perpendicular to the wires and of resistance 10 Ω, moves on these wires when a magnetic field of 0·02 Tesla is acting perpendicular to the plane containing the bar and the wires. Find the steady state velocity of the bar, If the mass of the bar is 0·002 kg, find its velocity as a function of time.*

Solution. When battery of emf E is connected, a current flows in the metallic bar due to which a force Bil acts on the bar, resulting the motion of the bar. When the bar moves, the magnetic flux changes and a current is induced in the bar which opposes the motion. At a stage when emf (Bvl) across the bar equals the emf of battery, the bar attains steady velocity given by

$$E = Bvl$$

or

$$v = \frac{E}{Bl} = \frac{1 \cdot 0}{0 \cdot 02 \times 0 \cdot 5}$$

$$= \mathbf{100\ m/s}$$

If v is instantaneous velocity, then net instantaneous current

$$i = \frac{E - Bvl}{R}$$

$\therefore$ Net force

$$F = Bil = B\left(\frac{E - Bvl}{R}\right)l$$

or

$$F = \frac{BEl}{R} - \frac{B^2vl^2}{R}$$

or

$$m \cdot \frac{dv}{dt} = \frac{BEl}{R} - \frac{B^2vl^2}{R}$$

$$\frac{dv}{dt} = \frac{BEl}{mR} - \frac{B^2l^2}{mR}v$$

or

$$\frac{dv}{\frac{BEl}{mR} - \frac{B^2l^2}{mR}v} = dt$$

Integrating

$$\frac{\log_e\left(\frac{BEl}{mR} - \frac{B^2l^2}{mR}v\right)}{-\frac{B^2l^2}{mR}} = t + C$$

C being constant of integration.

At $t = 0$, $v = 0$

$$\therefore \qquad \frac{\log_e\left(\frac{BEl}{mR}\right)}{-\frac{B^2l^2}{mR}} = C$$

$$\therefore \qquad \frac{\log_e\left(\frac{BEl}{mR} - \frac{B^2l^2}{mR}v\right)}{-\frac{B^2l^2}{mR}} = t + \frac{\log_e\left(\frac{BEl}{mR}\right)}{\left(-\frac{B^2l^2}{mR}\right)}$$

or

$$\log_e\left(\frac{BEl}{mR} - \frac{B^2l^2}{mR}v\right) = -\frac{B^2l^2}{mR}t + \log_e\left(\frac{BEl}{mR}\right)$$

or
$$\log_e\left[\frac{\left(\frac{BEl}{mR}-\frac{B^2l^2v}{mR}\right)}{\frac{BEl}{mR}}\right]=-\frac{B^2l^2}{mR}t$$

or
$$\log_e\left(1-\frac{Blv}{E}\right)=-\frac{B^2l^2}{mR}t$$

or
$$1-\frac{Blv}{E}=e^{-B^2l^2\,t/mR}$$

or
$$\frac{Blv}{E}=1-e^{-B^2l^2\,t/mR} \quad \text{or} \quad v=\frac{E}{lB}\left[1-e^{-B^2l^2\,t/mR}\right]$$

Substituting given values
$$v=\frac{1}{0.5\times 0.02}\left[1-e^{(0.02)^2\times(0.5)^2\,t/(0.002)\times 10}\right]$$
$$=100\,[1-e^{-0.005\,t}]$$

This is required expression.

Ex. 14. *Two parallel vertical metallic rails AB and CD are separated by 1m. They are connected at the two ends by resistances R_1 and R_2 as shown in the Figure. A horizontal metallic bar L of mass 0·2 kg slides without friction, vertically down the rails under the action of gravity. There is a uniform horizontal magnetic field of 0·6 T perpendicular to the plane of the rails. It is observed that when the terminal velocity is attained, the powers dissipated in R_1 and R_2 are 0·76 W and 1·2 W respectively. Find the terminal velocity of the bar L and the values of R_1 and R_2.*

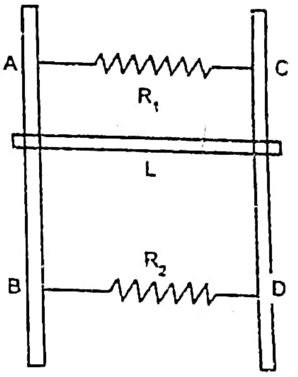


Fig. 7.15 (a)

Solution. Let e be the induced emf and i the induced current in the bar. The forces acting on the bar are

1. Weight mg acting vertically downward.
2. Magnetic force BiL acting vertically upward.

When these forces become numerically equal, the bar is in dynamical equilibrium and attains terminal velocity v (say),

i.e.
$$Mg=BiL$$

Substituting $M=0{\cdot}2$ kg, $L=1$ m, $B=0{\cdot}6$ Tesla, we get
$$0{\cdot}2\times 9{\cdot}8=0{\cdot}6\,i\times 1$$

or
$$i=\frac{0{\cdot}2\times 9{\cdot}8}{0{\cdot}6}=\frac{49}{15}\text{ A} \qquad \text{...(1)}$$

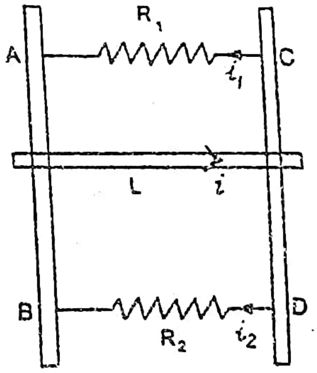


Fig. 7.15 (b)

Let i_1 and i_2 be currents in R_1 and R_2 respectively. Then
$$i=i_1+i_2$$

or
$$i_1+i_2=\frac{49}{15}\text{ A} \qquad \text{...(2)}$$

As R_1 and R_2 are in parallel with the bar

∴
$$e=R_1i_1=R_2i_2 \qquad \text{...(3)}$$

Power in R_1, $P_1=e\,i_1=0{\cdot}76$ watt ...(4)

Power in R_1, $P_2=e\,i_2=1{\cdot}2$ watt ...(5)

Dividing (4) by (5), we get

$$\frac{i_1}{i_2}=\frac{0{\cdot}76}{1{\cdot}2}=\frac{19}{30}$$

$$i_1=\frac{19}{30}i_2 \qquad \text{...(6)}$$

Then equation (2) gives

$$\frac{19}{30}i_2+i_2=\frac{49}{15} \quad \text{or} \quad \frac{49}{30}i_2=\frac{49}{15}$$

or

$$i_2=2A$$

From (6)

$$i_1=\frac{19}{30}\times 2A=\frac{19}{15}A$$

Now from (4)

$$ei_1=0{\cdot}76$$

or

$$e=\frac{0{\cdot}76}{i_1}=\frac{0{\cdot}76\times 15}{19}=0{\cdot}6\text{ V}$$

Induced emf $e = BvL$

Terminal velocity $v=\frac{e}{BL}=\frac{0{\cdot}6}{0{\cdot}6\times 1}=1\text{ m/s}$

Again from (3)

$$R_1=\frac{e}{i_1}=\frac{0{\cdot}6\times 15}{19}=\frac{9}{19}\,\Omega=0{\cdot}47\,\Omega$$

$$R_2=\frac{e}{i_2}=\frac{0{\cdot}6}{2}=0{\cdot}3\,\Omega$$

Ex. 15. *A rectangular frame ABCD made of a uniform metal wire has astraight connection between E and F made of the same wire, as shown in the figure 7.16. AEFD is a square of side 1 m and EB = FC = 0·5 m. The entire circuit is placed in a steadily increasing, uniform magnetic field directed into the plane of the paper and normal to it. The rate of change of the magnetic field is 1 T/s. The resistance per unit length of the wire is 1 Ω/m. Find the magnitudes and directions of the currents in the segments AE, BE and EF.*

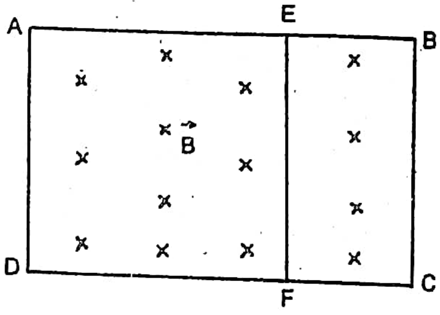


Fig. 7.16

Solution. Given $\frac{dB}{dt}=+1\text{ T/s}$

According to Faraday law, induced emf

$$e=\frac{d\phi}{dt}=\frac{d}{dt}(BA)$$

In the given problem, areas of two circuits remain constant.

$$\therefore \qquad e=A\frac{dB}{dt}$$

For circuit AFED, $\quad e_1=(1\times 1)\,m^2\times 1\text{ T/s}=1\text{ volt}$

For circuit EBCF $\quad =(0{\cdot}5\times 1\,m^2)\times 1\text{ T/s}=0{\cdot}5\text{ volt}$

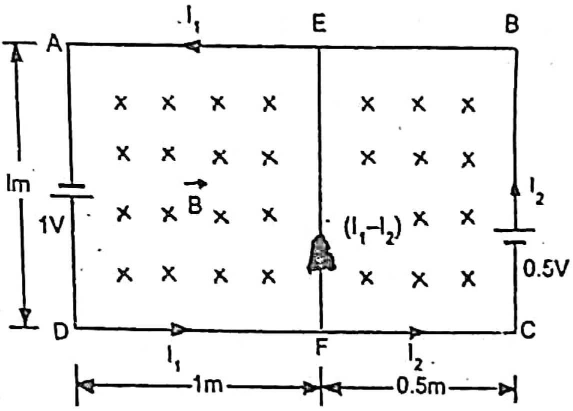


Fig. 7.17

According to Lenz's law, the current induced opposes the increase of current, hence current induced will tend to produce magnetic field in upward direction perpendicular to the plane of paper. Hence currents induced are shown in fig.

As resistance of wire = 1 Ω/m

$$R_{BC} = R_{AE} = R_{EF} = R_{DF} = R_{AD} = 1 \times 1 = 1\ \Omega$$

$$R_{BE} = R_{FC} = 0{\cdot}5\ \text{m} \times 1\ \Omega/m = 0{\cdot}5\ \Omega$$

Applying Kirchoff's second law to mesh AFEDA (induced emf = 1 V)

$$I_1 \times 1 + (I_1 - I_2) \times 1 + I_1 \times 1 + I_1 \times 1 = 1\ \text{V}$$

or $$4I_1 - I_2 = 1 \qquad ...(1)$$

Applying Kirchoff's II law to mesh BEFCB (induced emf = 0·5 V)

$$I_1 \times 0{\cdot}5 + I_2 \times 1 + I_2 \times 0{\cdot}5 - (I_1 - I_2) \times 1 = 0{\cdot}5$$

$$-I_1 + 3I_2 = 0{\cdot}5 \qquad ...(2)$$

Solving (1) and (2)

$$I_1 = \frac{7}{22} A, \quad I_2 = \frac{3}{11} A$$

Hence current in segment, *AE*, $I_1 = \frac{7}{22}$ A, directed from *E* to *A*

Current in segment *BE*, $I_2 = \frac{3}{11}$ A, directed from *B* to *E*

current in segment *EF* $= (I_1 - I_2) = \frac{7}{22} - \frac{3}{11}$

$= \frac{1}{22}$ A directed from *F* to *E*

7.6. Varying Magnetic Field

The magnetic flux linked with a circuit

$$\phi_m = N\mathbf{B} \cdot \mathbf{A} = NBA \cos\theta$$

where *B* is magnetic field induction, *A* is area and θ is angle between **B** and **A.** Clearly the magnetic flux linked with the circuit may change due to a number of ways : one of them is to vary the magnetic field with time. If a conducting loop is placed in time-varying magnetic field, the magnetic flux linked with the loop changes and an induced e.m.f. is developed in the loop. This induced e.m.f. will set the charge carriers in motion, thereby inducing a current.

From the microscopic point of view we can say that the varying magnetic field **B** causes an induced electric field **E** at all points around the loop. This electric field will exert a force **F** = q_0 **E** on a test charge of charge q_0 tangentially at each point of loop. Thus the electric lines of force that are set up by the varying magnetic field are concentric circles Fig. 7.18 represents a magnetic field of induction **B** perpendicular to the plane of paper downward. Let **B** be increasing with time at the same rate $\frac{dB}{dt}$ at each point. This can be arranged by increasing the current in windings of electro-magnet that maintains the increase of field with

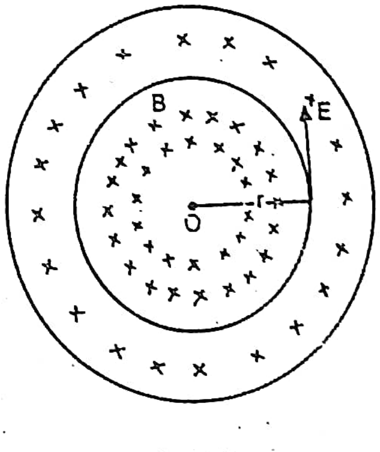


Fig. 7.18

time in a proper way. Consider a circle of radius r. The work done in carrying the test charge q_0 around this circle

$$W = \oint \mathbf{F}\cdot dl = \oint q_0 \mathbf{E}\cdot dl = q_0 E\, 2\pi r$$

$\therefore$ EMF induced, $e = \dfrac{W}{q_0} = E\cdot 2\pi r$

In general form, the induced e.m.f., may be expressed as induced e.m.f., $e = \oint \mathbf{E}\cdot dl$

$$\boxed{e = -\frac{d\phi_B}{dt}}$$

Thus Faraday's law of electromagnetic induction $e = -\dfrac{d\phi_\beta}{dt}$ may be expressed as

$$e = \oint \mathbf{E}\cdot dl = -\frac{d\phi_\beta}{dt}$$

Thus for varying magnetic fields the Faraday law may be restated as :

A changing magnetic field produces electric field. This electric field is not associated with charges but with a changing magnetic field. Although both kinds of electric fields exert forces on charges but there is a difference between them which is worth mentionable. The field associated with charges is **conservative** since line integral of this field for closed path is zero, while the electric field produced by changing magnetic field is **non-conservative**, since here.

$$\oint \mathbf{E}\cdot dl \neq 0.$$

Ex. 15. *A magnetic field of induction B exists perpendicular to plane of paper in a cylindrical volume of radius R. If B is increasing at the rate (dB/dt), calculate the electric field E at any radius (i) $r < R$ (ii) $r > R$.*

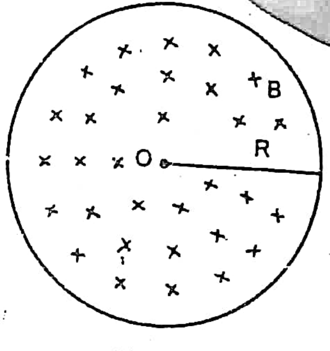


Fig. 7.19

Solution. (i) For $r < R$, the magnetic flux ϕ through the loop of radius r is

$$\phi = B\,(\pi r^2)$$

Now induced emf

$$e = \oint \mathbf{E}\cdot dl = -\frac{d\phi}{dt}$$

where E is electric field at radius r. This gives

$$E\, 2\pi r = -\frac{d}{dt}(B\pi r^2)$$

or

$$E\, 2\pi r = -\pi r^2\left(\frac{dB}{dt}\right)$$

or

$$E = -\frac{1}{2} r \frac{dB}{dt} \qquad \ldots(1)$$

Negative sign shows that the induced electric field acts opposite the change of magnetic flux.

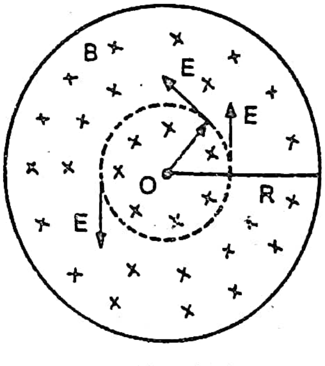


Fig. 7.20

(ii) If $r > R$, the magnetic flux through the loop is

$$\phi = B\,\pi R^2$$

$$\therefore \quad e = \oint \mathbf{E}\cdot dl = -\frac{d\phi}{dt}$$

gives
$$E\,2\pi\,r = -\pi R^2 \frac{dB}{dt}$$

or
$$E = -\frac{1}{2}\frac{R^2}{r}\frac{dB}{dt} \qquad ...(2)$$

7.7. Faraday's Laws in Universal Form : Differential Form of Faraday's Laws.

We have seen that e.m.f. induced in a coil depends merely on the rate of change of magnetic flux linked with the coil. Thus the electric field **E** generated in a coil depends upon the changing magnetic field **B** in which the coil is placed. Here we shall establish a relation between the electric field **E** and the changing magnetic field **B** with the help of Faraday's law.

Let C be a coil (or closed loop) of surface S enclosing the area A. If **B** is the magnetic field induction at any instant t at a point $P\,(x, y, z)$ in the magnetic field, then the magnetic flux linked with an infinitesimal area $\delta\mathbf{A}$ around the point (x, y, z) is

$$\delta\phi = \mathbf{B} \cdot \delta\mathbf{A}$$

The total magnetic flux linked with the coil,

$$\phi = \iint_S \mathbf{B} \cdot d\mathbf{A} \qquad ...(1)$$

If the magnetic field changes with the time due to which $d\phi$ is the change in magnetic flux linked with the coil in time dt, then according to Faraday's laws, e.m.f. induced in the coil is

$$e = -\frac{d\phi}{dt} \qquad ...(2)$$

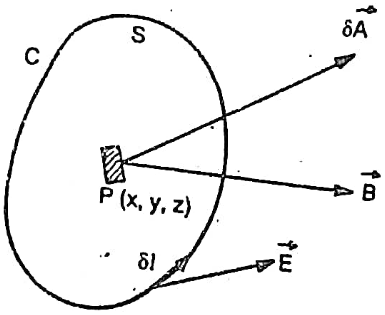


Fig. 7.20

But according to definition of induced e.m.f.

$$e = \int_C \mathbf{E} \cdot d\mathbf{l} \qquad ...(3)$$

where **E** is the electric field generated due to changing magnetic field **B**,

Substituting values of e and ϕ from equation (3) and (1) in equation (2), we get

$$\int_S \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\iint_C \mathbf{B} \cdot d\mathbf{A} \qquad ...(4)$$

According to Stoke's law, we have

$$\int_C \mathbf{E} \cdot d\mathbf{l} = \iint_S \text{curl } \mathbf{E} \cdot d\mathbf{A} \qquad ...(5)$$

Comparing equations (4) and (5), we get

$$\iint_S \text{curl } \mathbf{E} \cdot d\mathbf{A} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{A}$$

or
$$\iint_S \left(\text{curl } \mathbf{E} + \frac{d\mathbf{B}}{dt}\right) \cdot d\mathbf{A} = 0 \qquad ...(6)$$

As the surface is arbitrary *i.e.* this equation holds for any surface area, therefore

$$\text{curl } \mathbf{E} + \frac{d\mathbf{B}}{dt} = 0$$

i.e.
$$\text{curl } \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

As **B** is function of x, y, z and t; hence its variation with respect to time alone may be represented by partial derivative $\frac{\partial}{\partial t}$ instead of $\frac{d}{dt}$

Hence
$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This equation is called differential form of Faraday's laws.

7.8. Self-Induction : Self Inductance

Whenever the electric current flowing through a circuit changes, the magnetic flux linked with that circuit also changes. As a result an induced e.m.f. is set up in the circuit. According to Lenz's law the direction of induced e.m.f. is such as to oppose the change in the current. Thus the e.m.f. is against the current when the current in the circuit is increasing and along the direction of the current when the current is decreasing. *The phenomenon of the preduction of an induced e.m.f. in a circuit itself due to change in current through it is called self induction and the induced e.m.f. is called back e.m.f.*

Let C be a circuit in which the current I is flowing. The current will set up a magnetic field and hence a magnetic flux ϕ is linked with the circuit. As the magnetic field strength at any point is proportional to the current I flowing through a circuit, so that the magnetic flux ϕ linked with the circuit at any instant is proportional to the current I flowing through the same circuit at that instant *i.e.*

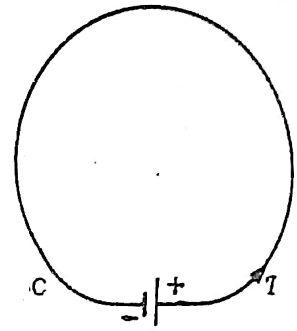


Fig. 7.22

$$\phi \propto I$$

or
$$\phi = LI \qquad \text{...(1)}$$

where L is a constant of proportionality, called the *coefficient of self induction* or *self inductance* of the coil. The practical unit of self inductance is henry.

When the current I flowing through a circuit is changed, the magnetic flux linked with the circuit also changes and an induced e.m.f. is set up in the circuit.

The rate of change of flux linked with the circuit is given by
$$\frac{d\phi}{dt} = L\frac{dI}{dt}$$

Therefore from Faraday's law the e.m.f. induced in the circuit is
$$e = -\frac{d\phi}{dt}$$

i.e.
$$e = -L\frac{dI}{dt} \qquad \text{...(2)}$$

Now when the current in the circuit is first switched on, the back e.m.f. opposes the growth of current, so that the current flows against the back e.m.f. and does work against it. If the growing current at any instant be I, then *work done against back e.m.f. in a short time dt will be,*
$$dW = -eIdt = +L\frac{dI}{dt}.Idt \text{ [using eq. (2)]}$$

Hence the *total work done in bringing the current from zero to a steady maximum value I_0 is*
$$W = +L\int_0^{I_0} I\frac{dI}{dt}\,dt = L\int_0^{I_0} I\,dI = \tfrac{1}{2}LI_0^2. \qquad \text{...(3)}$$

This work is stored as the energy of the magnetic field.

Equations (1), (2) and (3) enable us to define the self-inductance L of a circuit in the following three ways :

(1) From equation (1), $\phi = LI$
$$L = \frac{\phi}{I}.$$

If I = 1 amp, $L = \phi$ henry.

Thus ***the self inductance of a circuit (in henry) is defined as the magnetic flux (in webers) linked with the circuit when 1 ampere current flows through it.***

(2) From equation (2),
$$e = -L\frac{dI}{dt}.$$

or
$$L = \frac{e}{-\frac{dI}{dt}}$$

when $\frac{dI}{dt} = -1$ Amp/s,
$$L = e$$

Thus ***the self inductance of a circuit is defined as the e.m.f. induced in the circuit when the rate of decay of current in the circuit in unity.***

Here e.m.f. is in volts, current amp. and L in henries,

(3) From equation (3), $W = \frac{1}{2}LI_0^2$

or
$$L = \frac{2W}{I_0^2}.$$

Thus, ***the self inductance of a circuit is defined as twice the work done against the induced e.m.f. in establishing unit current in the coil.***

Here work is in joules, current in amperes and L in henries.

These three definitions are identical only when the permeability of the medium round which the coil is wound remains constant. The role of self inductance in an electrical circuit is same as that of inertia in mechanical motion. Thus the self inductance of a coil is a measure of its ability to oppose the change in current through it.

Self inductance of a solenoid :

Let us consider a long air cored solenoid having length l, cross-sectional area A and number of turns per unit length n.

If I amp is the current flowing through the solenoid, then the magnetic field inside the solenoid is
$$A = \mu_0 nI \text{ weber/m}^2.$$

The magnetic flux linked with the solenoid, is
$$\phi = \mu_0 nI \times \text{effective area of solenoid}$$
$$= \mu_0 nI\,(nl.A) \text{ webers}$$

(since effective area of solenoid = area of each turn × total number of truns in solenoid)

i.e.
$$\phi = \mu_0 n^2 Al\, I \quad ...(1)$$

But accoding to definition
$$\phi = LI \quad ...(2)$$

where L is the self inductance of the solenoid.

Compaing equations (1) and (2), we get
$$L = \mu_0 n^2 Al \text{ henry} \quad ...(3)$$

where $\mu_0 = 4\pi \times 10^{-7}$ weber/amp-metre.

Solved Examples

Ex. 17. *An air solenoid of 80 cm. length has 500 turns and its circular cross-section has a diameter of 2 cm. Calculate*

(a) the self-inductance of the solenoid.

(b) the self linked flux when the current in the solenoid is 2 amp,

(c) the rate of change of current in the solenoid that will produce a self-induced e.m.f. of 0·3 volt.

Sol. (a) The self inductance of the coil. $L = \mu_0 n^2 Al$

Here pemittivity of medium, $\mu_0 = 4\pi \times 10^{-7}$ weber/amp–m.

Length of solenoid, $l = 80$ cm $= 0{\cdot}80$ metre.

Number of turns per metre length, $n = \dfrac{550}{0{\cdot}8} = 625$ turns/metre.

Cross-sectional area of solenoid,

$$A = \pi r^2 = \pi \times \left(\frac{1}{100}\right)^2 \text{ metre}^2 = \pi \times 10^{-4}\text{ m}^2$$

$$\therefore \quad L = \mu_0 n^2 Al$$

$$= (4\pi \times 10^{-7}\text{ weber/amp–m})\,(625\text{ turns/m})^2 \times (\pi \times 10^{-4}\text{ m}^2)\,.\,(0{\cdot}80\text{ m.})$$

$$= 125\,\pi^2 \times 10^{-7}\text{ henries} = \mathbf{1{\cdot}232 \times 10^{-4}\ henries}$$

(b) Self-linked flux with the solenoid,

$$\phi = LI = (1{\cdot}232 \times 10^{-4}\text{ henries}).(2\text{ amp.}) = \mathbf{2{\cdot}464 \times 10^{-2}\ webers}$$

(c) Self-induced e.m.f. in the solenoid, $e = -\,L\,\dfrac{dI}{dt}$.

$\therefore$ Rate of change of current in the solenoid,

$$\frac{dI}{dt} = \frac{e}{L}\text{ (numerically)} = \frac{0{\cdot}3}{1{\cdot}232 \times 10^{-4}\text{ henry}} = \mathbf{2{\cdot}43 \times 10^{3}\ amp/sec.}$$

Ex. 18. *Calculate the self-inductance of a coil of 100 turns, if a current of 2A gives rise to magnetic flux of 5×10^{-5} weber through the coil. Also calculate the magnetic energy stored in the medium surrounding the coil for the above value of current.*

Solution. Usually flux linked with a coil is the flux linked per turn. If ϕ_B is the magnetic flux linked with each turn, then we have $N\phi_B = L\,i$

or
$$L = \frac{N\phi_B}{i} = \frac{100 \times (5 \times 10^{-5})}{2} = 2{\cdot}5 \times 10^{-3}\text{ N.}$$

The magnetic energy stored,
$$U = \tfrac{1}{2}Li^2 = \tfrac{1}{2} \times 2{\cdot}5 \times 10^{-3} \times 2^2$$

$$= \mathbf{5 \times 10^{-3}\ J.}$$

Ex. 19. *A solenoid having a core of 4 sq. cm. in cross-section, half air ($\mu_{r_1} = 1$) and half iron ($\mu_{r_2} = 500$) is 20 cm. long. If the number of turns on it is 1000 ; calculate its self-induction.*

Sol. The self inductance in this case is given by

$$L = \mu_0 n^2 l\,(\mu_{r_1} A_1 + \mu_{r_2} A_2)$$

Here $\mu_0 = 4\pi \times 10^{-7}$ weber/amp-m, $\mu_{r_1} = 1$, $\mu_{r_2} = 500$

$$l = 20\text{ cm} = 0{\cdot}20\text{ m},\; A_1 = A_2 = 2\text{ sq. cm} = 2 \times 10^{-4}\text{ sq m.}$$

and $$n = \frac{1000}{0{\cdot}20} = 5{,}000 \text{ turns/m}.$$

$$\therefore \quad L = (4\pi \times 10^{-7} \text{ weber/amp–m})\,(5000 \text{ turns/m})^2\,(0{\cdot}20 \text{ m}) \times \{1 \times 2 \times 10^{-4} + 500 \times 2 \times 10^{-4})\text{ m}^2\}$$

$$= 4\pi \times 10^{-7} \times (5000)^2 \times (0{\cdot}2) \times (501 \times 2 \times 10^{-4}) \text{ henry}$$

$$= \mathbf{0{\cdot}6292 \text{ henry}}.$$

Ex. 20. *Deive an expression for the self-inductance of a toroidal solenoid if its cross-section is (a) rectangular (b) circular.*

Sol. (a) Fig. 7·25 shows the section of a toroid (endless solenoid having rectangular cross-section of thickness h. Let b and a be the extrenal and internal radii of toroid respectively and N be the number of turns uniformly distributed over its circumference.

Consider a circular path of radius $r\,(r_1 < r < r_2)$ concentric with the toroid. If I amp is the current flowing in the winding of the toroid and B the magnitude of the magnetic field produced on every point of the circular path due to this current, then according to Ampere's law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu \times \text{current enclosed by the path}$$

Since the magnetic field is tangential around the circular path of radius r, we have

$$B.2\pi r = \mu\,(NI)$$

i.e. $$B = \frac{\mu NI}{2\pi r}.$$

where μ is the absolute permeability of the medium.

Now consider a rectangular strip of height h and width dr at a distance r from the centre, then area of this strip is

Therefore the magnetic flux linked with this strip is

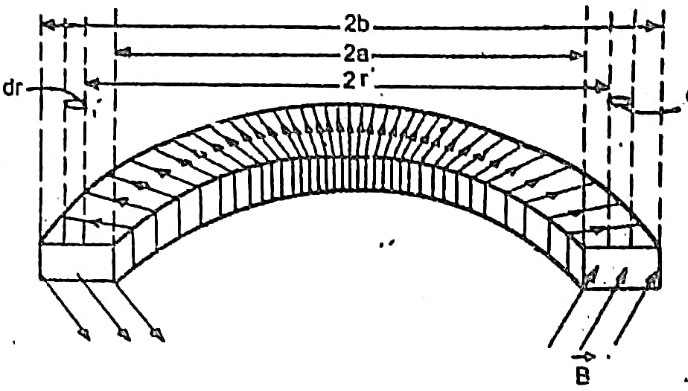


Fig. 7.25

$$d\phi = \mathbf{B} \cdot d\mathbf{A} = B\,hdr.$$

Therefore the flux ϕ' linked with the cross-section of one turn of the toroid is

$$\phi' = \int_a^b \frac{\mu NI}{\pi r}\,.\,hdr = \frac{\mu NIa}{2\pi}\int_a^b \frac{dr}{r}$$

i.e. $$\phi' = \frac{\mu NIh}{2\pi}\log_e \frac{b}{a}$$

Therefore the total flux ϕ linked with N turns of the toroid

$$\phi = N\phi' = \frac{\mu N^2 Ih}{2\pi}\log_e \frac{b}{c}$$

But according to definition

$$\phi = LI$$

Therefore the self-inductance L of the toroid is given by

$$L = \frac{\mu N^2 h}{2\pi}\log_e \frac{b}{a} \text{ henry}$$

(b) Fig. 7.26 shows a toroid having circular cross section. Let R be the mean radius of the toroid and N the number of turas in the toroid. If I amp. is the current flowing in the toroid and B the magnetic field produced due to this current on the circular path, then according to Ampere's law

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu \times \text{ current enclosed by the path.}$$

As the magnetic field $\mathbf{B}$ is tangential around the circular path of radius R, we have

$$B\, 2\pi R = \mu \times NI$$

i.e.
$$B = \frac{\mu NI}{2\pi R}$$

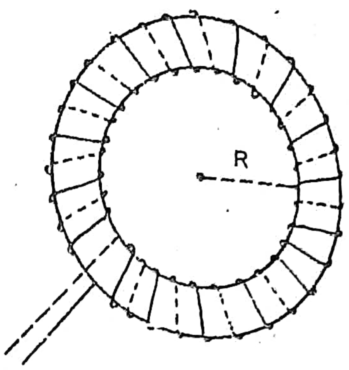


Fig. 7.26

If A is the area of cross-section of each turn of the toroid, therefore magnetic flux linked with the toroid is

$$\phi = NBA = N\frac{\mu NI}{2\pi R} \cdot A$$

$$= \frac{\mu N^2 A}{2\pi R} \text{ henry.}$$

Ex. 21. *Deduce an expression for the self-inductance of two long parallel wires.*

Sol. Let there be two infinitely long parallel wires each of radius r. Let d be the distance between their axes and i amp. the current flowing in each wire in opposite directions. Consider a thin strip of thickness dx between the wires at a distance x form the upper wire (fig. 7.27).

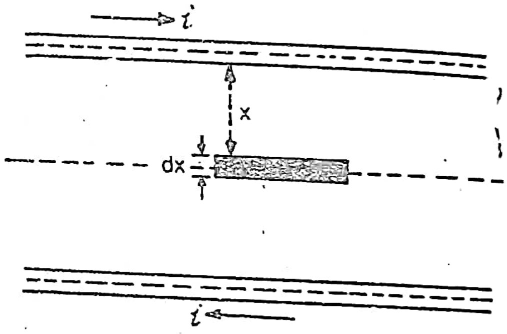


Fig. 7.27

($\because$ For length l, $d\phi = Blx$)

The magnetic field at any point on the strip is given by

$$B = \frac{\mu i}{2\pi x} + \frac{\mu i}{2\pi(d-x)}$$

Therefore the magnetic flux linked with unit lenght of the strip is

$$d\phi = B.dx$$

$$= \left\{\frac{\mu i}{2\pi x} + \frac{\mu i}{2\pi(d-x)}\right\} dx$$

Hence the magnetic flux linked with unit length of the system is

$$\phi = \frac{\mu i}{2\pi}\int_r^{d-r}\left(\frac{1}{x} + \frac{1}{x-d}\right) dx$$

$$= \frac{\mu i}{2\pi}\Big[\log_e x - \log_e(d-x)\Big]_r^{d-r}$$

$$= \frac{\mu i}{2\pi}\left[\log_e \frac{d-r}{r} - \log_e \frac{r}{d-r}\right]$$

$$= \frac{\mu i}{\pi}\log_e \frac{d-r}{r} \quad \text{...(1)}$$

But from definition

$$\phi = Li. \quad \text{...(2)}$$

Comparing (1) and (2), we get

$$L = \frac{\mu}{\pi}\log_e \frac{d-r}{r}. \quad \text{...(3)}$$

This respresents the expression for the self inductance per unit length of the system consisting two long parallel wires each of radius r and separated from each other by a distance d between their axes through a medium of absolute permeability μ.

7.9 Mutual Induction : Mutual Inductance

Consider a closed circuit C_2 placed in the magnetic field of closed circuit C_1 (fig. 7.28). The flux linked with circuit C_2 depends upon its position and the current flowing in the coil C_1. If the relative position of C_2 is fixed, the flux linked with C_2 is constant for a constant current in C_1. As the magnetic field at any point produced due to the current flowing in C_1 is directly proportional to the current flowing in circuit C_1; hence the flux linked with the circuit C_2 will depend upon the current flowing in circuit C_1. In general *whenever the current flowing through a circuit (C_1) changes, the magnetic flux linked with the neighbouring circuit (C_2) also changes. Hence an induced e.m.f. is set up in the neighbouring circuit. This phenomenon is called* **mutual induction.** The circuit in which the current changes is called the *primary circuit,* while the neighbouring circuit in which e.m.f. is induced is called the *secondary circuit.*

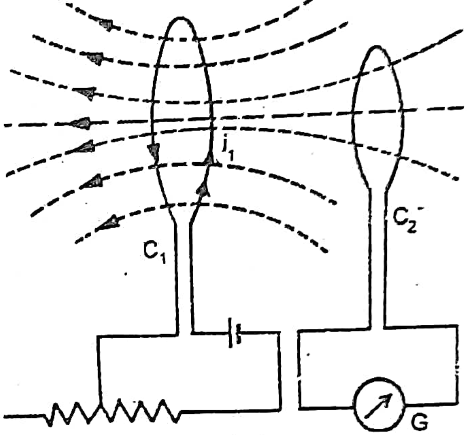


Fig. 7.28

Thus if i_2 is the current flowing in the circuit C_1 and $\mathbf{B_1}(x, y, z)$ is the magnetic field at any point (x, y, z) in the space surrounding the circuit C_1 due to this current, then the magnetic flux ϕ_2 linked with the coil C_2 due to this magnetic field is given by

$$\phi_2 = \iint_{S_2} \mathbf{B_1} \cdot \delta \mathbf{A_2}$$

where S_2 represents the effective surface area enclosed by the coil C_2

If the shapes, orientations and the positions of the two circuits C_1 and C_2 are fixed, then the magnetic flux ϕ_2 linked with circuit C_2 is proportional to the current i_1 flowing in circuit C_1 *i.e.*

$$\phi_2 \propto i_1$$

or

$$\phi_2 = M_{21}\, i_1 \qquad \text{...(1)}$$

where M_{21} is constant of proportionality and is termed as mutual inductance (or coefficient of mutual induction) of circuit C_2 with respect to circuit C_1.

If the current flowing in the coil C_2 changes with time, then the flux linked with the coil C_2 also changes and so an e.m.f. e_2 is induced in the circuit C_2 which is given by

$$e_2 = -\frac{d\phi_2}{dt} = -\frac{d}{dt}(M_{21} i_1) \quad \text{using (1)}$$

i.e.

$$e_2 = -M_{21}\frac{di_1}{dt} \qquad \text{...(2)}$$

Now suppose that the current i_2 is flowing in the coil C_2. If ϕ_1 is the flux linked with the coil C_2 due to this current, then

$$\phi_1 \propto i_2$$

or

$$\phi_1 = M_{12}\, i_2 \qquad \text{...(1')}$$

where M_{12} is the mutual inductance of circuit C_1 with respect to circuit C_2.

If the current flowing in the coil C_2 changes with time, then the flux linked with the coil C_1 also changes and so an e.m.f. e_1 is induced in the circuit C_1 which is given by

$$e_1 = -\frac{d\phi_1}{dt}$$

$$= -\frac{d}{dt}(M_{12}i_2)$$

i.e. $$e_1 = -M_{12}\frac{di_2}{dt}. \qquad ...(2')$$

It may be proved that $M_{21} = M_{12}$ *i.e.* if the shapes, orientations and the positions of the two coils (or circuits) are fixed, then the mutual inductance of the two given coils does not depend upon which circuit (C_1 or C_2) carries the current. Therefore we may represent $M_{12} = M_{21} = M$ (say) = Mutual inductance between the circuits. The practical (or M.K.S.A.) unit of mutual inductance is **henry.** Now equations (1) and (2) or (1′) and (2′) enable us to define the mutual inductance between two circuits in the following two ways :

(1) From equation (1) it $i_1 = 1$ amp., then $M = M_{21} = \phi_2$ henries *i.e. the mutual inductance* (in henry) *of two circuits is defined as the magnetic flux* (in webers) *linked with one circuit when unit current* (1 amp.) *flows through the other (neighbouring) circuit.*

(2) From equation (2) if $-\dfrac{di_1}{dt} = 1$, then $M = M_{21} = e_2$.

i.e. the mutual inductance (in henry) *of two circuits may be defined as the e.m.f.* (in volts) *induced one circuit when the rate of decay of current* (in amp./sec.) *in other (neighbouring) circuit is unity.*

Reciprocity Theorem of Mutual Inductances :

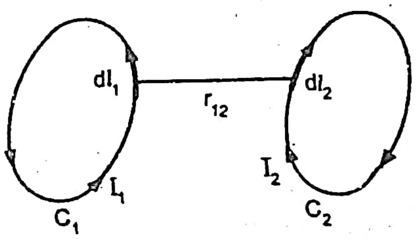


Fig. 7.29

Let there be two closed circuits (or coils) C_1 and C_2 situated close to each other (fig. 7.29) and carrying currents I_1 and I_2 amp. respectively. The current I_1 in C_1 will cause a magnetic flux linked with current C_2. If **B** is the magnetic field caused at a point in circuit C_2 due to current in circuit C_1, then magnetic flux linked with circuit C_2

$$\phi_{21} = \int_{S_2} \vec{B}_1 \cdot \vec{dS}_2$$

As $\text{div}\, \vec{B}_1 = 0 \Rightarrow \vec{B}_1 = \text{curl}\, \vec{A}_1 = \nabla \times \vec{A}_1$

where $\vec{A}_1$ is the vector potential at the point of circuit C_2 due to current in circuit C_1

$$\therefore \phi_{21} = \int_{S_2} (\nabla \times \vec{A}_1) \cdot \vec{dS}_2$$

Using stokes theorem

$$\int_{S_2} (\nabla \times \vec{A}_1) \cdot \vec{dS}_2 = \oint_{C_2} \vec{A}_1 \cdot \vec{dl}_2$$

$$\therefore \phi_{21} = \int_{C_2} \vec{A}_1 \cdot \vec{dl}_2$$

But $$A_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{dl_1}{r_{12}}$$

$$\therefore \quad \phi_{21} = \oint_{C_2}\left\{\frac{\mu_0 I_1}{4\pi}\oint_{C_2}\frac{\vec{dl}_1}{r_{12}}\right\}\cdot dl_2$$

$$= \frac{\mu_0 I_1}{4\pi}\oint_{C_1}\oint_{C_2}\frac{dl_1\cdot dl_2}{r_{12}} = M_{21} I_1$$

$$\Rightarrow M_{21} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{dl_1\cdot dl_2}{r_{12}}$$

Similarly flux linked with C_1 due to current flux in circuit C_2 is

$$\phi_{12} = M_{12} I_2$$

Proceeding as above $N_{12} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{dl_1\cdot dl_2}{r_{21}}$

As $$r_{12} = r_{21}$$

$$\therefore \quad M_{12} - M_{21}$$

This relatin is usually termed as the *reciprocity theorem of mutual induction.* On account of this theorem mutual inductance between two coils (or circuits) is represented by r in place of M_{12} or M_{21} .

A laboratory standard of mutual inductance (Mutual inductance of two given coils) :

Consider a long air-cored solenoid (primary coil) PP of cross-sectional area 'a' having n_1 turns per metre length. A short secondary coil SS of N_2 turns is wound closely over the central portion of the primary PP (fig. 7.30). If i amp is the current flowing in the solenoid, then the magnetic field within the solenoid is

$$B = \mu_0 n_1 i \text{ weber/m}^2 .$$

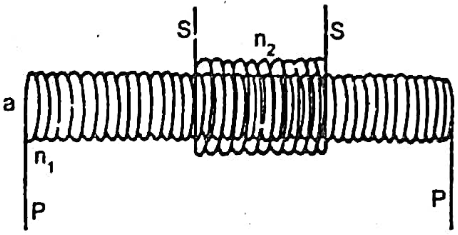


Fig. 7.30

$\therefore$ The magnetic flux linked with each turn of the central portion of the primary is

$$\phi' = Ba = \mu_0 n_1 i . a$$

Since the secondary is would closely over the central portion of the primary, the same flux is also linked with each turn of the secondary *i.e.* the magnetic flux linked with each turn of the secondary is

$$\phi' = \mu_0 n_1 i\, a \text{ weber} \qquad ...(1)$$

$\therefore$ Total flux linked with the secondary having n_2 turns is

$$\phi = N_2\phi' = \mu_0 n_1 N_2 a i$$

But according to definition, if M is the mutual inductance between the two coils, the total flux linked with the secondary is

$$\phi = M i \qquad ...(2)$$

Comparing equations (1) and (2), we get

$$M = \mu_0 n_1 N_2 a. \qquad ...(3)$$

The equation gives the mutual inductance between the two given coils.

If the coils are would round a core of constant relative permeability μ_r , then

$$M = \mu_0 \mu . n_1 N_2 a \qquad ...(4)$$

Solved Examples

Ex. 22. *In a car spark coil, an e.m.f. of 40,000 volts is induced in a secondary when the primary current changes from 4A to O in 10 μ-s. Find the mutual inductance between the primary and secondary winding of this spark coil.*

Solution. The e.m.f. induced

$$e = -M\frac{di}{dt}.$$

$$e = 40{,}000\text{ V}, \frac{di}{dt} = \frac{i_2 - i_1}{t_2 - t_1} = \frac{0-4}{10 \times 10^{-6}} = -4 \times 10^5 \text{ A/s}.$$

∴ Mutual inductance, $M = -\dfrac{e}{(di/dt)} = -\dfrac{40000}{(-4 \times 10^5)}$

$= \mathbf{0{\cdot}1}$ **Henry.**

Ex. 23. *A long solenoid of length 1 metre, cross-section 10 cm² having 1000 turns, has wound over its centre a small coil of 20 turns. Find the mutual inductance of this system in microhenries.*

(Rohilkhand 1996)

Sol. The mutual inductance of the system is

$$M = \mu_0 n_1 N_2 A$$

Here absolute permeability of air $\mu_0 = 4\pi \times 10^{-7}$ weber/amp-m, no. of turns per metre length of selenoid, $n_1 = 1000$ turns/m

Total no. turns of secondary coil, $N_2 = 20.$

Cross-sectional area of solenoid

$$A = 10\text{ cm}^2 = 10 \times 10^{-4}\text{ m}^2 = 10^{-3}\text{ m}^2$$

$$\therefore \quad M = (4\pi \times 10^{-7}\text{ weber/amp–m})\,(1000\text{ turns/m})\,(20) \times 10^{-3}\text{ m}^2$$

$$= 4\pi \times 10^{-7} \times 1000 \times 20 \times 10^{-3}\text{ henries}$$

$= 25{\cdot}12 \times 10^{-6}$ henries $=$ **25·12 microhenry.**

Ex. 24. *A circular coil P of 100 turns and radius 2 cm is placed co-axially at the centre of another circular coil Q of 1000 turns and radius 20 cm. Calculate*

(a) the mutual inductance of the coils,

(b) the induced e.m.f. in coil P when the current in the coil Q decreases from 5 amp. to 3 amp. in 0·04 sec;

(c) the rate of change of flux through the coil P at this instant, and

(d) the charge passing through coil P if its resistance is 8 ohms.

Sol. (a) The magnetic field at the centre of the coil Q is given by

$$B = \frac{\mu_0 N_1 i}{2a}.$$

Here $\mu_0 = 4\pi \times 10^{-7}$ weber/amp-m, $N_1 = 1000$, $a = 20\text{ cm} = 0{\cdot}2\text{ m}$

$$\therefore \quad B = \frac{4\pi \times 10^{-7}\text{ weber/amp m}) \times (1000) \times (i\text{ amp})}{(2 \times 0{\cdot}2\text{ m})}$$

$$= 3{\cdot}14 \times 10^{-2}\, i\text{ weber/m}^2.$$

The flux linked with the coil P, $\phi = B.AN_2 = B.\pi a^2.N_2$

$$= (3{\cdot}14 \times 10^{-3}\, i \text{ weber/m}^2) \times 3{\cdot}14 \times (0{\cdot}02 \text{ m})^2 \times 100$$
$$= 3{\cdot}944 \times 10^{-4}\, i \text{ webers.}$$

But according to definition, $\phi = Mi$

$$\therefore \qquad \mathbf{M = 3{\cdot}944 \times 10^{-4} \text{ henry.}}$$

(b) The e.m.f. induced in the coil P is given by

$$e = M \frac{di}{dt} \text{ (numerically)}$$
$$= (3{\cdot}944 \times 10^{-4} \text{ henries}) \times \left(\frac{5-2}{0{\cdot}04} \text{ amp/sec}\right)$$
$$= 19{\cdot}72 \times 10^{-3} \text{ volts} = \mathbf{19{\cdot}72 \text{ millivolt.}}$$

(c) The rate of change of flux through coil P,

$$\frac{d\phi}{dt} = e \text{ (numerically)}$$
$$= 19{\cdot}72 \times 10^{-3} \text{ weber/sec} = \mathbf{19{\cdot}72 \text{ milliweber/sec.}}$$

(d) The charge passing through coil P is given by

$$q = \int_0^t i dt = \int_0^t \frac{e}{R}\, dt = -\frac{1}{R}\int_0^t M \frac{di}{dt}\, dt \quad \left(\text{since } e = -M\frac{di}{dt}\right)$$
$$= -\frac{M}{R}\int_{i_2}^{i_2} di = \frac{M}{R}(i_1 - i_2)$$
$$= \frac{3{\cdot}944 \times 10^{-4}}{8} \times (5-3) \text{ coulomb}$$
$$= 9{\cdot}86 \times 10^{-5} \text{ coulomb} = \mathbf{98{\cdot}6 \text{ micro–coulomb.}}$$

Ex. 25. *A very small circular loop of area 5×10^{-4} m², resistance 2 ohm and negligible inductance is initially coplanar and concentric with a much larger fixed circular loop of radius 0·1 m. A constant current of 1 ampere is passed in the bigger loop and the smaller loop is rotated with angular velocity ω rad/sec about a diameter. Calculate (i) the flux lined with the smaller loop, (ii) induced e.m.f. and (iii) induced current in the smaller loop, as a function of time.* **(Roorkee 1992)**

Solution. Area of smaller loop $A_2 = 5 \times 10^{-4}$ m²

$$R_2 = 2\Omega$$

Radius of larger loop $r_1 = 0{\cdot}1$ m, current in larger loop, $I_1 = 1$A Angular velocity of smaller loop = ω rad/s, about a diameter. The magnetic field produced due to current $I_1 = 1$ A in the vicinity of smaller loop

$$B_1 = \frac{\mu_0 I_1}{2r_1} \text{ perpendicular to plane of loop.}$$
$$= \frac{4\pi \times 10^{-7} \times 1}{2 \times 0{\cdot}1} = 2\pi \times 10^{-6} \text{ weber/m}^2$$

(i) The magnetic flux linked with smaller loop

$$\phi_2 = B_1 A_2 \cos\theta$$

Initially area is directed perpendicular to the plane of loop

(at $t = 0$) and

$$\therefore \qquad \theta = 0 \text{ after time } t, \theta = \omega t$$

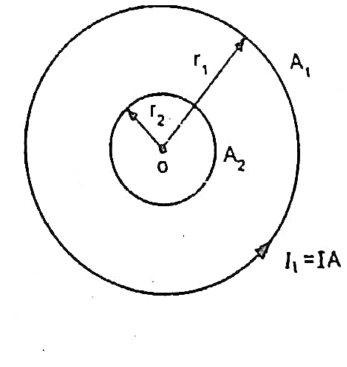


Fig. 7.31

∴ Flux $\phi_2 = B_1 A_2 \cos \omega t$

$$= 2\pi \times 10^{-6} \times 5 \times 10^{-4} \cos \omega t$$

∴ $\phi_2 = \pi \times 10^{-9} \cos \omega t$ weber

(ii) The induced emf, $e_2 = -\dfrac{d\phi_2}{dt}$

$$= -\frac{d}{dt}(\pi \times 10^{-9} \cos \omega t)$$

$$= \mathbf{\pi \times 10^{-9}\, \omega \sin \omega t \text{ volts}}$$

(iii) Induced current, $i_2 = -\dfrac{e}{R_2} = \dfrac{\pi \times 10^{-9}\, \omega \sin \omega t}{2}$ Amp

$$= \mathbf{1{\cdot}57 \times 10^{-9}\, \omega \sin \omega t \text{ Amp}}$$

Ex. 26. *A long solenoid of diameter* 0·1 m *has* 2×10^4 *turns per meter. At the centre of the solenoid a 100 turn coil of radius* 0·01 m *is placed with its axis coinciding with the solenoid axis. The current in the solenoid is decreased at a constant rate from* + 2A *to* – 2A *in* 0·05 *sec. Find the e.m.f. induced in the coil. Also find the total charge flowing through the coil during this time when the resistance of the coil is* $10\,\pi^2\,\Omega$. **(Roorkee 1993)**

Solution. The mutual inductance of solenoid–coil system

$$M = \mu_0 n_1 N_2 A_2 = \mu_0 n_1 N_2 \pi r_2^2$$

$$= 4\pi \times 10^{-7} \times 2 \times 10^4 \times 100 \times \pi \times (0{\cdot}01)^2$$

$$= 800\,\pi^2 \times 10^{-7} \text{ Henry} = 8\pi^2 \times 10^{-5} \text{ Henry}$$

EMF induced in the coil

$$e_2 = -M\frac{\Delta i}{\Delta t}$$

$$= -8\pi^2 \times 10^{-5} \times \left(\frac{-2-2}{0{\cdot}05}\right)$$

$$= 8\pi^2 \times 10^{-5} \times 80 = \mathbf{640\,\pi^2 \times 10^{-5} \text{ Volt}}$$

$$= 640 \times 9{\cdot}86 \times 10^{-5} \text{ Volt}$$

$$= 6310 \times 10^{-5} \text{ Volt} = \mathbf{63{\cdot}1 \text{ mV}}$$

Current in coil $i = \dfrac{e}{R}$

Charge induced $q = i\,\Delta t = \dfrac{e}{R}\Delta t$

$$= \frac{640 \times 8\pi^2 \times 10^{-5}}{10\,\pi^2} \times 0{\cdot}05$$

$$= 32 \times 10^{-6} \text{ coul} = \mathbf{32\,\mu C}$$

Ex. 27. *Find the mutual inductance between two coplanar concentric rings of radii* R_1 *and* R_2 *respectively.*

Sol. Let R_1 and R_2 be the radii of outher and inner rings respectively. In the current flowing in the outer ring is i_1 amp, then the magnetic field at the centre of the ring due to this current is

$$B_1 = \frac{\mu_0 i_1}{2R_1}$$

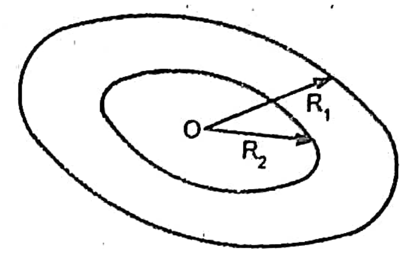


Fig. 7.32

The flux through the inner ring of radius R_2 is

$$\phi = \frac{\mu_0 i_1}{2R_1} \cdot \pi R_2^2 \quad \text{...(1)}$$

Here we have assumed $R_2 << R_1$ so that the magnetic field B_1 remains nearly uniform over this area.

According to definition, the flux through the inner ring is

$$\phi = M i_1 . \quad \text{...(2)}$$

where M is the mutual inductance between the two rings.

Comparing eqs. (1) and (2), we get

$$M = \frac{\mu_0 \pi R_2^2}{4R_1} \text{ henries.}$$

If outher ring contains N_1 turns of wire and the inner ring N_2 turns, then the magnetic field at the centre of the ring will be $N_1 B_1$ and the flux throught the inner ring would be $N_1 B_1 N_2 \pi R_2^2$. Therefore the mutual inductance between two rings would become

$$\mu_0 N_1 N_2 \pi R_2^2 / 2R_1 .$$

7.10. Combination of Two Inductances

1. Inductances in Series

Case (i) When inductors are at large distances. When two coils of inductances L_1 and L_2 are connected in series at large distances and current i is passed through them, the total flux linkage ϕ_{total} is the sum of the flux linkages $L_1 i$ and $L_2 i$. That is $\phi_{total} = L_1 i + L_2 i.$

If L be the equivalent inductance of the system, then

$$\phi_{total} = L i$$

$$\therefore \quad L i = L_1 i + L_2 i \quad \text{or} \quad L = L_1 + L_2 .$$

This is the required expression which holds provided the two coils are separated by a large distance so that there is no flux linkage in each coil due to the current in the other.

Case (ii) When inductors are at small distance : When the separation between the coils is small there wil be a mutual inductance M between them. In this case the resultant induced e.m.f. e in the coils is the sum of the e.m.f.'s e_1 and e_2 in the respective coil. That is

$$\begin{aligned} e &= e_1 + e_2 \\ &= \left(-L_1 \frac{di}{dt} \pm M \frac{di}{dt}\right) + \left(-L_2 \frac{di}{dt} \pm M \frac{di}{dt}\right). \end{aligned}$$

where (+) sign is for (+) coupling and (–) sign for (–) coupling.

But
$$e = -L \frac{di}{dt}$$

$$\therefore \quad -L \frac{di}{dt} = -L_1 \frac{di}{dt} - L_2 \frac{di}{dt} \pm 2M \frac{di}{dt}$$

i.e.
$$L = L_1 + L_2 \mp 2M$$

2. Inductances in parallel

Case (i) When inductors are at large distances : Suppose two coils L_1 and L_2 (at large distances) are connected in parallel between two points and a current i is divided between them. Then

$$i = i_1 + i_2 . \qquad ...(1)$$

Differentiating, we get

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} \qquad ...(2)$$

When the currents through the inductances were growing, induced e.m.f.'s were set up in them. As the potential difference across each coil is the same, the induced e.m.f. must also have the same value e (say). Thus

$$e = -L_1 \frac{di_1}{dt} = -L_2 \frac{di_2}{dt} \qquad ...(3)$$

If L be the equivalent inductance, then

$$e = -L \frac{di}{dt} \qquad ...(4)$$

In view of (3) and (4) equation (2) gives

$$\frac{e}{L} = \frac{e}{L_1} + \frac{e}{L_2}$$

or

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

$$L = \frac{L_1 L_2}{L_1 + L_2}$$

This is the required expression.

Case (ii) When inductors are at small distances, then there is a mutual inductance M between them. If i_1 and i_2 are instantaneous currents, then

$$i = i_1 + i_2$$

and in parallel e.m.f. is same, therefore

$$e = -L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}, \qquad e = -L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}$$

(+) sign occurs when coupling of inductors is positive and negative sign when coupling is negative

Solving these equations

$$e = -\frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M} \frac{di}{dt}$$

if L is equivalent inductance, then $e = -L \dfrac{di}{dt}$

Comparing last two equations, we get $L = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}$

7.11. Coupling of Two Coils With Flux Linkage : Coefficient of Coupling.

Let there be two coils C_1 and C_2 having self-inductance L_1 and L_2, and number of turns N_1 and N_2 respectively. Let the two coils be coupled together in such a way that whole of the flux from one coil links with the other coil. Let i_1 be the current flowing through coil C_1 and ϕ_1 the flux linked with each turn of C_1.

According to definition, self-inductance of coil C_1 is given by

$$N_1\phi_1 = L_1 i_1 \quad i.e. \quad L_1 = \frac{N_1\phi_1}{i_1}. \qquad \text{...(i)}$$

As whole of the flux from one coil is linked with the other coil; the flux linked with each turn of the coil C_2 is also ϕ_1. Therefore the flux linked with the coil C_2 due to the current flowing in coil C_1 is $N_2\,\phi_1$. Now according to definition, mutual inductance M between two coils is given by

$$N_2\phi_1 = Mi_1 \quad i.e. \quad M = \frac{N_2\phi_1}{i_1} \qquad \text{...(2)}$$

From eqs. (1) and (2),

$$M = \frac{N_2}{N_1}L_1 \qquad \text{...(3)}$$

Similarly if i_2 is the current fowing through the coil C_2 and ϕ_2 is the flux linked with each turn of coil C_2, then according to definitions.

Self-inductance L_2 is given by

$$N_2\phi_2 = L_2 i_2 \quad i.e. \quad L_2 = \frac{N_2\phi_2}{i_2} \qquad \text{...(4)}$$

Mutual inductance M is given by

$$N_1\phi_2 = Mi_2 \quad i.e. \quad M = \frac{N_1\phi_2}{i_2} \qquad \text{...(5)}$$

From eqs. (4) and (5),

$$M = \frac{N_1}{N_2}L_2. \qquad \text{...(6)}$$

Multiplying equation, (3) and (6), we get

$$M^2 = L_1L_2 \quad i.e. \quad M = \sqrt{(L_1L_2)} \qquad \text{...(7)}$$

Coefficient of Coupling. In practice, however, the condition that whole of the flux from one coil links with the other, is not satisfied. The quantity $\frac{M}{\sqrt{(L_1L_2)}} = K$ is termed as the *coefficient of coupling between* the coils. The closest coupling occurs when $K = 1$. When K is considerably less than unity, we say that the coils are loosely coupled.

7.12. Magnetic Energy Stored in Inductance

Consider a circuit containing a coil of inductance L and resistance R. Let E be the e.m.f. applied. Let i be the current flowing in the circuit at any time t. The potential difference across inductance $L\frac{di}{dt}$ opposes the growth of current in the circuit, therefore the equation of e.m.f.'s is

$$E - L\frac{di}{dt} = Ri \quad i.e. \quad E = L\frac{di}{dt} + Ri$$

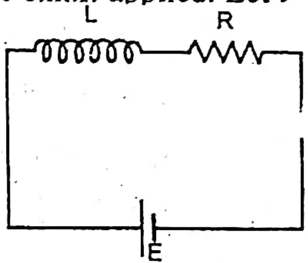


Fig. 7.38

The work done by the source of e.m.f. in increasing a small charge dq through the circuit in time dt is given by $dW = Edq = Eidt$

Using (1), we get

$$dW = \left(L\frac{di}{dt} + Ri\right)idt = Li\,di + i^2Rdt$$

$\therefore$ Total work done by the source of e.m.f. in establishing the current in circuit from 0 to I in time interval 0 to T is

$$W = \int_0^I Li\,di + \int_0^T i^2 R\,dt = \frac{1}{2}LI^2 + R\int_0^T i^2\,dt$$

The first tem in R.H.S. denotes the energy stored in inductance and the second term represents the energy dissipated as heat in the resistance.

That is the magnetic energy stored in inductance $= \frac{1}{2}LI^2$.

Magnetic Energy stored in a series of Inductances :

Let us consider a number of coils connected in series. The initial current in each coil is zero and final equilibrium current in k th circuit is I_k. Let i_k be the current in k th circuit at any time t. The current in each circuit $i_k(t)$ and the flux through it $\phi_k(t)$ will be some fraction α of their final values I_k and Φ_k.

i.e. $\qquad i_k(t) = \alpha I_k \quad \text{and} \quad \phi_k(t) = \alpha\Phi_k$

The induced e.m.f. in kth circuit $e_k = \dfrac{d\phi_k(t)}{dt}$

∴ Total work done by applied source for k th circuit.

$$W_k = \int_0^T e_k\, i_k(t)\,dt = \int_0^T \frac{d(\alpha\Phi_k)}{dt}\alpha I_k\,dt = I_k\Phi_k\int_0^T \alpha\frac{d\alpha}{dt}\,dt$$

$$= I_k\Phi_k\int_0^1 \alpha\,d\alpha = \frac{1}{2}I_k\Phi_k \qquad \text{...(2)}$$

On summing for all circuits $\qquad W = \dfrac{1}{2}\sum\limits_k I_k\Phi_k \qquad$ ...(3)

The flux linked through kth circuit is $\qquad \Phi_k = L_k I_k + \sum\limits_{k-1} M_{kj} I_j \qquad$ ...(4)

where M_{kj} is mutual inductance between jth and kth coils. Substituting value of Φ_k from (4) in (3), we get

$$W = \tfrac{1}{2}L_k I_k^2 + \tfrac{1}{2}\sum_j\sum_k M_{kj} I_k I_j \qquad \text{...(5)}$$

For a pair of coils '1' and '2', we have

$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M I_1 I_2 \qquad \text{...(6)}$$

where $\qquad M = M_{12} = M_{21}$

Equation (5) may be put in a more standard form as follows :

For k th circuit, the flux linked $\qquad \Phi_k = \int_k \mathbf{B}\cdot d\,\mathbf{S}_k$

If $\mathbf{A}$ is the magnetic vector potential, $\mathbf{B} = \text{curl}\,\mathbf{A}$

∴ $\qquad \Phi_k = \int_k \text{curl}\,\mathbf{A}\cdot d\,\mathbf{S}_k$

Using Stoke's theorem to change surface integral into line-intgral, we get

$$\Phi_k = \oint_k \mathbf{A}\cdot d\,\mathbf{l}_k \qquad \text{...(7)}$$

∴ Equation (3) takes the form

$$W = \tfrac{1}{2}\sum_k I_k\Phi_k = \tfrac{1}{2}\sum_k I_k\oint_k \mathbf{A}\cdot d\,\mathbf{l}_k \qquad \text{...(8)}$$

Now if make a transition from discrete to continuum using the relation $I d\,\mathbf{l} = \mathbf{J}\,d\tau$, $d\tau$ being a small volume element, we get

$$W = \tfrac{1}{2}\int_V (\mathbf{A}\cdot\mathbf{J})\,d\tau \qquad \text{But} \qquad \mathbf{J} = \nabla\times\mathbf{H}$$

∴
$$W = \tfrac{1}{2}\int_V [\mathbf{A}\cdot(\nabla\times\mathbf{H})]\,d\tau \qquad \text{...(9)}$$

Using vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{H}$$

we get
$$\mathbf{A}\cdot(\nabla\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{A} - \nabla\cdot(\mathbf{A}\times\mathbf{H})$$

So that equation (9) may be expressed as

$$W = \tfrac{1}{2}\int_V \{\mathbf{H}\cdot(\nabla\times\mathbf{A})\}\,d\tau - \tfrac{1}{2}\int_V \{\nabla\cdot(\mathbf{A}\times\mathbf{H})\}\,d\tau \qquad \text{...(10)}$$

Using Gauss divergence theorem to change volume integral in second term on R.H.S. of (10), we get

$$W = \tfrac{1}{2}\int_V (\mathbf{H}\cdot\mathbf{B})\,d\tau - \tfrac{1}{2}\int_V (\mathbf{A}\times\mathbf{H})\cdot d\mathbf{S} \qquad \text{...(11)}$$

As volume integral is taken for all space, extending the volume to infinity, the surface integral must be taken over the sphere at infinity. Because **H** and **A** fall off repidly at large distance $\left(H \propto \frac{1}{r^3}, A \,\alpha\, \frac{1}{r^2}\right)$, the surface integral vanishes as $r \to \infty$.

Hence equation (11) becomes

$$W = \tfrac{1}{2}\int_V (\mathbf{H}\cdot\mathbf{B})\,d\tau \qquad \text{...(12)}$$

If magnetic energy is distributed uniformly throughout in the region under consideration, then the magnetic energy stored per unit volume *i.e.* magnetic energy density

$$u_m = \frac{W}{V} = \tfrac{1}{2}\mathbf{H}\cdot\mathbf{B} \qquad \text{...(13)}$$

As $\mathbf{B} = \mu\mathbf{H}$ for linear medium

∴
$$u_m = \tfrac{1}{2}\mu H^2 \quad \text{or} \quad \frac{B^2}{2\mu} \qquad \text{...(14)}$$

This relation has been found by assuming that medium is linear. If the medium is non-linear, then equation (13) should be used.

Ex. 28. *A current of* 5 *amp produces a flux of* 2×10^{-3} *weber through a coil of* 500 *turns. Calculate the energy stored in the magnetic field.* **(Rohilkhand 1988)**

Sol. The self-inductance of coil

$$L = \frac{N\phi}{I} = \frac{500\times2\times10^{-3}\ \text{weber}}{(5\ \text{Amp})} = 0{\cdot}2\ \text{Henry}$$

∴Energy stored in magnetic field $= \frac{1}{2}LI^2 = \frac{1}{2}\times0{\cdot}2\times5^2 =$ **2·5 joule.**

Ex. 29. *The current in a coil of self-inductance* 2·0 *henry is increasing according to* $L = 2\sin t^2$ *ampere. Find the amount ofenergy spent during the period when the current changes from* 0 *to* 2 *ampere.* **(Roorkee 91)**

Solution.
$$i = 2\sin t^2 \qquad \text{...(1)}$$

Induced e.m.f. $= -L\dfrac{di}{dt}$

Work done during small time interval dt

$$dW = e\,i\,dt$$

∴ Total work done in raising the current against induced e.m.f. given time interval 0 to t

$$W = -\int_0^t ei\,dt = +\int_0^t L\frac{di}{dt}i\,dt$$

$$= \int_0^2 L\,i\,dt = L\left[\frac{i^2}{2}\right]_0^2 = \frac{1}{2}L(2)^2 = 2L = 2\times 2 = 4 \textbf{ joule.}$$

Ex. 30. *A circular loop of wire of radius 4 cm carries a current of 80 Amp. Calculate the energy at the centre of the loop.*

Sol. The magnetic field at the centre of loop is

$$B = \frac{\mu_0 i}{2r}$$

∴ Energy density at the centre of loop is

$$u_m = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2r}\right)^2 = \frac{\mu_0 i^2}{8r^2}$$

Here current $i = 80$ Amp, radius $r = 4$ cm $= 0{\cdot}04$ m.

∴ $$u_m = \frac{4\pi\times 10^{-7}\times(80)^2}{8\times(0{\cdot}04)^2} = \textbf{0·628 Joules.}$$

QUESTIONS AND PROBLEMS

1. (a) Give a brief account of the phenomenon of electromagnetic induction describing two experiments to demonstrate it.
   (b) State and explain Faraday's laws of electromagnetic induction.
   (Rohilkhand 1988; Meerut 82; Lucknow 72; Banaras 71)
2. State and explain the laws of Faraday and Lenz relating to electromagnetic induction. Deduce the relation between the quantity of electricity flowing through a circuit and the magnetic flux change producing it. **(Kanpur 1992)**
3. A conducting rod is moving with a velocity through a uniform magnetic field of induction **B**. Show that there appears inside the rod an electric field.
   $\mathbf{E} = -(\mathbf{v}\times\mathbf{B})$ **(Rohilkhand 1984)**
4. State the laws of electromagnetic induction. Deduce Faradays laws from Lorentz force.
5. (a) Deduce an expression for e.m.f. induced in varying magnetic field. What is the difference between electric field so produced and that produced by charges.
   (b) Can an electric field for which curl $E \neq 0$ be produced ? If yes how can it be done ? Explain.
6. Deduce Faraday's law of electromagnetic induction in the form curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. **(Meerut 1986; Lucknow 96)**
7. (a) Explain the meaning of self and mutual inductance ?
   (b) Derive an expression for self induction of a solenoid.
   (c) In figure, there are shown two conductors *A* and *B* in the form of wire loops located in uniform magnetic field which acts in a direction perpendicular to the plane of loops and directed away from the reader. The magnetic field now starts diminishing. Find the direction of induced current in these loops.
   (Rohilkhand 1986)

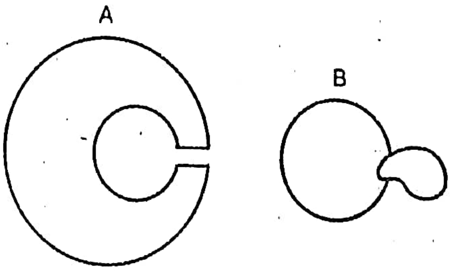


Fig. 7.39

8. Define coefficient of self-induction. Discuss the physical significance of coefficient of self-induction of two parallel wires. **(Kanpur 1997)**

9. A conducting rod is moving with velocity v through a uniform magnetic field B. Find expressions for

   (i) electric field produced inside the rod.

   (ii) Induced e.m.f. between the ends of the rod of length l. **(Agra 1992)**

10. Find the self inductance of a long solenoid and calculate the energy stored per unit volume in magnetic field. **(Meerut 1985)**

11. What do you mean by self and mutual inductances ? Calculate the energy stored in an inductor of inductance L., carrying a steady current I. **(Agra 1992)**

12. (a) Derive an expression for the energy stored in a magnetic field of induction B per unit volume in free space

    Or

    Show that the energy stored in a magnetic field per unit volume in free space is $(B^2/2\mu_0)$. **(Rohilkhand 1990)**

    (b) Derive an expression for the energy stored in the inductances in series.

13. A railway line 1.2 metre wide runs along the magnetic meridian. The vertical component of earth's magnetic field is 0·5 oersted. Calculate the e.m.f. in volts that will exist between rails when a train runs on the line at a speed of 60 km/hour.

    [Ans. 1 mV] **(Lucknow 1972)**

14. A circus trapeze consists of a metal bar 70 cm. long suspended by two ropes, each of length 700 cm. The bar is drawn back until it is at the same level as the supporting hooks and is released. Compute in volts the maximum e.m.f. induced in the bar if the vertical component of earth's magnetic field is 0·4 gauss at the locality in question. **(Agra 1972)**

    [Ans. 0·33 mV]

15. A car having a vertical aerial of 1·0 metre is moving with a uniform speed of 80 km/hour in the east-west direction. If the value of $H = 0{\cdot}36$ oersted, calculate the e.m.f. induced in the aerial. **(Kanpur 1973)**

    [Ans. 8×10^{-4} V]

16. A long solenoid of cross-sectional area 4×10^{-4} m^2 is wound with 1000 turns/metre and the current in its windings is increased at the rate of 100 amp/sec. Deduce the e.m.f. induced in it.

    [Ans. $16\,\pi \times 10^{-6}$ volts]

17. A metal air-plane propeller rotates at 20000 rotations per minute. What potential difference exists between the hub and the tip of the propeller when the air plane is flying horizontally in the direction due magnetic north ? The propeller is 3 m in total length. Take the horizontal component of earth's magnetic field to be 10^{-5} weber/m^2.

    [Ans. $2{\cdot}4 \times 10^{-2}$ volts]

18. A copper disc of radius 10 cm. rotates 20 times per second with is plane perpendicular to a uniform magnetic field. If the e.m.f. induced between the centre and edge of the disc in 3·14 milli volt, calculate the magnetic field strength.

    [Ans. 5×10^{-3} weber/m^2]

19. A disc of radius 8 cm is rotated inside a long solenoid of 5000 turns/metre, the axis of rotation coinciding with the axis of the solenoid. It the disc makes 600 r.p.m. and the current in the solenoid is 1 amp., find the potential difference between the centre and the circumference of the disc.

    [Ans. 1·203 milli-volts]

20. A metal disc of radius 10 cm. rotates at 3000 revolutions/minute perpendicular to a magnetic field of induction 0·1 weber/m^2. An external circuit is connected by brushes to the centre and rim of this disc. Calculate the induced e.m.f.

    [Ans. 0·151 volts]

21. A flat square coil of 10 turns has sides of length 12 cm. The coil rotates in a magnetic field whose flux density is 0·025 weber/m^2. Calculate (i) the angular velocity of the coil if the maximum e.m.f. produced is 20 milli-volts, (ii) average e.m.f. at this velocity.

    [Ans. (i) 0·885 rev/sec. (ii) zero]

22. Calculate the e.m.f. induced between the ends of an axle, 1·75 metre long of a railway carriage travelling on level ground with a uniform speed of 50 km/hour. The vertical component of earth's field is 0·5 oersted.
[**Ans.** 1·21 milli-volts] **(Meerut 1982)**

23. The magnetic flux through a circular coil, perpendicular to its plane is varying according to the following relation

$$\phi = (6t^2 + 7t + 1)$$

where ϕ is in milli-webers and t is in seconds. Calculate the e.m.f. induced in the coil when $t = 2$ sec.
[**Ans.** 0·031 volts]

24. A metal rod one meter long rotates about an axis through one end perpendicular to the rod, with an angular velocity of 12 rad/sec. The plane of rotation of the rod is perpendicular to a uniform magnetic field of 3 weber/m^2. What is the e.m.f. induced between the ends of the rod ?
[**Ans.** 216 volts]

25. Calculate the coefficient of self-induction of a coil having 1000 turns and a length of one metre. The area of cross-sections is 7 sq. cm and the permeability of the core is 1000.
[**Ans.** 0·88 henry]

26. Find self inductance of a solenoid of 1000 turns would on a paper tube of 25 cm length and 4 cm radius.
[**Ans.** 25 milli henry] **(Rohilkhand 1984)**

27. A solenoid of length 6·28 metre, area of cross-section 8·0 sq. cm and number of turns 1400 has a core of magnetic pemeability 900. Calculate the coefficient of self-induction of the solenoid. **(Kanpur 1998, 71)**
[**Ans.** 0·28 H]

28. If the current in a coil of self-inductance 5 milli henry increases uniformly from 0 to 1 amp. in 0·1 second, find the magnitude and direction of the self-induced e.m.f.
[**Ans.** 0·005 volt]

29. A straight coil of circular cross-section having 1000 tums and 4 sq. cm. face aea is placed with its axes parallel to a magnetic field which decreases by 100 gauss (= 100×10^{-4} weber/m^2) in 0·001 sec. Determine the e.m.f. induced in the coil. If the length of the coil be 20 cm, determine its self-inductance.
[**Ans.** 4 mV 0·0025 henry]

30. For a toroidal solenoid of rectangular cross-section, it is given that the total number of tums in the solenoid is 1000 and the outer and inner diameters of the ring are 20 cm. and 10 cm. respectively. If the thickness of the cylindrical surface is 1·0 cm, find the self inductance of the toroid.
[**Ans.** 1·386 milli henries]

31. Two inductances L_1 and L_2 of values 0·01 and 0·03 henry respectively are given. When connected in series with coils 'aiding' the resultnat inductance is 0·06 henry. What is the value of mutual inductance ? Had they been connected with the coils 'opposed', what would have the inductance of the combination ?
[**Ans.** $M = 0{\cdot}01H$, $L = 0{\cdot}02H$]

32. The ends of two fixed coils are A, B and C, D. If B is connected to C, the inductance of the combination is 10 mH. If again A is connected to D the inductance of the combination decreases to 8 mH. Calculate the coefficient of mutual inductance after establishing the required relation.
[**Ans.** 0·5 mH]

33. When the current in one circuit changes from 2 amp. to 1 amp. in 0·1 sec., an induced e.m.f. of 0·1 volt is produced in the nearby circuit. Calculate the value of the coefficient of mutual inductance of the two circuits. **(Agra 1993)**
[**Ans.** $M = 0{\cdot}01H$]

34. A coil of 100 tums and 1 cm radius is kept co-axially within a long solenoid having 8 turns per cm. and 5 cm. radius. Calculate the mutual inductance in henry. **(Agra 1990)**
[**Ans.** 31·54 micro henry]

35. For a toroidal solenoid of circular cross-section having two windings, it is given that the total number of tums in one winding are 1000 and in the other are 400. If the circumference of the ring is 40 cm. and the area of cross-section 3 cm^2, find the mutual inductance of the solenoid.
[**Ans.** $3{\cdot}768 \times 10^{-4}$ henry]

36. A solenoid of length 50 cm. wound with 50 turns of wire has radius of 4 cm. A coil of 700 turns is wound on the middle part of the solenoid. Calculate—(i) self inductance of the solenoid, and (ii) mutual inductance of the solenoid and coil.

    [**Ans.** 316·6 mH, 44·18 milli-henry]

37. What is the magnetic energy stored in a uniform magnetic field of induction 1 weber/m^2 in a volume of 10^{-3} metre^3?

    [**Ans.** 3·94 joules]

38. A current of 190 amperes produces a flux of 10 webers through a closely wound coil of 200 turns. Calculate the energy stored in the magnetic field.

    [**Ans.** 10^4 joules]

39. Calculate the energy in joules stored in a magnetic field of a solenoid 30 cm. long, 3 cm. in diameter, wound with 100 turns of wire and carrying a current of 10 amp.

    [**Ans.** $0{\cdot}148 \times 10^{-2}$ joules]

40. A magnetic field of intensity 5×10^{-5} weber/m^2 acts at right angles to a coil of radius 10 cms. with 200 turns. The resitance of coil is 5 ohm. Find the induced e.m.f. and the charge when the coil is removed from the field in one-tenth of a second.

    [**Ans.** e = 3·14 mV, 62·8 μC] **(Rohilkhand 1988)**

41. A solenoid of length 1 m. cross-sectional area 10 cm^2, having 1000 turns, has wound over its centre a small coil of 20 turns. Find the mutual inductance of this system in micro-henries.

    [**Ans.** 25·12 μ*H*] **(Rohilkhand 1987)**

42. A railway track has two parallel rails at separation 100 cm. If a train passes at a speed of 90 km/hour. Calculate the e.m.f. induced between the rails. Horizontal component of earth's field is 0·5 gauss and angle of dip $\phi = 45°$.

    [**Ans.** 1·25 mV] **(Garhwal 1995)**

43. What is the maximum electromotive force induced in a coil of 4000 turns and average radius 12 cm, rotating at 30 rev/sec in earth's magnetic field when the field intensity is $0{\cdot}5 \times 10^{-4}$ weber/m^2.

    [**Ans.** 0·567 V] **(Meerut 1986)**

44. A fan blade of length 2*a* rotates with a frequency *f* cycles/sec perpendicular to magnetic field *B*. Find the potential difference between the centre and end of the blade.

    [**Ans.** $4\pi B f a^2$] **(Meerut 1985)**

8

Maxwell's Equations and Electromagnetic Waves

8.1. Introduction

In the preceding chapters we have dealt with *steady state problems* in electrostatics and magnetostatics treating electric and magnetic phenomenon independent of each other. The only link between them was the fact that electric currents which produced magnetic fields are basically electric in nature, being charges in motion. Now if we wish to consider more general problems in which field quantities may depend upon time, the almost independent nature of electric and magnetic phenomenon disappears. Time-varying magnetic fields give rise to electric field and vice-versa. We then must speak of *electromagnetic fields* rather than electric and magnetic fields. The behaviour of time dependent electromagnetic fields is described by a set of equations known as *Maxwell's equations.* These equations are mathematical abstractions of experimental results.

In this chapter we shall steady the formulation of *Maxwell's equations* along with their general properties and the basic conservation law of charge and energy.

8.2. Equation of Continuity

Conservation of charge : According to principle of conservation of charge *the net amount of charge an isolated system remains constant.* For generality let us assume that the charge density is a function of time. Then the principle of conservation of charge may be stated as follows :

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in such a manner that the time rate of increase of charge within the volume equals the net rate of flow charge into the volume. This statement of conservation of charge in a medium may be expressed by the equation of continuity which may be derived as follows :

Let S be the surface enclosing a volume V and let $d\mathbf{S}$ be a small element of this surface. The direction of $d\mathbf{S}$ is taken to be that of the outward normal. If $\mathbf{J}$ is the current density (*i.e.* current per unit area placed normal of direction of current flow) at a point on surface element $d\mathbf{S}$, then $\mathbf{J} \cdot d\mathbf{S}$ represents the charge per unit time leaving volume V across $d\mathbf{S}$. Therefore the *time rate at which charge leaves the volume* V bounded by entire surface S is given by

$$\iint_S \mathbf{J} \cdot d\mathbf{S}.$$

If q is charge contained is V, then according to charge conservation law, the above integral must be equal to $-dq/dt$, where dq/dt represents the time rate of flow of charge into V, thus

$$\iint \mathbf{J} \cdot d\mathbf{S} = -\frac{dq}{dt}. \qquad \ldots(1)$$

But $$q = \iint_V \int \rho \, dv,$$

where ρ is the charge density and dv is an element of volume.

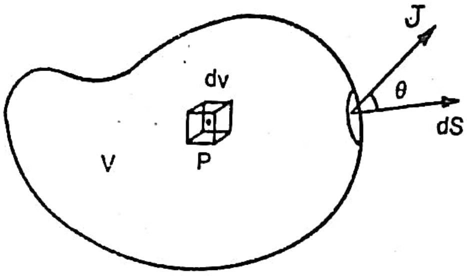


Fig. 8.1

Therefore equation (1) takes the form

$$\int_S \int \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iint_V \int \rho \, dv.$$

Since the order of differentiation and integration is interchangeable, therefore

$$\int_S \int \mathbf{J} \cdot d\mathbf{S} = -\iiint \frac{\partial \rho}{\partial t} \, dv, \qquad \ldots(2)$$

But from Gauss divergence

$$\int_S \int \mathbf{J} \cdot d\mathbf{S} = \iint_V \int \text{div } \mathbf{J} \, dv \qquad \ldots(3)$$

Comparing (2) and (3), we get

$$\int_V \iint \text{div } \mathbf{J} \, dv = -\iiint \frac{\partial \rho}{\partial t} \, dv.$$

or $$\int_V \iint \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dv = 0.$$

Since volume is arbitrary, therefore integrand must be zero

i.e. $$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

This is the required *equation of continuity and expresses the conservation of charge.*

The current is called stationary if there is no accumulation of charge at any point *i.e.* for stationary current $\partial \rho / \partial t = 0$ at all points. Therefore the criterion for stationary flow is

$$\text{div } \mathbf{J} = \nabla \cdot \mathbf{J} = 0. \qquad \ldots(5)$$

8.3. Maxwell's Postuiate ; Displacement Current.

From Amperes circuital law, we have

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \qquad \ldots(1)$$

(refer equation (3) of section 5.8)

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}. \qquad \ldots(3)$$

∴ $$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}. \qquad \ldots(4)$$

But from Stoke's theorem

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \text{curl } \mathbf{H} \cdot d\mathbf{S}$$

Comparing (3) and (4), we get

$$\int_S \text{curl } \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

i.e. $$\int_S (\text{curl } \mathbf{H} - \mathbf{J}) \cdot d\mathbf{S} = 0. \qquad \ldots(5)$$

As the surface is arbitrary, therefore integrand must vanish *i.e.*

$$\text{curl}\,\mathbf{H} - \mathbf{J} = 0$$

or

$$\text{curl}\,\mathbf{H} = \mathbf{J}. \qquad ...(6)$$

Let us examine the validity of this equation for time-varying fields. Since div of curl of any vector quantity is always zero. therefore div curl **H** = 0. Then equation (6) implies

$$\text{div}\,\mathbf{J} = 0. \qquad ...(7)$$

Now continuity equation is

$$\text{div}\,\mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \qquad ...(8a)$$

i.e.

$$\text{div}\,\mathbf{J} = -\frac{\partial \rho}{\partial t}. \qquad ...(8b)$$

According to this equation div $\mathbf{J} = 0$ only if $\partial\rho/\partial t = 0$ *i.e.*, charge density is static. Thus we conclude that Ampere's equation (1) is valid only for steady state conditions and is insufficient for the cases of time-varying fields. Hence to include time-varying fields Ampere's law must be modified. Maxwell investigated mathematically how one could alter Ampere's equaion (1) so as the make it consistent with the equation of continuity. Maxwell assumed that the definition for current densiy **J** is incomplete and hence some thing say, $\mathbf{J}_d$ must be added to it. Then total current density which must be solenoidal, becomes $C = \mathbf{J} + \mathbf{J}_d$.

Using this postulate, equation (6) becomes

$$\text{curl}\,\mathbf{H} = \mathbf{C} = \mathbf{J} + \mathbf{J}_d \qquad ..(9)$$

In order to identify $\mathbf{J}_d$ let us take the divergence of equation (9) :

$$\text{div curl}\,\mathbf{H} = \text{div}\,(\mathbf{J} + \mathbf{J}_d)$$

But div curl **H** = 0 since div of curl of any vector is always zero ; therefore we get

$$\text{div}\,(\mathbf{J} + \mathbf{J}_d) = 0$$

or

$$\text{div}\,\mathbf{J} + \text{div}\,\mathbf{J}_d = 0$$

or

$$\text{div}\,\mathbf{J}_d = -\text{div}\,\mathbf{J}. \qquad ...(10)$$

But div $\mathbf{J} = -\partial\rho/\partial t$ from equation of continuity, hence equaion (10) becomes

$$\text{div}\,\mathbf{J}_d = \frac{\partial \rho}{\partial t} \qquad ...(11)$$

But Gauss theorem in differential form gives

$$\text{div}\,\mathbf{D} = \rho \qquad ...(12)$$

Using this equation (11) may be written as

$$\text{div}\,\mathbf{J}_d = \frac{\partial}{\partial t}(\text{div}\,\mathbf{D})$$

$$= \text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

This gives

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \qquad ...(13)$$

Therefore the modified from the Ampere's law is

$$\boxed{\text{curl}\,\mathbf{H} = \mathbf{J} + \mathbf{J}_d = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \qquad ...(14)$$

The term which Maxwell added to Ampere's law to include time varying fields is known as *displacement current* because it arises when electric displacement vector **D** changes with time. By addition this term

Maxwell assumed that this term (displacement current) is as effective as the *conduction current* **J** for producing magnetic field.

Characteristics of displacement current

(i) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current since it is not linked with the motion of charges. For example displacement current has a finite value even in a perfect vaccum where there are no charges at all.

(ii) The magnitude of displacement current is equal to rate of change of electric displacement vector *i.e.* $\mathbf{J}_d = \partial\mathbf{D}/\partial t$.

(iii) Displacement current serves the purpose to make the total current continuous across the discontinuity in a conduction current. As an example, a battery charging a capacitor produces a closed current loop in terms of total current $\mathbf{J}_{total} = \mathbf{J} + \mathbf{J}_d$.

(iv) Displacement current in a good conductor is negligible as compared to the conduction current at any frequency less than optical frequencies ($\approx 10^{15}$ Hertz).

With the postulate of displacement current Maxwell was able to derive his theory of electromagnetic waves. We may consider the experimental observation of such waves, with the properties predicted, as the experimental basis for *Maxwell's* postulate. Furthermore we shall show that this postulate has a reasonable physical interpretation.

8.4. Physical Interpretation of Maxwell's Postulate

Maxwell's original explanation of displacement current was puzzling, therefore for convenience we shall consider another point of view.

The modified form of Ampere's law may be expressed as

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}. \qquad ...(1)$$

Just as $\oint \mathbf{E}\cdot d\mathbf{l}$ represents electromotive force in electrostatics, the magnetomotive force (m.m.f.) around the path C is

$$\text{m.m.f.} = \oint_C \mathbf{H}\cdot d\mathbf{l} \qquad \text{(refer section 5.29).} \; ...(2)$$

Substituting this into (1), we get

$$\text{m.m.f.} = \oint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}. \qquad ...(3)$$

Equations (1) and (2) indicate that there are two ways of producing a magnetic intensity, one with a ordinary conduction current density **J**, as observed by Oersted and postulated by Ampere, and the other by means of time varying electric displacement, as postulated by Maxwell. Since $\mathbf{D} \propto \mathbf{E}$ for air or vacuum, we may say that *a changing electric field gives rise to a magnetic field,* This is the converse of Faraday's discovery that a changing magnetic field gives rise to an electric field.

If we take the case where $\mathbf{J} = 0$ everywhere, then equation (1) becomes

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \oint_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S}, \quad (\mathbf{J} = 0). \qquad ..(4)$$

This equaion indicates that time varying electric displacement produces a magnetic field : thus varifying Maxwell's postulate.

Direct observation of a magenetic field produced by a changing electric field is difficult. We can not look for an *induced magnetic current* because there are neither free poles nor conductors for magnetic currents. Also we cannot maintain a constant value of $\partial\mathbf{D}/\partial t$ long enough to measure the resulting magnetic field, as we would that due to a steady current. Thus Maxwell's postrlate is not as susceptible to direct experimental verification as is Faraday's law. That is why it was the last fundamental law of classical electromagnetism to discover.

If Maxwell's postulate is converse of Faraday's law, then the question arises why is there not converse of Ampere's law ? If we put this quesiton in slight different manner we may say, why does equation (3) contains two terms, while the corresponding equation for e.m.f.

$$e = -\frac{d\phi}{dt} = -\iint \mathbf{B} \cdot d\mathbf{S} \qquad ...(5)$$

has but one term ?

The answer is that the term missing in equation (5) involves *a current density of magnetic current* or a flow of magnetic poles of one sign and since isolated poles of one sign and magnetic currents due to them have no physical significance, therefore the term analogous to **J** in equation (3) and the converse of Ampere's law do not exist. Therefore we must realise the fact that the fundamental role of electric charges leads to certain lack of symmetry in our equations.

8.5. Maxwell's Equations and Their Empirical Basis.

There are four fundamental equations of electromagnetism known as *Maxwell's equations* which may be written in *differential form* as

1. $\nabla \cdot \mathbf{D} = \rho$ (Differential form of Gauss law in electrostatics)
2. $\nabla \cdot \mathbf{B} = 0$ (Differential form of Gauss law in magnetostatics)
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Differential form of Faraday's law of electromagnetic induction
4. $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt}$ Maxwell's modification of Ampere's law.

In above equations the notation have the following meanings :

D = electric displacement vector in coulomb/m^2.
ρ = charge density of coul/m^3.
B = magnetic induction in weber/m^2.
E = electric field intensity in volt/m or n/coul.
H = magnetic field intensity in amp/m-turn.

Each of Maxwell's equations represents a generalisation of certain experimental observations : Equations (1) represents the differential form of Gauss's law in electrostatics which in turn derives from Coulomb's law. Equation (2) represents Gauss's law in magnetostatics which is usually said to represent the fact that isolated magnetic poles do not exist in our physical world. Equation (3) represents differential form of Faraday's law of electromagnetic induction and finally equation (4) represents Maxwell's modification of Ampere's law to include time varying fields.

It is clear that the Maxwell's equations represent mathematical expression of certain experimental results. As already pointed out these equations can not be verified directly, however their application to any situation can be verified. As a result of extensive experimental work, Maxwell's equations are now known to apply to almost all macroscopic situations and they are usually used, must like conservation of momentum, as guiding principles.

8.6. Derivation of Maxwell's Equation

1. Derivation of first Equation Div $\mathbf{D} = \nabla \cdot \mathbf{D} = \rho$

Let us consider a surface S bounding a volume V in a dielectric medium. In a dielectric medium total charge consists of free charge plus polarisation charge. If ρ and ρ_p are the charge densities of free charge and polarisation charge at a point in a small volume element dV, then *Gauss' law* can be expressed as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V (\rho + \rho_p)\, dV$$

But polarisation charge density $\rho_p = -\operatorname{div} \mathbf{P}$, therefore above equaions takes the form

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V (\rho - \operatorname{div} \mathbf{P})\, dV$$

i.e.

$$\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_V \rho\, dV - \int_V \operatorname{div} \mathbf{P}\, dV$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \operatorname{div} (\varepsilon_0 \mathbf{E})\, dV = \int_V \rho\, dV - \int_V \operatorname{div} P\, dV$$

i.e.

$$\int_V \operatorname{div} (\varepsilon_0 \mathbf{E} + \mathbf{P})\, dV = \int_V \rho\, dV \qquad \ldots(1)$$

But $\varepsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{D}$ = electric displacement vector.

Therefore equation (1) becomes

$$\int_V \operatorname{div} \mathbf{D}\, dV = \int_V \rho\, dV$$

$$\int_S \operatorname{div} (\mathbf{D} - \rho)\, dV = 0$$

Since this equation is true for all volumes, therefore the integrand in this equation must vanish *i.e.*

$$\operatorname{div} \mathbf{D} - \rho = 0$$

or

$$\operatorname{div} \mathbf{D} = \rho \quad i.e.\ \nabla \cdot \mathbf{D} = \rho$$

2. Derivation of Second Equation div $\mathbf{B} = \nabla \cdot \mathbf{B} = 0$

Since isolated magnetic poles and magnetic currents due to them have no physical significance : therefore magnetic lines of force in general are either closed curves or go off to infinity. Consequently the number of magnetic lines of force entering any arbitrary closed surface is exactly the same as leaving it. It means that the flux of magnetic induction **B** across any closed surface is always zero, *i.e.*

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \operatorname{div} \mathbf{B}\, dV = 0$$

As the surface bounding the volume is arbitrary, therefore this equation holds only if the integrand vanishes *i.e.*

$$\operatorname{div} \mathbf{B} = 0 \text{ or } \nabla \cdot \mathbf{B} = 0.$$

Note. For an alternative derivation of div $\mathbf{B} = 0$ refer section 5.10 of chapter 5.

3. Derivation of third equation

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

According to Faraday's law of electromagnetic induction it is known that e.m.f. induced in a closed loop is defnied as negative rate of change of magnetic flux *i.e.*

$$e = -\frac{d\phi}{dt}$$

But magnetic flux $\phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ where S is any surface having loop as boundary

$$e = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \qquad \text{....(2)}$$

(Since surface is fixed in space, hence only **B** changes with time).

But e.m.f. 'e' can also be computed by calculating the work done in carrying a unit charge round the closed loop C. Thus if **E** is the electric field intensity at a small element $d\mathbf{l}$ of loop, we have

$$e = \int_C \mathbf{E} \cdot d\mathbf{l} \qquad \text{...(3)}$$

Comparing equations (2) and (3), we get

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \qquad \text{...(4)}$$

Using Stoke's theorem to change line integral into surface integral, we get

$$\int_S \text{curl } \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{E}$$

or

$$\int_S \left(\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} = 0. \qquad \text{...(5)}$$

Since surface is arbitrary, therefore equation (5) holds only if the intergrand vanishes *i.e.*

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

or

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \textit{i.e.} \quad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}.$$

4. Derivation of fourth equation

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

For its derivation refer section 8.3.

8.7. Maxwell's Equations in Integral Form

Physical significance of Maxwell's Equations.

By means of Gauss' and Stoke's theorems we can put the field equations integral form and hence obtain their physical significance.

i. Maxwell's first equation is $\nabla \cdot \mathbf{D} = \rho$.

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla \cdot \mathbf{D}\, dV = \int_V \rho\, dV.$$

Changing volume integral in L.H.S. of above equation into surface integral by Gauss divergence theorem. We get

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dV. \qquad \text{...(1)}$$

where S is the surface which bounds volume V. Equation (1) represents Maxwell's first equation $\nabla \cdot \mathbf{D} = \rho$ in integral form. Since $\int_V \rho \, dV = q$, the net charge contained in volume V, therefore Maxwell's first equation signifies that :

The net outward flux of electric displacement vector through the surface enclosing a volume is equal to the net charge contained within that volume.

2. Maxwell's second equation is $\nabla \cdot \mathbf{B} = 0$.

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla \cdot \mathbf{B} = 0.$$

Using Gauss divergence theorem to change volume integral into surface integral, we get

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0. \qquad ...(2)$$

where S is the surface which bounds volume V. Equation (2) represents Maxwell's second equation in integral form and signifies that :

The net outward flux of magnetic induction **B** *through any closed surface is equal to zero.*

3. Maxwell's third equation is $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

Integrating above equation over a surface S bounded by a curve C, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}.$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral along the boundary of C, we get

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}. \qquad ...(3)$$

Equation (3) represents Maxwell's third equation in integral form and signifies that

The electromotive force $\left(\textit{e.m.f. } e = \int_C \mathbf{E} \cdot d\mathbf{l} \right)$ *around a closed path is equal to negative rate of change of magnetic flux linked with the path* $\left(\textit{since magnetic flux } \phi = \int_S \mathbf{B} \cdot d\mathbf{S} \right)$.

4. Maxwell's fourth equation is

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking surface integral over surface S bounded by curve C, we obtain

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{H}}{\partial t} \right) \cdot d\mathbf{S}$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral, we get

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \qquad ...(4)$$

This equation represents Maxwell's fourth equation in integral form and signifies that

The magnetomotive force $\left(\textit{m.m.f.} = \oint_C \mathbf{H} \cdot d\mathbf{l} \right)$ *around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.*

8.8. Maxwell's Equations in Some Particular Cases.

Maxwell's equation in differential form are

$$\left.\begin{aligned}\nabla\cdot\mathbf{D} &= \rho\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \text{and}\qquad \nabla\times\mathbf{H} &= \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}.\end{aligned}\right\}\qquad ...(A)$$

Case (i) Maxwell's Equations in Free Space. In free space the volume charge density ρ and current density **J** are zero, hence Maxwell's equations (A) take the form

$$\nabla\cdot\mathbf{D} = 0 \qquad ...(1)$$

$$\nabla\cdot\mathbf{B} = 0 \qquad ...(2)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad ...(3)$$

and

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \qquad ...(4)$$

with

$$\mathbf{D} = \varepsilon_0\mathbf{E} \quad\text{and}\quad \mathbf{B} = \mu_0\mathbf{H} \qquad ...(5)$$

where ε_0 and μ_0 are absolute permittivity and permeability of free space respectively.

Case (ii) Maxwell's equations in linear isotropic medium. In a linear isotropic medium, we have

$$\mathbf{D} = \varepsilon\mathbf{E} \quad\text{and}\quad \mathbf{B} = \mu\mathbf{H} \qquad ...(6)$$

where ε and μ are absolute permitivity and permeability of medium respectively.

Using equations (6), *Maxwell's equation (A) for linear isotropic medium take the form*

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon \qquad ...(7)$$

$$\nabla\cdot\mathbf{B} = 0 \qquad ...(8)$$

$$\nabla\times\mathbf{E} = \mu\frac{\partial\mathbf{H}}{\partial t} \qquad ...(9)$$

and

$$\nabla\times\mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t}+\mathbf{J}. \qquad ...(10)$$

Case (iii) Maxwell's equations for harmonically varying fields. If electromagnetic fields vary harmonically with time that we may write

$$\mathbf{D} = \mathbf{D}_0 e^{i\omega t} \quad\text{and}\quad \mathbf{B} = \mathbf{B}_0 e^{i\omega t} \qquad ...(11)$$

where $\mathbf{D}_0$ and $\mathbf{B}_0$ are peak values of **D** and **B** respectively. Equations (11) yield

$$\left.\begin{aligned}\frac{\partial\mathbf{D}}{\partial t} &= \mathbf{D}_0\, i\omega\, e^{i\omega t} = i\omega\mathbf{D}\\ \frac{\partial\mathbf{B}}{\partial t} &= \mathbf{B}_0\, i\omega\, e^{i\omega t} = i\omega\mathbf{B}\end{aligned}\right\}\qquad ...(12)$$

Using (12) Maxwell's equations (A) take the form

$$\nabla\cdot\mathbf{D} = \rho \qquad ...(13)$$

$$\nabla\cdot\mathbf{B} = 0 \qquad ...(14)$$

$$\nabla\times\mathbf{E} + i\omega\mathbf{B} = 0 \qquad ...(15)$$

$$\nabla\times\mathbf{H} - i\omega\mathbf{D} = \mathbf{J}. \qquad ...(16)$$

Ex. 1. *Starting from Maxwell's equations, establish Coulomb's law.* **(Meerut 1997)**

Solution. From Maxwell's equation

$$\text{div } D = \rho \qquad ...(1)$$

Taking its volume integral over a sphere of radius r, we get

$$\int_V \text{div}\,\mathbf{D}\,dV = \int_V \rho\,dv$$

As $\int_V \rho\,dv =$ Net charge enclosed by sphere

$$= q\ (\text{say})$$

$$\therefore \int_V \text{div}\ \mathbf{D}\ dV = q$$

changing vaolume integral into surafce integral, we get

$$\int_S \mathbf{D}\ .\ d\mathbf{S} = q$$

But $\mathbf{D} = \varepsilon\,\mathbf{E}$

$$\therefore \int_S \varepsilon\,\mathbf{E}\cdot d\mathbf{S} = q \Rightarrow \varepsilon \int \mathbf{E}\cdot d\mathbf{S} = q$$

$\Rightarrow$

$$\varepsilon\,E\,4\pi r^2 = q \Rightarrow E = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}$$

In vector form

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{q\,\mathbf{r}}{r^3}$$

Force on charge q_0 will be $\mathbf{F} = q\mathbf{E}$

$$= \frac{1}{4\pi\varepsilon}\frac{qq_0}{r^3}r$$

This is Coulomb's law.

Ex. 2. *Show that equation of continuity div* $\mathbf{J} + \partial\rho/\partial t = 0$ *is contained in Maxwell's equations.*

or

Starting from Maxwell's equations, establish the equation of continuity. **(Meerut 1997)**

Solution. From Maxwell's fourth equation, we get

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \qquad \text{...(1)}$$

Taking divergence of either sides of (1), we get

$$\text{div curl}\,\mathbf{H} = \text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right). \qquad \text{...(2)}$$

But div curl $\mathbf{H} = 0$ since divergence of curl of any vector always vanishes, therefore equation (2) gives

$$\text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

i.e.

$$\text{div}\,\mathbf{J} + \text{div}\,\frac{\partial \mathbf{D}}{\partial t} = 0$$

i.e.

$$\text{div}\,\mathbf{J} + \frac{\partial}{\partial t}(\text{div}\,\mathbf{D}) = 0,$$

(since space and time operations are interchangeable). Also from Maxwell's first equation div $\mathbf{D} = \rho$, therefore equation (8) gives

$$\text{div}\,\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

This is the required result.

Ex. 3. *Starting from Maxwell's equations.*

Curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and curl $\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ *respectively, show that div* $\mathbf{B} = 0$ *and* div $\mathbf{E} = \rho$.

Solution. Given curl $\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

Taking divergence of both sides

$$\text{div curl } \mathbf{E} = -\text{div}\left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

As div curl of any vector is zero and space and time operations are interchangeable,

$$\text{div}\frac{\partial \mathbf{B}}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t}(\text{div } \mathbf{B}) = 0$$

$$\Rightarrow \text{div } \mathbf{B} = \text{constant}$$

As isolated magnetic poles do not exist in nature, therefore for $\boxed{\text{div } \mathbf{B} = 0}$

Given $\text{curl } \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$

Taking div of both sides

$$\text{div curl } \mathbf{H} = \text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$$

As $\text{div curl } \mathbf{H} = 0$

$\Rightarrow$ $\text{div } \mathbf{J} + \text{div}\left(\dfrac{\partial \mathbf{D}}{\partial t}\right) = 0$

$$\text{div } \mathbf{J} = -\text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

But $\text{div } \mathbf{J} + \dfrac{\partial q}{\partial t} = 0$ (equation of continuity)

$\therefore$ $-\text{div}\left(\dfrac{\partial \mathbf{D}}{\partial t}\right) + \dfrac{\partial \rho}{\partial t} = \Rightarrow \dfrac{\partial}{\partial t}(\text{div } \mathbf{D}) = \dfrac{\partial \rho}{\partial t}$

on integrating $\boxed{\text{div } \mathbf{D} = \rho}$

8.9. Electromagnetic Energy, Poynting Theorem.

In preceding chapters we have seen that

electromagnetic potential energy, $U_c = \frac{1}{2}\int_V \mathbf{E} \cdot \mathbf{B}\, dv$ ...(1)

and energy stored in a magnetic field $U_m = \frac{1}{2}\int_V \mathbf{H} \cdot \mathbf{B}\, dv$ ...(2)

Now let us see whether these expressions apply to non-static situations.

Maxwell's equations in differential form are

$$\nabla \cdot \mathbf{D} = \rho \quad \text{or} \quad \text{div } \mathbf{D} = \rho \qquad ...(3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{or} \quad \text{div } \mathbf{B} = 0 \qquad ...(4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{or} \quad \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad ...(5)$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{or} \quad \text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad ...(6)$$

Taking scalar product of equation (5) with $\mathbf{H}$ and equation (6) with $\mathbf{E}$, we get

$$\mathbf{H} \cdot \text{curl } \mathbf{E} = -\mathbf{H}\frac{\partial \mathbf{B}}{\partial t} \qquad ...(7)$$

and
$$\mathbf{E} \cdot \text{curl}\, \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(8)}$$

Now subtracting equation (8) from equation (7), we get

$$\mathbf{H} \cdot \text{curl}\, \mathbf{E} - \mathbf{E} \cdot \text{curl}\, \mathbf{H} = -\, \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J}$$

$$= -\left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right) - \mathbf{E} \cdot \mathbf{J} \quad \text{...(9)}$$

Using vector identity.

$$\text{div}\,(\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \text{curl}\, \mathbf{E} - \mathbf{E} \cdot \text{curl}\, \mathbf{H} \quad \text{...(10)}$$

equation (9) may be expressed as

$$\text{div}\,(\mathbf{E} \times \mathbf{H}) = -\left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right) - \mathbf{E} \cdot \mathbf{J} \quad \text{...(11)}$$

Now if the medium is *linear* so that the relations

$$\mathbf{B} = \mu \mathbf{H} \quad \text{and} \quad \mathbf{D} = \varepsilon \mathbf{E} \quad \text{...(12)}$$

apply, then we may write

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \frac{\partial}{\partial t}(\varepsilon \mathbf{E}) = \frac{1}{2}\, \varepsilon \frac{\partial}{\partial t}(\mathbf{E})^3 = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{E} \cdot \mathbf{D}\right)$$

and
$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \frac{\partial}{\partial t}(\mu \mathbf{H}) = \frac{1}{2}\, \mu \frac{\partial}{\partial t}(\mathbf{H})^2 = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{H} \cdot \mathbf{B}\right)$$

Using the relationships, equation (11) takes the form

$$\text{div}\,(\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\right] - \mathbf{J} \cdot \mathbf{E} \quad \text{..(13)}$$

Each term in above equation has certain physical significance which may be seen by integrating equation (13) over a volume *V* bounded by surface *S*. Thus

$$\int_V \text{div}\,(\mathbf{E} \times \mathbf{H})\, dv = -\int_V \left\{ \frac{\partial}{\partial t} \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \right\} dv - \int_V \mathbf{J} \cdot \mathbf{E}\, dv$$

Using Gauss divergence theorem to change volume integral L.H.S. of above equation into surface integral, we get

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \tfrac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv - \int_V \mathbf{J} \cdot \mathbf{E}\, dv$$

Rearranging this equation, we get

$$-\int_V \mathbf{J} \cdot \mathbf{E}\, dv = \frac{d}{dt}\int_V \tfrac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv + \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \quad \text{...(14)}$$

To understand the physical significance of above equation, let us now interpret each term in it.

Interpretation of $\int_V \mathbf{J} \cdot \mathbf{E}\, dv$. To understand the meaning of this term let us consider a charged particle *q* moving with velocity **v** under the combined effect of mechanical, electric and magnetic forces.

The electromagnetic force due to field vectors **E** and **B** acting on the charged particle is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

As the magnetic force $q(\mathbf{v} \times \mathbf{B})$ is always perpendicular to velocity, hence the magnetic field does no work. Therefore for a single charge *q* the rate of doing work by electromagnetic field **E** and **B** is

$$\frac{\partial W}{\partial t} = \mathbf{F} \cdot \mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\, \mathbf{E} \cdot \mathbf{v}$$

If $\mathbf{F}_m$ is the mechanical force, then work done by *mechanical force against electromagnetic field vectors* per unit time *i.e.* the rate at which mechanical work is done on the particle is

$$\frac{\partial W_m}{\partial t} = \mathbf{F}_m \cdot \mathbf{v} = -\mathbf{F} \cdot \mathbf{v} = -q\,\mathbf{E} \cdot \mathbf{v} \qquad \text{...(15)}$$

If the electromagnetic field consists of a group of charges moving with different velocities *e.g.* n_i charge carriers each of charge q_i moving with velocity $\mathbf{v}_i$ $(i = 1, 2, 3, ...)$; the equation (15) may be written as

$$\frac{\partial W_m}{\partial t} = -\sum_i n_i q_i \mathbf{v}_i \cdot \mathbf{E}_i \qquad \text{...(16)}$$

In this case current density $\mathbf{J} = \sum_i \mathbf{J}_i = \sum_i n_i q_i \mathbf{v}_i$

using this substitution, equation (16) becomes

$$\frac{\partial W_m}{\partial t} = -\Sigma\, \mathbf{J}_i \cdot \mathbf{E}_i = -\mathbf{J} \cdot \mathbf{E} \qquad \text{...(17)}$$

This equation represents the *power density* that it transferred into electromagnetic field.

Therefore the expression $-\int \mathbf{E} \cdot \mathbf{J}\, dv$ represents ***rate of energy transferred into the electromagnetic field through the motion of free charge in volume V.***

If there are no sources of e.m.f. in volume V, then the term

$$-\int \mathbf{E} \cdot \mathbf{J}\, dv = -\int \frac{\mathbf{J}^3}{\sigma}\, dv \qquad (\text{Since } \mathbf{J} = \sigma\mathbf{E})$$

is negative and represents negative of rate of heat energy produced

Interpretation of $\frac{d}{dt}\int \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv$; We know

$\int_V \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\, dv = U_e$, electrostatic potential energy in volume V

$\int_V \frac{1}{2}\mathbf{H} \cdot \mathbf{B}\, dv = U_m$, magnetic energy in volume V

∴ Obviously

$$U = \int_V \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv \qquad \text{...(18)}$$

represents some sort of potential energy of electromagnetic field. One need not ascrible this potential energy to the charged particles but consider this term as a *field energy*. This is known as *electromagnetic field energy* in volume *V*. *A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of electromagnetic theory.* Obviously $\frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$ represents energy density of electromagnetic field *i.e.*

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B}) \qquad \text{...(19)}$$

Consequently the term $\frac{d}{dt}\int \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\, dv$ *represents the* rate *of electromagnetic energy stored in volume V.*

Interpretation of $\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}$ **:** Since surface integral in this term involves only electric and magnetic fields, it is feasible to interpret this term as the rate of energy flow across the surface. It means that $(\mathbf{E} \times \mathbf{H})$ itself represents the energy flow per unit time per unit area. The latter interpretation, however, leads to certain difficulties : the only interpretation which survives is that the *surface integral of* $(\mathbf{E} \times \mathbf{H})$

over a closed surface represents the amount of electromagnetic energy crossing the closed surface per second. The vector (**E** × **H**) is known as the *Poynting vector* and usually represented by the symbol **S** *i.e.*

Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad ...(20)$$

Interpretation of Energy Equation (13) or (14)

In view of above interpretations, equation (13) may be expressed as

$$-\mathbf{J} \cdot \mathbf{E} = \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} \qquad ...(21)$$

The physical meaning of this equation is that the *time rate of change of electromagnetic energy with a certain volume plus time rate of the energy flowing out through the boundary surface is equal to the power transferred into the electromagnetic field.*

This is the statement of conservation of energy in electromagnetism and is known as **Poynting theorem.**

8.10. Poynting Vector

In preceding section we have seen that

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad ...(1)$$

is known as Poynting vector and is interpreted as the power flux *i.e.* amount of energy crossing unit area placed perpendicular to the vector, per unit time. The conception of energy of the electromagnetic field as residing in the medium is very fundamental one and has great advantage in the development of the theory. Maxwell thought of the medium as resembling an elastic solid, the electrical energy representing the potential energy of strain of the medium, the magnetic energy the kinetic energy of motion. Though such a mechanical view no longer exists, still the energy is regarded as being localised in space and as travelling in the manner indicated by Poynting vector. In a light wave there is certain energy per unit volume, proportional to the square of the amplitude (*E* or *H*). This energy travels along and Poynting vector is the vector that measures the rate of flow or the intensity of the wave. In a plane electromagnetic wave **E** and **H** are at right angles to each other and at right angles to the direction of flow ; thus **E** × **H** must be along the direction of flow. In more complicated waves as well, Poynting vector points along the direction of flow of radiation. For example if we have a source of light and we wish to find at what rate it is emitting energy, we surround it by a closed surface and integrate the normal component of Poynting vector over the surface. The whole conception of energy being transported in the medium is fundamental to the electromagnetic theory of light.

In case of time varying field **S** = **E** × **H** *gives the instantaneous value of the Poynting vector.* Let us find the form of Poyting vector for such cases. Let *the fields* **E** *and H be given by real parts of complex exponentials of the form*

$$\mathbf{E} = \mathbf{E}_0 (\mathbf{r})\, e^{i\omega t}$$

At a given point of space let us assume that **E** is given by the real part of $\mathbf{E}_0 e^{i\omega t}$ and **H** by real part of $\mathbf{H}_0 e^{i\omega t}$ where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex vector functions of position. Let the real and imaginary parts of $\mathbf{E}_0$ be denoted by $\mathbf{E}_r$ and $\mathbf{H}_{im}$ respectively. Similarly real and imaginary parts of $\mathbf{H}_0$ are $\mathbf{H}_r$ and $\mathbf{E}_{im}$. Then

$$\mathbf{E} = \text{Real part of } (\mathbf{E}_0 e^{i\omega t}) = \text{Re}\,(\mathbf{E}_0 e^{i\omega t},$$

where Re denotes '*real part of*'

$$\therefore\ \mathbf{E} = \text{Re}\,(\mathbf{E}_r + i\,\mathbf{E}_{im})(\cos \omega t + i \sin \omega t) = \mathbf{E}_r \cos \omega t - \mathbf{E}_{im} \sin \omega t$$

Similarly, $\mathbf{H} = \mathbf{H}_r \cos \omega t - \mathbf{H}_{im} \sin \omega t$

Then *Poynting vector is*

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = (\mathbf{E}_r \times \mathbf{H}_r)\cos^2 \omega t + (\mathbf{E}_{im} \times \mathbf{H}_{im})\sin^2 \omega t$$
$$- [(\mathbf{E}_r \times \mathbf{H}_{im}) + (\mathbf{E}_{im} \times \mathbf{H}_r)] \sin \omega t \cos \omega t$$

We notice that there are two types of terms in the above expression ; the first two whose time average is different from zero, since $\cos^2 \omega t$ and $\sin^2 \omega t$ average to $\frac{1}{2}$; and the last term whose time average is zero since $\sin \omega t \cos \omega t$ averages to zero. Thus the *time average* of Poynting vector (average being) taken over a complete cycle) is

$$<\mathbf{S}> = <\mathbf{E} \times \mathbf{H}> = \frac{1}{2}[(\mathbf{E}_r \times \mathbf{H}_r) + (\mathbf{E}_{im} \times \mathbf{H}_{im})] \qquad ...(2)$$

This equation can be written in a convenient way by using the notation of complex conjugates, where the complex conjugate of a complex number is the number obtained from the original one by changing the sign of i, wherever it appears is indicated by *a* (*) over the number. In terms of this notation

$$(\mathbf{E} \times \mathbf{H}^*) = (\mathbf{E}_0 e^{i\omega t}) \times (\mathbf{H}_0^* e^{-i\omega t})$$
$$= \mathbf{F}_0 \times \mathbf{H}_0^* = (\mathbf{E}_r + i\,\mathbf{E}_{im}) \times (\mathbf{H}_r - i\,\mathbf{H}_{im})$$
$$= (\mathbf{E}_r \times \mathbf{H}_r + \mathbf{E}_{im} \times \mathbf{H}_{im}) + i\,(\mathbf{E}_{im} \times \mathbf{H}_r - \mathbf{E}_r \times \mathbf{H}_{im}) \qquad ...(3)$$

Comparing (2) and (3), we note that, except for the factor $\frac{1}{2}$, the real part of equation (3) is just the same as the quantity appearing in equation (2). That is, we have

$$<\mathbf{S}> = <\mathbf{E} \times \mathbf{H}> = \frac{1}{2}\,Re\,(\mathbf{E} \times \mathbf{H}^*) \qquad ...(4)$$

where **E** and **H** appearing on the R.H.S. of above equation are the complex quantities whose real parts give the real **E** and **H** appearing on the L.H.S. of the equation.

Similarly we may show that

Average electrostatic energy density.

$$<u_e> = <\frac{1}{2}\varepsilon E^2> = \frac{1}{4}\varepsilon \mathbf{E} \cdot \mathbf{E}^* \qquad ...(5)$$

and average magnetic energy density,

$$<u_m> = <\frac{1}{2}\mu H^2> = \frac{1}{4}\mu \mathbf{H} \cdot \mathbf{H}^* \qquad ...(6)$$

Ex. 4. *Calculate the magnitude of Poynting vector at the surface of the sun. Given that power radiated by sun =* $3 \cdot 8 \times 10^{26}$ *watts and radius of sun =* 7×10^8 *m.*

Solution. From definition of Poynting vector S, it is the power radiated per unit area and surface area of sun is $4\pi R^2$, R being radius of sun.

∴ If P is the total power radiated by sun.

$$P = S 4\pi R^2$$

i.e.

$$S = \frac{P}{4\pi R^2} = \frac{3 \cdot 8 \times 10^{26}}{4 \times 3 \cdot 14 \times (7 \times 10^8)^2} = \mathbf{6 \cdot 175 \times 10^7\ watt/m^2}.$$

Ex. 5. *If the average distance between the sun and earth is* $1 \cdot 5 \times 10^{11}$ *m, show that the average solar energy incident on the earth is* $\approx 2\ cal/cm^2$*-min (called the solar constant).* **(Meerut 1969)**

Solution. If r is the distance between sun and earth, and Poynting vector at the surface of earth, then

$$S_e . 4\pi r^2 = P$$

$$\therefore \quad S_e = \frac{P}{4\pi R^2} = \frac{3\cdot8 \times 10^{26}}{4 \times 3\cdot14 \times (1\cdot5 \times 10^{11})^2} \text{ watt/m}^2$$

$$= \frac{3\cdot8 \times 10^{26} \times 60}{4 \times 3\cdot14 \times (1\cdot5 \times 10^{11})^3 \times 4\cdot2 \times 10^4} \text{ cal/cm}^2\text{–min}$$

$$= 2 \text{ cal/cm}^2\text{–min.}$$

Ex. 6. *A parallel plate capacitor has circular plates, each of radius a. It is being charged. Show that the rate at which energy flow into the capacitor from the surrounding space is equal to the rate at which the stored electrical energy increases.*

Solution. Let E be the electric field at any instant within the capacitor during the charging process. Its direction is perpendicular to plates and directed from (+) and (−) plate.

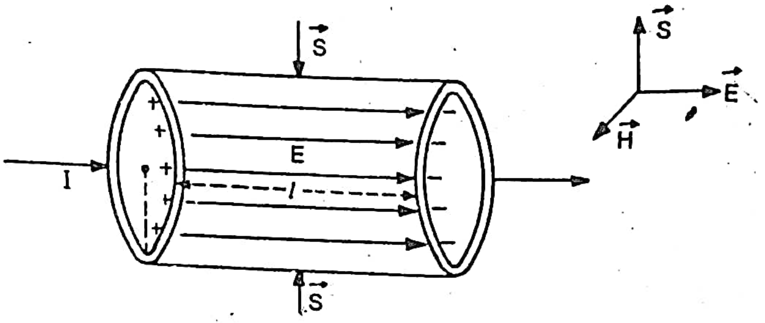


Fig. 8.2

As changing electric field produces a magnetic field given by

$$\oint \mathbf{B} \cdot dl = \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

But $\phi_E = \int_S \mathbf{E} \cdot d\mathbf{S}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ in free space

and
$$\int \mu_0 \mathbf{H} \cdot dl = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S}$$

$$\int \mathbf{H} \cdot dl = \varepsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S}.$$

As $\mathbf{H}$ is perpendicular to length of plates, we have

$$H(2\pi a) = \varepsilon_0 \frac{\partial}{\partial t} (E \pi a^2)$$

$$\Rightarrow \quad H = \frac{1}{2} \varepsilon_0 a \frac{\partial E}{\partial t} \quad \text{...(1)}$$

The directions of $\mathbf{S}$, $\mathbf{E}$ and $\mathbf{H}$ are shown in fig.

Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

In magnitude $S = EH \sin 90° = EH$

using (1)
$$S = E\left(\frac{1}{2} \varepsilon_0 a \frac{\partial E}{\partial t}\right)$$

∴ The rate of flow of field energy into the capacitor through its cylindrical surface will be

$$P = -\int_A \mathbf{S} \cdot d\mathbf{A} = \int_A S\, dA$$

(as S and area element dA are antiparallel)

$$P = \int_A \frac{1}{2} \varepsilon_0 a E \frac{\partial E}{\partial t} dA$$

$$= \frac{1}{2} \varepsilon_0 a E \frac{\partial E}{\partial t} (2\pi a l)$$

$$= \varepsilon_0 \pi a^2 l E \frac{\partial E}{\partial t} \quad \text{...(2)}$$

Energy stored in capacitor $U = \frac{1}{2}\varepsilon_0 E^2 \pi a^2 l$

Rate of increase of electrical energy

$$\begin{aligned} R &= \frac{dV}{dt} \\ &= \frac{1}{2}\varepsilon_0 2E \frac{\partial E}{dt}(\pi a^2 l) \\ &= \varepsilon_0 \pi a^2 l E \frac{\partial E}{\partial t} \qquad \text{...(3).} \end{aligned}$$

Clearly (2) and (3) are identical.

8.11. The Wave Equation

We shall now derive the equations for electromagnetic waves be the use of Maxwell's equations. This is one of the most important applications of Maxwell's equations.

Let us consider a uniform linear medium having permitivity ε, permeability μ and conductivity σ; but not any charge or any current other than that determined by Ohm's-law. Then

$$\mathbf{D} = \varepsilon\mathbf{E}\,;\ \mathbf{B} = \mu\mathbf{H}\,;\ \mathbf{J} = \sigma\mathbf{E} \text{ and } \rho = 0. \qquad \text{...(1)}$$

So the Maxwell's equations

$$\left.\begin{aligned} \operatorname{div}\mathbf{D} &= \rho \\ \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \text{and} \qquad \operatorname{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \end{aligned}\right\} \qquad \text{...(2)}$$

in this case take the form

$$\operatorname{div}\mathbf{E} = 0 \qquad \text{...(3)}$$

$$\operatorname{div}\mathbf{H} = 0 \qquad \text{...(4)}$$

$$\operatorname{curl}\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \qquad \text{...(5)}$$

and

$$\operatorname{curl}\mathbf{H} = \sigma E + \varepsilon\frac{\partial\mathbf{E}}{\partial t} \qquad \text{...(6)}$$

Taking curl of equation (5), we get

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\mu\frac{\partial}{\partial t}(\operatorname{curl}\mathbf{H})$$

Substituting curl **H** from equation (6), we get

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)$$

i.e.

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\sigma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad \text{...(7)}$$

Similarly, if we take the curl of equation (6) and substitute **E** from equation (5), we obtain

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = -\sigma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2} \qquad \text{...(8)}$$

Now using vector identity

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}$$

and keeping in view equations (3) and (4) (*i.e.*, div **E** = 0 and div **H** = 0) ; equation (7) and (8) take the form

$$\nabla^2 \mathbf{E} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and

$$\nabla^2 \mathbf{H} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad \text{...(10)}$$

Equations (9) and (10) represent wave equations which govern the electromagnetic field in a homogeneous, linear medium in which the charge density is zero; whether this medium is conducting or non-conducting. However, it is not enough that these equations be satisfied; Maxwell's equations must also be satisfied. It is clear that equations (9) and (10) are consequence of Maxwell's equations; but the converse is not true. Now the problem is to solve wave equations (9) and (10) in such a manner the Maxwell's equations are also satisfied. One method that works very well for *monochromatic wave* (*i.e.* waves characterised by a single frequency) is to obtain a solution for **E**. Then curl **E** will give time derivative of **B** $\left(\text{since curl } E = -\frac{\partial \mathbf{B}}{\partial t}\right)$ so that **B** can be computed.

It is more convenient to use the method of complex variable analysis for the solution of wave equations. The time dependence of the field (for certainty we take vector **E**) is taken to be $e^{-i\omega t}$, so that

$$\mathbf{E}(\mathbf{r}, t) = E_S(\mathbf{r})\, e^{-i\omega t} \qquad \text{...(11)}$$

It may be noted that the *physical electric field is obtained by taking the real part of* (11) : furthermore $\mathbf{E}_S(\mathbf{r})$ is in general complex so that the actual electric field is proportional to cos ($\omega t + \phi$), where ϕ is phase of $\mathbf{E}_S(\mathbf{r})$. Using equation (11), equation (9) (droping common factor $e^{-i\omega t}$) gives

that

$$\nabla^2 \mathbf{E}_S + \omega^2 \varepsilon\mu\, \mathbf{E}_S + i\omega\sigma\mu\, \mathbf{E}_S = 0 \qquad \text{...(12)}$$

Here the spatial electric field $\mathbf{E}_S$ depends on the space co-ordinates *i.e.*

$$\mathbf{E}_S = \mathbf{E}_S(\mathbf{r}).$$

For plane electromagnetic waves it is convenient to put

$$\mathbf{E}_S = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

where **k** is the propagation wave vector defined as

$$\mathbf{k} = \frac{2\pi}{\lambda}\,\mathbf{n} = \frac{\omega}{v}\,\mathbf{n}, \ \mathbf{n} \text{ being unit vector along } \mathbf{k}$$

and **r** is position vector from origin, **v** is the phase velocity of the wave.

With this in mind, equation (11) may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad \text{...(13)}$$

Here $\mathbf{E}_0$ is complex amplitude and is constant in space and time. *It is important to note that when field vector is in form* (13), *i.e.* operation of grad, div and curl on field vector is equivalent to

$$\left.\begin{aligned} &\text{grad} \rightarrow i\mathbf{k}\,;\ \text{div} = \nabla\cdot - i\,\mathbf{k}\cdot\,;\ \text{curl} = \nabla\times \rightarrow i\mathbf{k}\times \\ &\text{Also } \frac{\partial}{\partial t} \rightarrow -i\omega. \end{aligned}\right\} \qquad \text{...(14)}$$

Now we shall conside various cases of interest to determine field vectors **E** and **H** in electromagnetic field.

8.12. Plane Electromagnetic Waves in Free Space.

Maxwell's equaitons are

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \nabla\cdot\mathbf{D} = \rho \\ \text{div}\,\mathbf{B} &= \nabla\cdot\mathbf{B} = 0 \\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \qquad \text{curl}\,\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \text{ and } \begin{aligned} \mathbf{B} &= \mu\mathbf{H} \\ \mathbf{D} &= \varepsilon\mathbf{E} \\ \mathbf{J} &= \sigma\mathbf{E} \end{aligned} \qquad \text{...(1)}$$

Free space is characterised by

$$\rho = 0,\ \sigma = 0,\ \mu = \mu_0 \text{ and } \varepsilon = \varepsilon_0 . \qquad \text{...(2)}$$

Therefore Maxwell's equations reduce to

$$\left.\begin{aligned} \text{div}\,\mathbf{E} &= 0 && \text{...(a)} \\ \text{div}\,\mathbf{H} &= 0 && \text{...(b)} \\ \text{curl}\,\mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} && \text{....(c)} \\ \text{and} \qquad \text{curl}\,\mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\cdot && \text{...(d)} \end{aligned}\right\} \qquad \text{...(3)}$$

Taking curl of equation 3(c), we get

$$\text{curl curl}\,\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}(\text{curl}\,\mathbf{H})$$

Substituting curl **H** from [3(d)], we get

$$\text{curl curl}\,\mathbf{E} = \mu_0 \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{...(4)}$$

Now

$$\text{curl curl}\,\mathbf{E} = \text{grad div}\,\mathbf{E} - \nabla^2\mathbf{E}$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\nabla^2\mathbf{E} \qquad [\text{since div}\,\mathbf{E} = 0 \text{ from 3(a)}]$$

Making this substitution equation (4) becomes

$$\nabla^2\mathbf{E} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \text{...(5)}$$

Now taking curl of equation [3(d)], we get

$$\text{curl curl}\,\mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t}(\text{curl}\,\mathbf{E}).$$

Substituting curl from [3(c)], we get

$$\text{curl curl}\,\mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right) = -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad \text{...(6)}$$

Again using identity curl curl **H** = grad div **H** – ∇^2**H** and noting that div **H** = 0 from [3(b)], we obtain

$$\text{curl curl}\,\mathbf{H} = -\nabla^2\mathbf{H}.$$

Making this substitution in equation (6), we get

$$\nabla^2 \mathbf{H} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad \text{...(7)}$$

Equations (5) and (7) represent *wave equations governing electromagnetic fields* **E** *and* **H** *in free space.* It may be noted that these equations may be obtained by using (2) in equations (9) and (10) of preceding section. Equations (5) and (7) are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 u - \mu_0 \varepsilon_0 \frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{...(8)}$$

where u is a scalar *and can stand for one of the components of* **E** and **H**. It is obvious that equation (8) resembles with the general wave equation

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \qquad \text{...(9)}$$

where v is the velocity of wave.

Comparing (8) and (9), we see that *the field vectors* **E** *and* **H** *are propagated in free space as waves at a speed equal to*

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad \text{Since } \mu_0 = 4\pi \times 10^{-7} \text{ weber/Amp–m}$$

$$\varepsilon_0 = 8{\cdot}542 \times 10^{-12} \text{ farad/m}$$

$$= \sqrt{\left(\frac{4\pi}{\mu_0 . 4\pi\varepsilon_0}\right)}.$$

So that $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ m/farad.

$$= \sqrt{\left(\frac{4\pi}{4\pi \times 10^{-7}} \times 9 \times 10^9\right)}$$

$$= 3 \times 10^8 \text{ m/sec} = c, \textit{ the speed of light.}$$

Therefore it is reasonable to write c the speed of light in place of $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$; so equations (5) and (7) take the form

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{...(10)}$$

$$\nabla^2 \mathbf{H} - \frac{1}{c} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad \text{...(11)}$$

and

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0. \qquad \text{...(12)}$$

Now let us find the solution of above equations for plane electromagnetic waves. ***A plane wave is defined as a wave whose amplitude is the same at any point in a plane perpendicular to a specified direction.***

The plane wave solutions of above equations in well known form may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \, e^{i \mathbf{k} \cdot \mathbf{r} - i\omega t} \qquad \text{...(13)}$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad \text{...(14)}$$

$$u(\mathbf{r}, t) = u_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad \text{...(15)}$$

where $\mathbf{E}_0$, $\mathbf{H}_0$ and u_0 are complex amplitudes which are constant in space and time while **k** is a wave propagation vector denoted as

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi}{\lambda}\,\mathbf{n} = \frac{2\pi\nu}{c}\,\mathbf{n} = \frac{\omega}{c}\,\mathbf{n} \qquad \text{...(16)}$$

Here **n** is a unit vector in the direction of wave propagation. Now in order to apply the conditions $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{H} = 0$, let us first find $\nabla\cdot\mathbf{E}$ and $\nabla\cdot\mathbf{H}$.

$$\nabla\cdot\mathbf{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\; e^{i(k_x x + k_y y + k_z z) - i\omega t}]$$

$$[\text{since } \mathbf{k}\cdot\mathbf{r} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= [k_x x + k_y y + k_z z]$$

$$\therefore \quad \nabla\cdot\mathbf{E} = (E_{0x}\, ik_x + E_{0y}\, ik_z\, E_0)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i\mathbf{k}\cdot\mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} = i\mathbf{k}\cdot\mathbf{E}$$

Similarly $\qquad \nabla\cdot\mathbf{H} = i\mathbf{k}\cdot\mathbf{H}$

Thus the requirements $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{H} = 0$ demand that

$$\mathbf{k}\cdot\mathbf{E} = 0 \text{ and } \mathbf{k}\cdot\mathbf{H} = 0 \qquad \text{...(17)}$$

This means that *electromagnetic field vectors* **E** *and* **H** *are both perpendicular to the direction of propagation vector* **k**. ***This implies that electromagnetic waves are transverse in character.*** Further restrictions are provided by curl equations (3c) and 3(d) viz.

$$\text{curl}\,\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \text{ and } \text{curl}\,\mathbf{H} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

using equations (13) and (14), above equations yield

$$i\mathbf{k}\times\mathbf{E} = -\mu_0\cdot(-i\omega\mathbf{H}) \text{ or } \mathbf{k}\times\mathbf{E} = \mu_0\omega\mathbf{H} \qquad \text{...(18)}$$

and

$$i\mathbf{k}\times\mathbf{H} = \varepsilon_0\cdot(-i\omega\mathbf{E}) \text{ or } \mathbf{k}\times\mathbf{H} = -\varepsilon_0\omega\mathbf{E}. \qquad \text{...(19)}$$

From equation (18) it is obvious that field vector **H** is perpendicular to both **k** and **E** and according to equation (19) **E** perpendicular to both **k** and **H**. This simply means that *field vectors* **E** *and* **H** *are mutually perpendicular and also they are also perpendicular to the direction of propagation of wave.* This all in turn implies that in a plane electromagnetic wave, vectors (**E**, **H**, **k**) form a set of orthogonal vectors which form a right handed co-ordinate system in tha order (fig. 8.2).

Further from equation (18).

$$\mathbf{H} = \frac{1}{\mu_0\omega}(\mathbf{k}\times\mathbf{E}) = \frac{k}{\mu_0\omega}(\mathbf{n}\times\mathbf{E}) \qquad (\text{since } \mathbf{k} = k\mathbf{n})$$

$$= \frac{1}{\mu_0 c}(\mathbf{n}\times\mathbf{E}) \qquad \text{...(20)}$$

This equation in term of moduli

$$H = \frac{1}{\mu_0 c}\,E\left(\text{since } \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c\right)$$

Now the ratio of magnitude of **E** to the magnitude of **H** is symbolised as Z_0 *i.e.*

$$Z_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \left|\frac{\mathbf{E}_0}{\mathbf{H}_0}\right| = \mu_0 c = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}$$

$$\left(\text{since } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right)$$

$$= \sqrt{\left(\frac{4\pi \times 10^{-7}}{8{\cdot}85 \times 10^{-12}}\right)} = 376{\cdot}6 \text{ ohms} \qquad \text{...(21)}$$

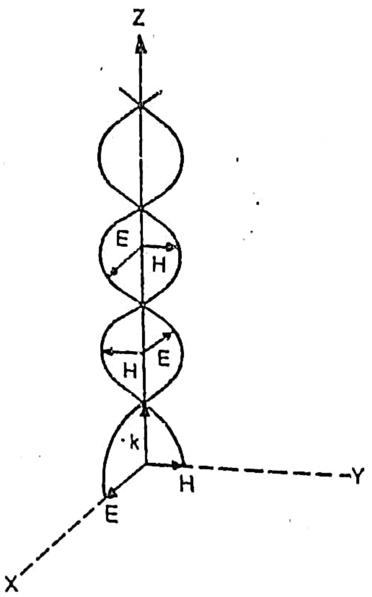


Fig. 8.2

where the units of $\mathbf{Z}_0$ are most easily seen from the fact that it measures a ratio of E in volt/m to H in amp-turn/m and therefore must equal volt/amp or Ohms. Because the units of E/H are the same as those of impedance, the value of $\mathbf{Z}_0$ is often referred to as the **wave impedance** of free space. Further since the ratio $\mathbf{Z}_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right|$ is real and positive ; this implies *that field vectors* **E** *and* **H** *are in the same phase i.e.* they have the same relative magnitude at all points at all times (fig. 8.2).

The Poynting vector (i.e. energy flow per unit area per unit time) for a plane electromagnetic wave is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \frac{\mathbf{n} \times \mathbf{E}}{\mu_0 c} \qquad \text{using (20)}$$

$$= \frac{1}{\mu_0 c}\,\mathbf{E} \times (\mathbf{n} \times \mathbf{E}) = \frac{1}{\mu_0 c}\,[(\mathbf{E}\cdot\mathbf{E})\,\mathbf{n} - (\mathbf{E}\cdot\mathbf{n})\,\mathbf{E}]$$

$$= \frac{1}{\mu_0 c}\,E^2\mathbf{n}$$

(since $\mathbf{E}\cdot\mathbf{n} = 0$, **E** being prependicular to **n**)

$$= \frac{E^2}{\mathbf{Z}_0}\,\mathbf{n} \qquad \text{[refer equation (21)]}$$

For a plane electromagnetic wave of angular frequency ω, the average value of **S** over a complete cycle is given by

$$<\mathbf{S}> = \frac{1}{\mathbf{Z}_0} <E^2> \mathbf{n}$$

$$= \frac{1}{Z_0} < \left(E_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)^2 >_{real} \mathbf{n}$$

$$= \frac{1}{Z_0}\,E_0^2 < \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) > \mathbf{n}$$

$$= \frac{1}{Z_0}\,\frac{E_0^2}{2}\,\mathbf{n} \qquad \left[\text{since } (\cos^2 < \omega t - \mathbf{k}\cdot\mathbf{r}) > = \tfrac{1}{2}\right]$$

$$= \frac{1}{Z_0} E_{rms}^2 \, \mathbf{n} \qquad \ldots(22)$$

$$\left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}}\right)$$

It is obvious that the direction of Poynting vector is along the direction of propagation of electromagnetic wave. This means that the *flow of energy in a plane electromagnetic wave in free space is along the direction of wave.*

Ratio of Electrostatic and magnetic energy densities is given by

$$\frac{u_0}{u_m} = \frac{\frac{1}{2}\varepsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\varepsilon_0}{\mu_0}\frac{E^2}{H^2} = \frac{\varepsilon_0}{\mu_0}\cdot\frac{\mu_0}{\varepsilon_0} = 1. \qquad \ldots(23)$$

$$\left[\text{since } \frac{E}{H} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\right]$$

i.e. the electromagnetic energy density is equal to magnetostatic energy density. Total *electromagnetic energy density*

$$\therefore \qquad u = u_e + u_m = 2u_e = 2 \times \tfrac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2$$

$\therefore$ Time average of *energy density*

$$\langle u \rangle = \langle \varepsilon_0 E^2 \rangle = \varepsilon_0 \langle (E_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t})^2 \rangle_{real}$$

$$= \varepsilon_0 E_0^2 \langle \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) \rangle = \tfrac{1}{2}\varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2 \qquad \ldots(24)$$

Dividing (22) by (24), we obtain

$$\frac{\langle \mathbf{S} \rangle}{\langle u \rangle} = \frac{1}{Z_0 \varepsilon_0}\mathbf{n} = \frac{1}{\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\,\varepsilon_0}\mathbf{n} = \frac{n}{\sqrt{\mu_0 \varepsilon_0}} = c\,\mathbf{n} \qquad \ldots(25)$$

Thus we obtain

$$\langle \mathbf{S} \rangle = \langle u \rangle c\,\mathbf{n} \qquad \ldots(26a)$$

i.e. energy flux = energy density × c. ...(26b)

This equation implies that the *energy density associated with an electromagnetic wave in free space propagates with the speed of light with which the field vectors do.*

Summarising we may say for electromagnetic waves in free space that :

1. In free space the electromagnetic waves travel with the speed of light.

2. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic waves. Thereby indicating the electromagnetic waves are transverse in nature.

3. The field vectors **E** and **H** are in same phase.

4. The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented by

$$\langle \mathbf{S} \rangle = \frac{E_{rms}^2}{Z_0}\mathbf{n} = \langle u \rangle \, c\,\mathbf{n}.$$

5. The electrostatic energy density is equal to the magnetic energy density and the energy density associated with the electromagnetic wave in free space propagates with the speed of light.

Ex. 7. *If the earth receives 2 cal* min^{-1} cm^{-2} *solar energy, what are the amplitudes of electric and magnetic field of radiation.*

(Rohilkhand 1997; U.P.C.S. 1979)

Solution. From Poynting theorem the energy flux per unit area per second is

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin 90 = EH \qquad \text{...(1)}$$

The energy flux per unit area per second at earth is (given)

$$2 \text{ cal min}^{-1} \text{ cm}^{-2} = \frac{2 \times 4\cdot 2 \times 10^4}{60} \text{ joules m}^{-2} \text{ sec}^{-2} \qquad \text{...(2)}$$

Comparing (1) and (2), we get

$$EH = \frac{2 \times 4\cdot 2 \times 10^4}{60} = 1400 \qquad \text{...(3)}$$

But

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\left(\frac{4\pi \times 10^{-7}}{8\cdot 85 \times 10^{-12}}\right)} = 376\cdot 6. \qquad \text{...(4)}$$

Multiplying equations (3) and (4), we get

$$E^2 = 1400 \times 376\cdot 6$$

$$\therefore \quad E = \sqrt{(1400 \times 376\cdot 6)} = \mathbf{726\cdot 1 \text{ volt/m.}}$$

Substituting this value of E in (3), we get

$$H = \frac{1400}{E} = \frac{1400}{726\cdot 1} = \mathbf{1\cdot 928 \text{ amp-turn/m}}.$$

∴ Amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 726\cdot 1 \times \sqrt{2} = \mathbf{1027 \text{ volt/m.}}$$

$$H_0 = H\sqrt{2} = 1\cdot 928\sqrt{2} = \mathbf{2\cdot 730 \text{ amp-turn/m}}.$$

Ex. 8. *Assuming that all the energy from a 1000 watt lamp is radiated uniformly, calculate the average values of the intensities of electric and magnetic fields of radiation at a distance of 2m from the lamp.*

Solution. Considering the lamp as a point source, the total flux energy over a sphere drawn round the lamp as centre is 1000 watt = 1000 joule/sec.

This energy flux falls on area $4\pi r^2 = 4\pi \times 2^2 = 16\pi^2 m^2$; therefore the energy flux per unit area per second = $\frac{1000}{16\pi^3}$ watt/m^2.

Hence from Poynting theorem

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH = \frac{1000}{16\pi^2} \qquad \text{...(5)}$$

and

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376\cdot 6. \qquad \text{...(6)}$$

Multiplying equations (5) and (6), we get

$$E^2 = \frac{1000}{16\pi^2} \times 376\cdot 6,$$

$$\therefore \quad \mathbf{E} = \sqrt{\left[\frac{1000}{16\pi^2} \times 376\cdot 6\right]} = \mathbf{48\cdot 87 \text{ volt/m.}}$$

Substituting the value of E in equation (5), we get

$$\mathbf{H} = \frac{1000}{16\pi^2 E} = \frac{1000}{16\pi^2 \times 48{\cdot}87} = 0{\cdot}1297 \textbf{ amp-turn/m}.$$

8.14. Plane Electromagnetic Waves in a Non-conducting Isotropic medium. (*i.e.* Isotropic Dielectric)

A non-conducting medium which has same properties in all directions is called an isotropic dielectric. Maxwell's equations are

$$\left.\begin{aligned} \text{div } \mathbf{D} &= \nabla \bullet \mathbf{D} = \rho \\ \text{div } \mathbf{B} &= \nabla \bullet \mathbf{B} = 0 \\ \text{curl } \mathbf{E} &= \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \text{and} \qquad \text{curl } \mathbf{H} &= \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \qquad \text{...(7)}$$

In an *isotropic dielectric* (or non-conducting isotropic medium)

$$\mathbf{D} = \varepsilon\,\mathbf{E},\ \mathbf{B} = \mu\mathbf{H},\ \mathbf{J} = \sigma\mathbf{E} = 0 \text{ and } \rho = 0$$

Therefore Maxwell's equations in this case take the form

$$\left.\begin{aligned} \text{div } \mathbf{E} &= \nabla \bullet E = 0 && \text{...(a)} \\ \text{div } \mathbf{H} &= \nabla \bullet \mathbf{H} = 0 && \text{...(b)} \\ \text{and} \qquad \text{curl } \mathbf{E} &= -\mu\frac{\partial \mathbf{H}}{\partial t} && \text{...(c)} \\ \text{curl } \mathbf{H} &= \varepsilon\frac{\partial \mathbf{E}}{\partial t} && \text{...(d)} \end{aligned}\right\}$$

Taking curl of equation (2c), we get

$$\text{curl curl } \mathbf{E} = -\mu\frac{\partial}{\partial t}(\text{curl } \mathbf{H})$$

Substituting curl **H** from (2d) in above equation

$$\text{curl curl } \mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl } \mathbf{E} = -\mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{...(3)}$$

Similarly if we take curl of (2*d*) and substitute curl **E** from (2c), we get

$$\text{curl curl } \mathbf{H} = -\mu\varepsilon\frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad \text{...(4)}$$

using vector identity

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2\mathbf{A}$$

and keeping in mind equations (2a) and (2b) *i.e.*, div **E** = 0 and div **H** = 0 equations (3) and (4) give

$$\nabla^2\,\mathbf{E} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{...(5)}$$

and

$$\nabla^2\mathbf{H} - \mu\varepsilon\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad \text{...(6)}$$

These equations are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 u - \mu\varepsilon \frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{...(7)}$$

where u is a scalar and can stand for any one of components of **E** and **H**. It is obvious that equation (6) resembles with the general wave equation

$$\nabla^2 u - \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{...(8)}$$

where v is the speed of wave.

This means that the field vector **E** and **H** are propagated in isotropic dielectric as waves with speed v given by

$$v = \frac{1}{\sqrt{(\mu\varepsilon)}} = \frac{1}{\sqrt{(K_m \mu_0 K_e \varepsilon_0)}}$$

where K_m is relative permeability of medium and K_e is relative permittivity (or dielectric constant) of th medium.

As $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$, speed of electromagnetic waves in free space.

$$\therefore \qquad v = \frac{c}{\sqrt{K_m K_e}} \qquad \text{...(10)}$$

Since $K_m > 1$ and $K_e > 1$: thereby indicating ***that the speed of electromagnetic waves is an isotropic dielectric is less than the speed of electromagnetic waves in free space.***

As

$$n = \frac{c}{v} \quad i.e.\ v = \frac{c}{n} \qquad \text{...(11)}$$

∴ Comparing (10) and (11) we note that the refractive index n *in this particular case is*

$$n = \sqrt{(K_m K_e)} \qquad \text{...(12)}$$

For a non-magnetic material $K_m = 1$; therefore

$$n = \sqrt{K_e} \quad i.e.\ n^2 = K_e \qquad \text{...(13)}$$

This relation is known as Maxwell's relation and has been verified by a number of experiments.

Replacing $\mu\varepsilon$ by $\frac{1}{v^2}$, wave equations (5) and (6) may be expressed as

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{...(14)}$$

and

$$\nabla^2 \mathbf{H} - \frac{1}{v^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad \text{...(15)}$$

The *plane-wave solutions* of equations (14) and (15) in well known from may be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad \text{...(16)}$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad \text{...(17)}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex amplitudes which are constant in space and time : while **k** is wave propagation vector given by

$$\mathbf{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\hat{n} \qquad ...(18)$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation.

Relative directions of E and H. The requirement $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$, demand that

$$\mathbf{k} \cdot \mathbf{E} = 0 \quad \text{and} \quad \mathbf{k} \cdot \mathbf{H} = 0 \qquad ...(19)$$

Comparing (7) and (8), we see

$$v = \frac{1}{\sqrt{(\mu\varepsilon)}} \qquad ...(19a)$$

This means that the field vector E and H are both perpendicular to the direction of propagation vector **k**. This implies that *electromagnetic waves in isotropic dielectric are transverse in nature* Further restrictions are provided by curl equations (2c) and (2b) *viz.*

$$\text{curl } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \text{curl } \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Using (16) and (17), these equations yield

$$\mathbf{k} \times \mathbf{E} = \mu\omega\mathbf{H} \qquad ...(20)$$

and

$$\mathbf{k} \times \mathbf{H} = -\varepsilon\omega\mathbf{E} \qquad ...(21)$$

From these equations it is obvious that *field vectors* E *and* H *are mutually perpendicular and also they are perpendicular to the direction of propagation vector* **k**. *This in turn implies that in a plane electromagnetic wave in isotropic dielectric, vector* (**E, H, k**) *from a set of orthogonal vectors which form a right handed coordinate system in given order (fig. 8.2).*

Phase of E and H and Wave Impedance. From equation (20)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k} \times \mathbf{E}) = \frac{k}{\mu\omega}(\hat{n} \times \mathbf{E})$$

$$= \frac{1}{\mu v}(\hat{n} \times \mathbf{E}) = \sqrt{\frac{\varepsilon}{\mu}}\,(\hat{n} \times \mathbf{E}) \qquad ...(22)$$

$$\left(\text{since } k = \frac{\omega}{v} \text{ and } v = \frac{1}{\sqrt{(\mu\varepsilon)}}\right)$$

Now the ratio of magnitude of E to the magnitude of H is symbolised by Z *i.e.*

$$Z = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\mathrm{E}_0}{\mathrm{H}_0} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\left(\frac{K_m\mu_0}{K_e\varepsilon_0}\right)} = \text{real quantity} \qquad ...(23)$$

This implies *that the field vectors* E *and* H *are in the same phase, i.e., they have same relative magnitudes at all points at all time. The unit of Z comes out to be ohm, since*

$$\mathbf{Z} = \frac{\mathrm{E}}{\mathrm{H}} = \frac{\text{volt/m}}{\text{amp–turn/m}} = \frac{\text{volt}}{\text{amp}} = \text{ohm};$$

hence the value of Z is referred to as *wave impedance of isotropic dielectric medium.* The *wave impedance of medium is related to that of free space by the relation*

$$Z = \sqrt{\left(\frac{K_m\mu_0}{K_e\varepsilon_0}\right)} = \sqrt{\left(\frac{K_m}{K_e}Z_0\right)} \qquad ...(24)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the wave impedance of free space.

Poynting vector for a plane electromagnetic wave in an isotropic dielectric is given by

$$\mathrm{S} = \mathrm{E} \times \mathrm{H} = \mathrm{E} \times \left(\sqrt{\frac{\varepsilon}{\mu}}\,\hat{n} \times \mathrm{E}\right)$$

$$= \frac{\mathbf{E}\times(\hat{n}\times\mathbf{E})}{Z} \quad \left(\text{since } Z = \sqrt{\frac{\mu}{\varepsilon}}\right)$$

$$= \frac{(\mathbf{E}\cdot\mathbf{E})\,\hat{n}-(\mathbf{E}\cdot\hat{n})\,\mathbf{E}}{Z}$$

$$= \frac{\mathbf{E}^2}{Z}\,\hat{n} \quad (\text{since } \mathbf{E}\cdot\hat{n} = 0 \text{ because } \mathbf{E} \text{ is perpendicular to } \hat{n})$$

The time average of *Poynting vector* is

$$<\mathbf{S}> = <\mathbf{E}\times\mathbf{H}> = <\frac{\mathbf{E}^2}{Z}\,\hat{n}>$$

$$= \frac{1}{Z} < \left(\mathbf{E}_0\, e^{i\,\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^2 >_{real}\, \hat{n}$$

Since for finding actual physical fields we often take real parts of complex exponentials.

∴

$$<\mathbf{S}> = \frac{1}{Z}\,E_0^2\, <\cos^2(\omega t - \mathbf{k}\ \mathbf{r})>\,\hat{n}$$

$$= \frac{1}{Z}\,E_0^2\cdot\tfrac{1}{2}\,\hat{n} = \frac{1}{2Z}\,E_0^2\,\hat{n}$$

$$= \frac{1}{Z}\,E_{rms}^2\,\hat{n} \quad \left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}}\right) \qquad \text{...(25a)}$$

$$= \sqrt{\left(\frac{K_e}{K_m}\right)}\,\frac{1}{Z_0}\,E_{rms}^2\,\hat{n}$$

$$= \frac{\sqrt{(K_e K_m)}}{K_m}\,\frac{1}{Z_0}\,E_{rms}^2\,\hat{n} = \frac{n}{K_m}\,\frac{1}{Z_0}\,E_{rms}^2\,\hat{n}. \qquad \text{...(25b)}$$

[becuase refractive index $n = \sqrt{(K_e K_m)}$]

$$= \frac{n}{K_m} <\mathbf{S}>_{\text{free space}} \qquad \text{...(25c)}$$

$$\left[\text{because } <\mathbf{S}>_{\text{free space}} = \frac{1}{Z_0}\,E_{rms}^2\,\hat{n}\right]$$

Equations (25a) and (25b), show that the *flow of energy is along the direction of propagation of electromagnetic wave.* Equation (25c) shows that the Poynting vector for electromagnetic wave in isotropic dielectric is $\sqrt{\left(\frac{K_e}{K_m}\right)}$ or $\frac{n}{K_m}$ times of the Poynting vector if the same electromagnetic wave were propagated through free space. It may be noted that the average of Poynting vector may also be obtained as

$$<\mathbf{S}> = <\mathbf{E}\times\mathbf{H}> = \tfrac{1}{2}\,.\,\text{Real part of } (\mathbf{E}\times\mathbf{H}^*)$$

Power flow and Energy density. Let us find the ratio of electrostatic and magnetostatic energy densities in an electromagnetic wave field *i.e.*

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\varepsilon E^2}{\frac{1}{2}\mu H^2} = \frac{\varepsilon}{\mu}\,\frac{E^2}{H^2} = \frac{\varepsilon}{\mu}\,Z^2 = \frac{\varepsilon}{\mu}\cdot\frac{\mu}{\varepsilon} = 1. \qquad \text{...(26)}$$

$$\left[\text{since } Z = \frac{E}{H} = \sqrt{\left(\frac{\mu}{\varepsilon}\right)}\right]$$

This implies that for the case of electromagnetic waves in an isotropic dielectric the electrostatic energy density (u_e) is equal to the magnetostatic energy density (u_m).

Therefore total *electromagnetic energy density*

$$u = u_e + u_m = 2u_e \text{ (since } u_e = u_m)$$
$$= 2 \cdot \frac{1}{2} \varepsilon E^2 = \varepsilon E^2$$

Therefore *time average of energy density*

$$< u > = < \varepsilon E^2 > = \varepsilon < E^2 > = \varepsilon < \left(E_0 \, e^{i \mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^2_{real} >$$
$$= \varepsilon E^2 < \cos^2 (\omega t - \mathbf{k} \cdot \mathbf{r} > = \frac{\varepsilon E_0^2}{2}$$
$$= \varepsilon E_{rms}^2 \qquad ...(27)$$

Dividing equaion (23a) with equation (27), we obtain

$$\frac{< \mathbf{S} >}{< \mu >} = \frac{(E_{rms}^2 \hat{n}/Z)}{\varepsilon E_{rms}^2} = \frac{1}{Z\varepsilon} \hat{n} = \frac{2}{\sqrt{\left(\frac{\mu}{\varepsilon}\right)} \, \varepsilon} \hat{n} \text{ since } Z = \sqrt{\frac{\mu}{\varepsilon}}$$
$$= \frac{1}{\sqrt{\mu\varepsilon}} \hat{n} = v\hat{n} \left(\text{since } v = \frac{1}{\sqrt{\mu\varepsilon}} \right)$$

Thus we obtain

$$< \mathbf{S} > = < u > v\hat{n}$$

or $$< \mathbf{S} >_{av} = u_{av} \; v\hat{n} \qquad ...(28a)$$

or in words **energy flux = v × energy density.** ...(28b)

This equation has a simple meaning. If the energy were flowing with velocity v (= phase velocity of electromagnetic wave with which electromagnetic field vectors propagate), in the direction of propagation of wave, all the energy contained in a cylinder of unit cross-section and height equal to v would cross unit cross-section per second, forming the flux. This in turn implies *that the energy density associated with an electromagnetic wave in a stationary homogeneous nonconducting medium propagates with the same speed with which the field vectors do.*

Summarising we may say for the case of electromagnetic waves in isotropic dielectric that :

1. In isotropic dielectric the electromagnetic waves travel with a speed less than the speed of light.

2. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic wave. Thereby indicating that electromagnetic waves are transverse in nature.

3. The field vectors **E** and **H** are in the same phase.

4. The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented as

$$< S > = \frac{E_{rms}^2 \, \hat{n}}{Z} = < u > v\hat{n}$$

5. The electrostatic energy density is equal to the magnetostatic energy density and the total energy density is given by

$$< u > = \varepsilon E_{rms}^2 .$$

6. The energy density associated with an electromagnetic wave propagates with the phase velocity of the wave.

Ex. 8. *A plane electromagnetic wave travelling in positive z-direction in an unbounded lossless dielectric medium with relative permeability* $\mu_r = 1$ *and relative permittivity* $\varepsilon_r = 3$ *has a peak electric field intensity* $E_0 = 6$ *V/m. Find*

(i) the speed of the wave,

(ii) the intrinsic impedance of the medium,

(iii) the peak magnetic field intensity (H_0),

and (iv) the peak Poynting vector $\mathbf{S}(z, t)$

(Banaras 1990)

Solution. $E_0 = \sqrt{(E_{0x}^2 + E_{0y}^2)} = 6 \text{ V/m}, \varepsilon_r = 3, \mu_r = 1$

(i) The speed of electromagnetic wave

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r \mu_0 \varepsilon_r \varepsilon_0}}$$

$$= \frac{1}{\sqrt{(\mu_0\varepsilon_0)}} \cdot \frac{1}{\sqrt{(\mu_r\varepsilon_r)}} = \frac{c}{\sqrt{(\mu_r\varepsilon_r)}}$$

$$= \frac{3 \times 10}{\sqrt{(1 \times 3)}} = \mathbf{1.73 \times 10^8 \text{ m/s.}}$$

(ii) Impedance of medium

$$Z = \sqrt{\left(\frac{\mu}{\varepsilon}\right)} = \sqrt{\left(\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}\right)} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)} \cdot \sqrt{\left(\frac{\mu_r}{\varepsilon_r}\right)}$$

$$= \sqrt{\left(\frac{4\pi \times 10^{-7}}{8.86 \times 10^{-12}}\right)} \cdot \sqrt{\left(\frac{1}{3}\right)} = \frac{376.6}{\sqrt{3}}$$

$$= \mathbf{217.6\ \Omega.}$$

(iii) Peak value of magnetic field

$$H_0 = \frac{E_0}{Z} = \frac{6}{217.6} = \mathbf{2.76 \times 10^{-2} \text{ A/m.}}$$

(iv) Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

$$\text{Peak Poynting vector} = E_0 H_0 = \frac{E_0^2}{Z}$$

$$= \frac{6^2}{217.6} = \mathbf{0.165 \text{ W/m}^2}.$$

8.14. Plane Electromagnetic Waves in Anisotropic Non-conducting Medium (*i.e.* Anisotropic Dielectric) :

In Anisotropic medium in one in which electromagnetic field properties depend on direction. Let us consider a non-magnetic non-conducting homogeneous anisotropic medium. In such a medium

$$\mathbf{J} = 0; \ \rho = 0 \ \text{ and } \ \mu = \mu_0 \qquad \text{...(1)}$$

Moreover the permitivity ε is no longer a scalar; but it is a tensor; so that components of electric displacement **D** are in general related to components of **E** by the equations

$$\left.\begin{aligned} D_x &= \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z \\ D_y &= \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_y \\ D_z &= \varepsilon_{zx} E_z + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z \end{aligned}\right] \qquad ...(2)$$

where the coefficients are scalar constants for a homogeneous medium.

If we choose the co-ordinate axes as the principal axes of Fresnel ellipsoid for the medium, then **D** and **E** are related by rather simple equations

$$\left.\begin{aligned} D_x &= \varepsilon_x E_x = K_x \varepsilon_0 E_x \\ D_y &= \varepsilon_y E_y = K_y \varepsilon_0 E_y \\ \text{and} \qquad D_z &= \varepsilon_z E_z = K_z \varepsilon_0 E_z \end{aligned}\right] \qquad ...(3)$$

So Maxwell's equations in an anisotropic dielectric medium take the form

$$\left.\begin{aligned} &\text{div}\,\mathbf{D} = 0 && \text{(a)} \\ &\text{div}\,\mathbf{H} = 0 && \text{(b)} \\ &\text{curl}\,\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} && \text{(c)} \\ \text{and} \qquad &\text{curl}\,\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} && \text{(d)} \end{aligned}\right] \qquad ...(4)$$

It may be noted that div **E** is not zero in this case by virtue of (4a) since **D** and **E** do not, in general, possess the same direction.

Now consider a plane wave advancing with angular frequency ω and phase velocity v along the direction of propagation vector **k.** Then

$$\mathbf{E} = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(5)$$

where **r** is the radius vector from origin and

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{\omega}{v}\mathbf{n},$$

n being unit vector along **k.**

From equation (4c)

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial t} &= -\frac{1}{\mu_0}\,\text{curl}\,\mathbf{E} \\ &= -\frac{1}{\mu_0}\,\text{curl}\,(\mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}). \end{aligned} \qquad ...(6)$$

Using vector identity

$$\text{curl}\,(\phi\mathbf{A}) = \phi\,\text{curl}\,\mathbf{A} - \mathbf{A} \times \text{grad}\,\phi \qquad ...(7)$$

and keeping in view that curl $\mathbf{E}_0 = 0$ (since $\mathbf{E}_0$ is spatially constant), equation (6) gives

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{E_0}{\mu_0} \times \text{grad}\,(e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}).$$

Now $\qquad \text{grad}\,(e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}) = i\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}.$

$$\therefore \qquad \frac{\partial \mathbf{H}}{\partial t} = \frac{E_0}{\mu_0} \times (i\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}) = \frac{ie^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}}{\mu_0}\,\mathbf{E}_0 \times \mathbf{k}.$$

Integrating,

$$\mathbf{H} = -\frac{i}{\mu_0}\,\frac{e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}}{\omega}\,\mathbf{E}_0 \times \mathbf{k}$$

$$= -\frac{1}{\mu_0\omega}\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\,\mathbf{E}_0 \times \mathbf{k}\ [= \mathbf{H}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\ \text{(say)}] \qquad ...(9)$$

$$= -\frac{1}{\mu_0\omega}\,\mathbf{k} \times \mathbf{E} \text{ using (5)}$$

$$= \frac{1}{\mu_0\omega}\,\mathbf{E} \times \mathbf{k} \qquad ...(9)$$

This shows that **H** is normal to the plane of **E** and **k**.

Now from (4d)

$$\frac{\partial \mathbf{D}}{\partial t} = \text{curl } \mathbf{H}$$

$$= \text{curl } (\mathbf{H}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}).$$

Using vector identity (7) and noting that curl $\mathbf{H}_0 = 0$, we obtain

$$\frac{\partial \mathbf{D}}{\partial t} = -\,\mathbf{H}_0 \times \text{grad } (e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t})$$

$$= -\,\mathbf{H}_0 \times i\,\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

Integrating

$$\mathbf{D} = -\,\mathbf{H}_0 \times i\,\mathbf{k}\,\frac{\mathbf{r}^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}}{(-i\omega)}$$

$$= \frac{1}{\omega}\,\mathbf{H} \times \mathbf{k} \text{ using (8)} \qquad ...(10)$$

This shows that **D** is normal to **k** and **H**. Thus equation (9) and (10) imply that both **H** and **D** are normal to the direction of propagation vector **k**. Therefore *the electromagnetic wave in anisotropic non-conducting medium is transverse with respect to* **H** *and* **D**.

Now substituting value of **H** from (9) in (10), we get

$$\mathbf{D} = \frac{1}{\omega}\left(\frac{1}{\mu_0\omega}\,\mathbf{k} \times \mathbf{E}\right) \times \mathbf{k}$$

$$= -\frac{1}{\mu_0\omega^2}\,\mathbf{k} \times (\mathbf{k} \times \mathbf{E})$$

$$= -\frac{1}{\mu_0\omega^2}\,[\mathbf{k}\cdot\mathbf{E})\,\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\,\mathbf{E}]$$

$$= -\frac{1}{\mu_0\omega^2}\,[k^2\mathbf{E} - (\mathbf{k}\cdot\mathbf{E})\,\mathbf{k}]. \qquad ...(11)$$

This equation *shows that vectors* **D, B** *and* **n** *all lie in the same plane*, (fig. 8.3).

Now the *Poynting vector* **S** = **E** × **H** is normal to the plane containing **E** and **H**. This implies that vector **D, E, k** and **S** are coplanar. Also since **k** is normal to **D** ; and **S** is normal to **E** ; therefore the angle between **S** and **k** is equal to the angle between **E** and **D**. In other words **S** is not in the direction of **k**. This leads to an important result that *in an anisotropic medium energy is not propagated in general, in the direction of wave*

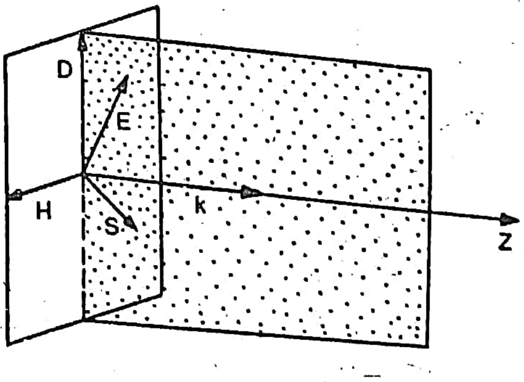


Fig. 8.3

propagation : since poynting vectors **k** represents the direction of energy flow.

Fresnel's law for phase velocity v. Equation (11) may be expressed as

$$\mathbf{D} = \frac{k^2}{\mu_0\omega^2}\,(\mathbf{E} - (\mathbf{n}\cdot\mathbf{E})\,\mathbf{n}] \quad [\text{since } \mathbf{k} = k\mathbf{n}]$$

or

$$\mathbf{D} = \frac{1}{\mu_0 v^2}\,[\mathbf{E} - (\mathbf{n}\cdot\mathbf{E})\,\mathbf{E}] \qquad ...(12)$$

$$\left(\text{since } \mathbf{k} = \frac{\omega}{v} \text{ or } v = \frac{\omega}{k}\right)$$

If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosines of unit wave vector **n** ; then

$$\mathbf{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

So that equation (12) in terms of components may be written as

$$D_x = \frac{1}{\mu_0 v^2}\,[E_x - (\mathbf{n}\cdot\mathbf{E})\cos\alpha] \qquad ...(13a)$$

$$D_y = \frac{1}{\mu_0 v^2}\,[E_y - (\mathbf{n}\cdot\mathbf{E})\cos\beta] \qquad ...(13b)$$

$$D_z = \frac{1}{\mu_0 v^2}\,[E_z - (\mathbf{n}\cdot\mathbf{E})\cos\gamma] \qquad ...(13c)$$

Now from equation (3)

$$E_x = \frac{D_x}{K_x\varepsilon_0},\; E_y = \frac{D_y}{K_y\varepsilon_0} \text{ and } E_z = \frac{D_z}{K_z\varepsilon_0}$$

But from definition of *refractive index*

$$n_x = \sqrt{K_x} = \frac{c}{v_x},\; n_y = \sqrt{K_y} = \frac{c}{v_y} \text{ and } n_z = \sqrt{K_z} = \frac{c}{v_z};$$

Also

$$c = \frac{1}{\sqrt{(\mu_0\varepsilon_0)}}$$

Using these equations, we get

$$E_x = \mu_0 v_x^2\, D_x;\; E_y = \mu_0 v_y^2\, D_y \text{ and } E_z = \mu_0 v_z^2\, D_z$$

So that equations (13) give

$$\left.\begin{aligned} D_x &= \frac{1}{\mu_0}\,\frac{(\mathbf{n}\cdot\mathbf{E})\cos\alpha}{v_x^2 - v^2} \\ D_y &= \frac{1}{\mu_0}\,\frac{(\mathbf{n}\cdot\mathbf{E})\cos\beta}{v_y^2 - v^2} \\ \text{and} \quad D_z &= \frac{1}{\mu_0}\,\frac{(\mathbf{n}\cdot\mathbf{E})\cos\gamma}{v_z^2 - v^2} \end{aligned}\right\} \qquad ...(14)$$

Since vector **D** is normal to **n**, we get

$$\mathbf{D}\cdot\mathbf{n} = D_x\cos\alpha + D_y\cos\beta + D_z\cos\gamma = 0.$$

Using equation (14), we obtain

$$\frac{\cos^2\alpha}{v_x^2 - v^2} + \frac{\cos^2\beta}{v_y^2 - v^2} + \frac{\cos^2\gamma}{v_z^2 - v^2} = 0 \qquad ...(15)$$

This is well known *Fresnel's law for phase velocity.* This equation indicates that the phase velocity v, in general, can have two values in any given direction. However for most anisotropic media, it is found that there are two directions for which equation (15) has only one solution *i.e.*, it gives only one phase velcoity. These directions are called *optic axes* and the medium is said to be *biaxial.* As a spacial case, there are also *uni-axial crystals* in which there is only one optic axis. It may be noted that optic axis is a direction in the medium and not a line.

8.15. Plane Electromagnetic Waves in a Conducting Medium.

Maxwell's equations are

$$\left.\begin{aligned} \text{div}\,\mathbf{D} &= \nabla\cdot\mathbf{D} = \rho \\ \text{div}\,\mathbf{B} &= \nabla\cdot\mathbf{B} = 0 \\ \text{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \text{curl}\,\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}\right\} \quad \text{...(1)}$$

Let us assume that medium is linear and isotropic and is characterised by permitivity ε, permeability μ and conductivity σ, but not any charge or any current other than that determined by Ohm's law. Then

$$\mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H},\ \mathbf{J} = \sigma\mathbf{E} \ \text{ and } \ \rho = 0.$$

So that Maxwell's equation (1) in this case take the form

$$\left.\begin{aligned} \text{div}\,\mathbf{E} &= 0 && \text{...(a)} \\ \text{div}\,\mathbf{H} &= 0 && \text{...(b)} \\ \text{curl}\,\mathbf{E} &= -\mu\frac{\partial \mathbf{H}}{\partial t} && \text{...(c)} \\ \text{curl}\,\mathbf{H} &= \sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t} && \text{...(d)} \end{aligned}\right\} \quad \text{...(2)}$$

Taking curl of equation [2(c)], we get

$$\text{curl curl}\,\mathbf{E} = -\mu\frac{\partial}{\partial t}(\text{curl}\,\mathbf{H})$$

Substituting curl **H** from [2(d)], we get

$$\text{curl curl}\,\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl}\,\mathbf{E} = -\sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{...(3)}$$

Similarly, if we take the curl of [2 (d)] and substitute curl **E** from [2 (c)], we obtain

$$\text{curl curl}\,\mathbf{H} = -\sigma\mu\frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{...(4)}$$

Now using vector identity

$$\text{curl curl}\,\mathbf{A} = \text{grad div}\,\mathbf{A} - \nabla^2\mathbf{A}$$

and keeping in view equations [2(a)] and [2(b)] (*i.e.* div $\mathbf{E} = 0$ and div $\mathbf{H} = 0$), equations (3) and (4) take the form

$$\nabla^2\mathbf{E} - \sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \text{...(5)}$$

$$\nabla^2\mathbf{H} - \sigma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t} = 0 \qquad \text{...(6)}$$

These equations represent wave equaitons governing electromagnetic fields **E** and **H** in a homogeneous isotropic conducting medium of conductivity σ. It is apparent that these equations are vector equations of identical form; which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2\psi - \sigma\mu\frac{\partial\psi}{\partial t} - \varepsilon\mu\frac{\partial^2\psi}{\partial t} = 0 \qquad \text{...(7)}$$

where ψ is a scalar and can stand for any one of components of **E** and **H**.

In an *isotropic dielectric* we have seen that the *time varying fields are transverse i.e.* the field vectors **E** and **H** are perpendicular to the direction in which the spatial variation occurs. In the limit of zero frequency we know from electrostatics and magnetostatics that the *static fields in a dielectric are longitudinal* in the sense that the fields are derivable from scalar potentials and so point in the direction of spatial variation.

If the conductivity is not zero, modifications are necessary. For simplicity let us assume that the fields vary in only one spatial variable x_α. Therefore decomposing the fields into longitudinal and transverse parts :

$$\left.\begin{aligned}\mathbf{E}(x_\alpha, t) &= \mathbf{E}_l(x_\alpha, t) + \mathbf{E}_t(x_\alpha, t)\\ \mathbf{H}(x_\alpha, t) &= \mathbf{H}_l(x_\alpha, t) + \mathbf{H}_t(x_\alpha, t)\end{aligned}\right\} \qquad \text{...(8)}$$

where subscript l and t denoted longitudinal and transverse parts respectively. Then, because of properties of curl operation, we find that the transverse parts of **E** and **H** satisfy the two curl equations [2(c)] and (2(b)] ; leading to transverse wave, while the longitudinal parts satisfy the equations :

From equation [2(a)]
$$\frac{\partial\mathbf{E}_l}{\partial x_\alpha} = 0 \qquad \text{...[9(a)]}$$

From equation [2(b)]
$$\frac{\partial\mathbf{H}_l}{\partial x_\alpha} = 0 \qquad \text{...[9(b)]}$$

From equation [2(c)]
$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right)\mathbf{E}_l = 0 \qquad \text{...[9(c)]}$$

[since curl $\mathbf{E}_l$ = curl grad $\phi_l = 0$]

From equation [2(d)]
$$\frac{\partial\mathbf{H}_l}{\partial t} = 0 \qquad \text{...[9(d)]}$$

[since curl $\mathbf{H}_l$ = curl grad $\phi_l = 0$]

Equations [9(c)], and [9(d)], show that the only longitudinal magnetic field is possible in a static uniform field. This is the same situation as the case of a dielectric. But equations 9(a) and 9(c) show that the longitudinal electric field is uniform in space, while possesses the *time variation* given by

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right)E_l = 0 \quad i.e. \quad \frac{\partial E_l}{\partial_l} = -\frac{\sigma}{\varepsilon}E_l$$

i.e.
$$\frac{\partial E_l}{E_l} = -\frac{\sigma}{\varepsilon}dt$$

Integrating, we get

$$\log E_l = -\frac{\sigma}{\varepsilon}t + \log E_0 \qquad \text{(where } \log E_0 \text{ is constant of integration)}$$

i.e.
$$E_l(\alpha, t) = E_0 e^{-(\sigma/\varepsilon)i} \qquad \text{...(10)}$$

Consequently, ***no static electric fields can exist in a conducting medium in the absence of an applied current density.*** For good conductors like copper ; $\sigma \approx 10^7$ mho/m, so that disturbances are damped out in an extremely short time.

Therefore we shall consider the transverse field in the conducting medium. Let us assume that the fields vary as $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, then solutions of equations (5), (6) and (7) may expressed as

$$\mathbf{E} = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \qquad ...(11)$$

$$\mathbf{H} = \mathbf{H}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \qquad ...(12)$$

$$\psi = \psi_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \qquad ...(13)$$

where $\mathbf{k}$ is a wave vector, may be complex, while $\mathbf{E}_0$, $\mathbf{H}_0$ and ψ_0 are complex amplitudes which are constant in space and time.

Substituting value of ψ from (13) in eqn. (7), we obtain

$$(-k^2 + i\sigma\mu\omega + \mu\varepsilon\omega^2)\,\psi = 0.$$

Since ψ is arbitrary, therefore this equation holds only if

$$(-k^2 + i\sigma\mu\omega + \mu\varepsilon\omega^2) = 0.$$

This means that the *propagation wave vector* $\mathbf{k}$ is complex given by

$$k^2 = \mu\varepsilon\omega^2\left(1 + \frac{i\sigma}{\omega\varepsilon}\right) \qquad ...(14)$$

In above equation the first term corresponds to the displacement current and the second to the conduction current contribution. As k is complex we may write assuming that σ is real.

$$k = \alpha + i\beta \qquad ...(15)$$

So that

$$k^2 = \alpha^2 - \beta^2 + 2i\alpha\beta. \qquad ...(16)$$

Comparing equation (14) and (16), we get

$$\left.\begin{aligned} \alpha^2 - \beta^2 &= \mu\varepsilon\omega^2 \\ \text{and} \qquad 2\alpha\beta &= \mu\omega\sigma \end{aligned}\right\} \qquad ...(17)$$

$$\left.\begin{aligned} \alpha &= \sqrt{\mu\varepsilon}\cdot\omega\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}+1}{2}\right]^{1/2} \\ \beta &= \sqrt{\mu\varepsilon}\cdot\omega\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}-1}{2}\right]^{1/2} \end{aligned}\right\} \qquad ...(18)$$

Now in terms of α and β, the field vectors $\mathbf{E}$ and $\mathbf{H}$ take the form

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\alpha+i\beta)\,\mathbf{n}\cdot\mathbf{r}-i\omega t} = \mathbf{E}_0\, e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}\, e^{i(\alpha\mathbf{n}\cdot\mathbf{r}-\omega t)} \qquad ...(19)$$

and

$$\mathbf{H} = \mathbf{H}_0\, e^{i(\alpha+i\beta)\,\mathbf{n}\cdot\mathbf{r}-i\omega t} = \mathbf{H}_0\, e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}\, e^{i(\alpha\mathbf{n}\cdot\mathbf{r}-\omega t)} \qquad ...(20)$$

From equations (19) and (20) it is obvious that *field amplitudes are spatially attenuated due to the presence of term* $e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}$. The quantity β is a measure of attenuation and is known as absorption coefficient. Also in last exponential terms in (19) and (20) the usual notation $\mathbf{k}$ has been replaced by α, therefore we conclude that the field vectors are propagated in the conducting medium with speed ($v = \omega/k$) given by

$$v = \frac{\omega}{\alpha} = \frac{1}{\sqrt{\mu\varepsilon}}\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}+1}{2}\right]^{-1/2} \qquad \text{using (18)} \quad ...(21)$$

Now let us consider the form of propagation vector $k = \alpha + i\beta$ in the two particular cases :

Case (i) *For a poor conductor* $\frac{\sigma}{\omega} \ll 1$, then we get

$$\alpha = \sqrt{\mu\varepsilon\omega} \text{ and } \beta = \sqrt{\frac{\mu}{\varepsilon}}\cdot\frac{\sigma}{2}$$

$$\therefore \qquad k = \alpha + i\beta = \sqrt{\mu\varepsilon\omega} + i\,\frac{\sigma}{\varepsilon}\sqrt{\frac{\mu}{\varepsilon}} \qquad ...(22)$$

This is correct to first order in $\sigma/\omega\varepsilon$. In this limit $\alpha \gg \beta$ and the attenuation of wave determined by β is independent of frequency, aside from the possible variation of conductivity.

Case (ii) *For a good conductor* $\sigma/\omega\varepsilon \gg 1$, so that α and β are approximately equal *i.e.*

$$\alpha \approx \beta = \mu\varepsilon\cdot\omega\sqrt{\frac{(\sigma/\omega\varepsilon)}{2}} = \sqrt{\frac{\mu\sigma\omega}{2}} \qquad ...(23)$$

$$\therefore \qquad k \approx \alpha + i\beta = (1+i) = \sqrt{\frac{\mu\sigma\omega}{2}} \qquad ...(24)$$

where only lowest order term in $\omega\varepsilon/\sigma$ have been kept.

Skin Depth or Penetration Depth. The waves given by equation (19) & (20) show an exponential damping or attenuation with distance. Greater is the value of β, greater is the attenuation. The term $1/\beta$ measures, the depth at which electromagnetic wave entering a conductor is damped to $1/e = 0{\cdot}369$. of its initial amplitude at the surface. This depth is known as the *skin depth* or the *penetration depth* and is usually represented by δ,

$$\delta = \frac{1}{\beta} = \frac{1}{\omega\sqrt{\mu\varepsilon}}\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)\right\}^2}-1}{2}\right]^{-1/2} \qquad ...(25a)$$

$$= \sqrt{\frac{2}{\mu\sigma\omega}} \left[\text{for good conductors characterised by } \frac{\sigma}{\omega\varepsilon} \ll 1\right] \qquad ...(25b)$$

It is obvious that the skin depth decreases with increasing frequency. For copper at 60 cycles δ is 0·86 cm, but at 1 megacycle, it has dropped to 0·0067. That is why in high frequency circuits current flows only on the surface of the conductors. The major importance of the skin depth is that it measures the depth to which an electromagnetic wave can penetrate a conducting medium. Therefore, the conducting sheets which are used as electromagnetic shields must be thicker than the skin depth.

Remarks (i) *For silver* $\sigma \approx 10^7$ *mho/m at a typical microwave frequency* $\approx 10^8$ *c/s, the skin depth* $\approx 10^{-4}$ *cm. Thus at microwave frequencies the skin depth in silver is very small and consequently performance of a pure silver component and a silver plated brass component would be expected to be indistinguishable.*

(ii) *For see water* $\sigma = 4{\cdot}3$ *mho/m at a frequency of 60 kc/s ; so that* $\delta \approx 1$ *meter. That is why radiocommunication with submerged submarine becomes increasing difficult at several skin depths.*

Relative directions of E and H. Substituting **E** and **H** from (11) and (12) in Maxwell's divergence equation, we obtain

$$i\mathbf{k}\cdot\mathbf{E} = 0 \quad \text{or} \quad \mathbf{k}\cdot\mathbf{E} = 0 \qquad \text{...(26)}$$

$$i\mathbf{k}\cdot\mathbf{H} = 0 \quad \text{or} \quad \mathbf{k}\cdot\mathbf{H} = 0 \qquad \text{...(27)}$$

These equations imply that field vectors **E** and **H** are both perpendicular to the direction of propagation vector **k**. This implies that electromagnetic wave in a conducting medium is transverse. Further restrictions on **E** and **H** is imposed by curl equations. Using (11) and (12) curl equations demand

$$i\mathbf{k}\times\mathbf{E} = i\mu\omega\mathbf{H} \quad i.e. \quad \mathbf{k}\times E = \mu\omega\,\mathbf{H} \qquad \text{...(28)}$$

and

$$i\mathbf{k}\times\mathbf{H} = (\sigma - i\varepsilon\omega)\,\mathbf{E} \quad i.e. \quad \mathbf{k}\times\mathbf{H} = -(\varepsilon\omega + i\sigma)\,\mathbf{E} \qquad \text{...(29)}$$

These equations imply that electromagnetic field vectors **E** and **H** are mutually perpendicular and also they are perpendicular to the direction of propagation vector **k**.

Phase of E and H. From equation (28)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k}\times\mathbf{E}) = \frac{1}{\mu\omega}\,k\,(\mathbf{n}\times\mathbf{E})$$

$$= \frac{1}{\mu\omega}(\alpha + i\beta)\,(\mathbf{n}\times\mathbf{E}) \qquad \text{...(29)}$$

This implies that

$$\left|\frac{\mathbf{H}}{\mathbf{E}}\right| = \frac{H_0}{E_0} = \frac{\alpha + i\beta}{\mu\omega} = \text{complex quantity} \qquad \text{...(30)}$$

i.e. field vectors **H** *and* **E** *are out of phase in a conductor.* The magnitude and phase of complex k, written as $k = |k|\,e^{i\phi}$, may be defined as

$$|k| = |\alpha + i\beta| = \sqrt{(\alpha^2+\beta^2)} \times \sqrt{\mu\varepsilon\,\omega}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} \qquad \text{...(31)}$$

and

$$\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right) \qquad \text{...(32)}$$

so equation (29) may be expressed as

$$\mathbf{H} = \frac{1}{\mu\omega}\,\omega\sqrt{\mu\varepsilon}\left[1 + \left(\frac{\sigma}{\mu\omega}\right)^2\right]^{1/4} e^{i\phi}\,(\mathbf{n}\times\mathbf{E})$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{-i(-\phi)}\,(\mathbf{n}\times\mathbf{E}) \qquad \text{...(33)}$$

This interpretation of this equation is that **H** lags behind **E** in time by the phase angle ϕ given by equation (32) and has a relative magnitude

$$\left|\frac{\mathbf{H}}{\mathbf{E}}\right| = \frac{H_0}{E_0} = \sqrt{\frac{\varepsilon}{\mu}}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} \qquad \text{...(34)}$$

Poynting Vector. The Poynting vector is given by

$$\mathbf{S} = (\mathbf{E}\times\mathbf{H})$$

the *time average of Poynting vector* may expressed as

$$S_{av} = \frac{1}{2}\,\text{Real part of }(\mathbf{E}\times H^*) = \frac{1}{2}\,Re\,(\mathbf{E}\times\mathbf{H}^*)$$

where $\mathbf{H}^*$ (denotes complex conjugate of **H** and *Re* denotes *real part of.*

$$\mathbf{S}_{av} = \frac{1}{2}\,Re\left[\mathbf{E}\times\left\{\sqrt{\frac{\varepsilon}{\mu}}\left[\left(1 + \frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{-i\phi}\,\mathbf{n}\times\mathbf{E}^*\right\}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} Re\,\{\mathbf{E}\times(\mathbf{n}\times\mathbf{E}^*)\,e^{-i\phi}\}$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} Re\ [\{(\mathbf{E}\cdot\mathbf{E}^*)\,\mathbf{n})-(\mathbf{E}\cdot\mathbf{n})\,\mathbf{E}^*\}\,e^{-i\phi}]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[\left(1+\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} E_0^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}\ \mathbf{n}\cos\phi \qquad ...(35)$$

$$[\text{Since } (\mathbf{E}\cdot\mathbf{E}^*) = E_0^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \text{ and } Re\,(e^{-i\phi}) = \cos\phi]$$

For good conductors $\sigma/\varepsilon\omega \gg 1$ so that $\phi = \pi/4$ and also $E_{rms} = \frac{E_0}{\sqrt{2}}$ hence,

$$\mathbf{S}_{av} = \sqrt{\left\{\left(\frac{\sigma}{2\mu\omega}\right)\right\}}\,E_{rms}^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}\ \mathbf{n}$$

Energy density. The total energy density of electromagnetic field is given by

$$u = u_e + u_m$$

where electrostatic energy

$$u_e = \frac{1}{2}Re\,\frac{1}{2}\,(\mathbf{E}\cdot\mathbf{D}^*)$$

$$= \frac{1}{4}\varepsilon\,Re\,(\mathbf{E}\cdot E^*)$$

$$= \frac{1}{4}\,\varepsilon\,E_0^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \qquad ...(37a)$$

$$= \frac{1}{2}\,\varepsilon\,E_{rms}^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \qquad ...(37b)$$

and magnetic energy density

$$u_m = \frac{1}{2}\,Re\,\frac{1}{2}\,(\mathbf{H}\cdot\mathbf{B}^*)$$

$$= \frac{1}{4}\,\mu\ Re\ (\mathbf{H}\cdot\mathbf{H}^*)$$

$$= \frac{1}{4}\,\mu\,H_0^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}$$

$$= \frac{1}{4}\,\mu\,\frac{\varepsilon}{\mu}\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}^{1/2} E_0^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}$$

[using (34)]

$$= \frac{1}{4}\,\varepsilon\left\{1+\left(\frac{\sigma}{\varepsilon\,\omega}\right)^2\right\}^{1/2} E_0^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}}$$

$$= \left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{1/2} u_e \qquad ...(38)$$

[using (37a)]

$$\therefore\quad \text{Total energy density } u = u_e + u_m = u_e + \left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} u_e$$

$$= \left[1+\left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}\right]^{1/2} u_e$$

$$= \left[1+\left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}\right]^{1/2} \times \frac{1}{2}\,\varepsilon\,E_{rms}^2\,e^{-2\beta\,\mathbf{n}\cdot\mathbf{r}} \qquad ...(39)$$

From equations (36) and (39) it is obvious that the ***energy flux and energy density are damped as the electromagnetic wave propagates in a conducting medium.*** This energy lose is due to Joule heating of the medium. From equations (37) and (38) it is also obvious that ***in a conducting medium the electrostatic and magnetic energy densities are different ; the magnetic energy density being greater than electrostatic energy density.***

Summarising we say for the case of electromagnetic waves in a conducting medium that :

1. The electromagnetic field vectors **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic wave. Thereby indicating that electromagnetic waves are transverse in nature.

2. The field vectors **E** and **H** are damped exponentially as the wave penetrates the conducting medium,

3. The field vectors **E** and **H** are not in phase : but **H** lags behind **E** by angle ϕ given by

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\varepsilon\omega}\right)$$

and moreover the magnitude of **H** is much greater than that of **E**.

4. The energy flow is along the direction of propagation of electromagnetic wave ; but is damped off exponentially.

5. The magnetic energy density is much greater than electric energy density and both are damped off exponentially.

Ex. 9. *The constitution parameters of copper are given by*

$$\mu_r = 1,\ \varepsilon_r = 1,\ \sigma = 5{\cdot}8 \times 10^7\ mho/m.$$

Calculate the intrinsic impedance of copper at a frequency

$$\nu = 3 \times 10^9\ c/s.$$

Sol. The intrinsic impedance of copper is given by

$$Z = Z_r + i\,Z_{im} = \sqrt{\left(\frac{\mu\omega}{2\sigma}\right)} + i\sqrt{\left(\frac{\mu\omega}{2\sigma}\right)} + \frac{(1+i)}{\sqrt{2}}\sqrt{\left(\frac{\mu\omega}{\sigma}\right)} = \frac{1+i}{2}\sqrt{\left(\frac{\mu}{\varepsilon}\cdot\frac{\omega\varepsilon}{\sigma}\right)}$$

$$= \frac{1+i}{\sqrt{2}} \times \sqrt{\left(\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}\right)}\sqrt{\left(\frac{\omega\varepsilon}{\sigma}\right)}$$

Here $\mu_r = \varepsilon_r = 1$, $\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)} = Z_0 = 376{\cdot}6\ \Omega$

$$\therefore \quad Z = \frac{1+i}{\sqrt{2}}(376{\cdot}6)\sqrt{\left(\frac{\omega\varepsilon}{\sigma}\right)} = \frac{1+i}{\sqrt{2}}(37{\cdot}6)\sqrt{\left(\frac{2\pi\nu\varepsilon_0}{\sigma}\right)}$$

Here $\sqrt{\left(\frac{\omega\varepsilon}{\sigma}\right)} = \sqrt{\left(\frac{\omega\varepsilon_0}{\sigma}\right)} = \sqrt{\left(\frac{2\pi \times 3 \times 10^9 \times 8{\cdot}86 \times 10^{-12}}{5{\cdot}8 \times 10^7}\right)}$

$$= 5{\cdot}36 \times 10^{-5}$$

$$\therefore \quad Z = \frac{1+i}{\sqrt{2}} \times 376{\cdot}6 \times 5{\cdot}36 \times 10^{-5}$$

$$= (0{\cdot}02)\,\frac{1+i}{\sqrt{(2)}} = (0{\cdot}02)\,e^{i\pi/4}\ \Omega$$

Ex. 10. *The constitution parameters of aluminium are given by*

$$\mu_r = 1,\ \varepsilon_r = 1\ \text{and}\ \sigma = 3{\cdot}54 \times 10^7\ mho/m.$$

Find the frequency for which the skin depth of in aluminium is 0·01 *mm.*

Sol. $\delta = \sqrt{\left(\frac{2}{\mu\sigma\omega}\right)} = \sqrt{\left(\frac{2}{\mu\sigma\,2\pi\nu}\right)} = \frac{1}{\sqrt{(\mu\sigma\pi\nu)}}$

$$\therefore \qquad \text{Frequency } \nu = \frac{1}{\delta^2 . \pi \mu \sigma}$$

$$= \frac{1}{(0{\cdot}01 \times 10^{-3})^2 \times \pi \times 4\pi \times 10^{-7} \times 3{\cdot}54 \times 10^7}$$

$$= 71{\cdot}56 \times 10^6 \text{ Hz} = \mathbf{71{\cdot}56 \text{ MHz.}}$$

8.16. A Simple Model for Dynamic Conductivity.

If the frequency of alternating field established across a conductor is increased, the conducting electrons owing to their inertia, follow the field with increasing difficulty. This suggests a decrease in the conductivity with increasing frequency. For a quantitative value of conductivity at a given frequency, let us consider the simplest model given by Drude in 1900; according to which a metal contains a certain number n_0 of electrons per unit volume free to move under the action of applied electric fields, but subject to damping force due to collision. The collisions occur between electrons and lattice vibrations, lattice imperfections and impurities. If b is the damping constant, the damping force may be expressed as

$$\mathbf{F}_{\text{damping}} = -bm\mathbf{v}.$$

Here damping constant b is assumed to be some sort of a average rate of collisions involving appreciable momentum transfer.

If **E** *is electric field strength,* then according to Newton's 2nd law, the equation of motion of *conducting electron*

$$m \frac{d\mathbf{v}}{dt} = e\mathbf{E}(\mathbf{r}, t) - bm\mathbf{v} \qquad \text{...(1)}$$

For *rapidly oscillating fields* the displacement of the electron is small compared to a wavelength, so that we may write approximately

$$\mathbf{E} = \mathbf{E}_s e^{-i\omega t}$$

where $\mathbf{E}_s$ is the electric field at the average position of the electron.

Therefore equation (1) becomes

$$m \frac{d\mathbf{v}}{dt} = e\,\mathbf{E}_s e^{-i\omega t} - bm\mathbf{v}. \qquad \text{...(2)}$$

Writing this equations in terms of current density **J** keeping in mind that

$$\mathbf{J} = n_0 e\mathbf{v} \quad \text{or} \quad \mathbf{v} = \frac{\mathbf{J}}{n_0 e}, \text{ we get}$$

$$m \frac{d}{dt}\left(\frac{\mathbf{J}}{n_0 e}\right) = e\,\mathbf{E}\,e^{-i\omega t} - bm\left(\frac{\mathbf{J}}{n_0 e}\right)$$

Simplifying and rearranging, we obtain

$$m \frac{d\mathbf{J}}{dt} + bm\mathbf{J} = n_0 e^2\,\mathbf{E}\,e^{-i\omega t} \qquad \text{...(3)}$$

To find a *steady state solution of current density* **J**, let us write

$$\mathbf{J} = \mathbf{J}_s e^{-i\omega t}$$

where $\mathbf{J}_s$ represents the *current density* at the average position of the electron.

So that equation (3) becomes

$$m \frac{d}{dt}(\mathbf{J}_s e^{-i\omega t}) + bm\mathbf{J}_s e^{-i\omega t} = n_0 e^2\,\mathbf{E}_s e^{-i\omega t}$$

i.e.
$$mJ_s(-i\omega)\,e^{-i\omega t} + bmJ_s\,e^{-i\omega t} = n_0\,e^2\,E_s\,e^{-i\omega t}$$

i.e.
$$-im\omega\mathbf{J} + bm\mathbf{J} = n_0e^2\mathbf{E}.$$

This gives

$$\therefore \quad \mathbf{J} = \frac{n_0e^2\,\mathbf{E}}{m\,(b - i\omega)} \qquad \text{...(4)}$$

If we compare this equation with $\mathbf{J} = \sigma\mathbf{E}$, we find that the conductivity σ is given by

$$\sigma = \frac{n_0e^2}{m\,(b - i\omega)}. \qquad \text{...(5)}$$

Assuming one free electron per atom, a metal such a copper ($n_0 \approx 1 \times 10^{28}$ electrons/m^3, $\sigma \approx 5 \times 10^7$ mho/m has an empirical damping constant $b \approx 3 \times 10^{13}$ sec^{-1}. This shows that for frequencies of the order of or smaller than, *microwave frequencies* ($\sim 10^{10}$ sec^{-1}) *the magnetic conductivity is essentially real i.e.,* current is in phase with the applied field) *and* independent of frequency ; so that expression (5) takes the form

$$\sigma = \frac{n_0e^2}{mb}. \qquad \text{...(6)}$$

This is well-known ***Lorentz-Drude expression for conductivity***. At higher frequencies (in the infrared and beyond), however, the conductivity is compelx and depends markedly on frequency in a manner quantitatively described by simple result (5).

8.17. Propagation of Electromagnetic Waves in Ionised Gases.

In certain cases of ionised gases when the pressure is quite low such as the ionosphere or a tenuous plasma, we may assume that there are no collisions and hence no energy losses *i.e.* damping constant $b = 0$, so that the *conductivity* σ (equation (5) of preceding section) becomes purely imaginary and is given by

$$\sigma = -\frac{n_0e^2}{mi\omega} \approx \frac{in_0e^2}{m\omega} \qquad \text{...(1)}$$

Now let us use usual Maxwell's equations

$$\left.\begin{aligned} \operatorname{div}\mathbf{D} &= \nabla\cdot\mathbf{D} = 0 \\ \operatorname{div}\mathbf{B} &= \nabla\cdot\mathbf{B} = 0 \\ \operatorname{curl}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \operatorname{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \end{aligned}\right\} \quad \text{with} \quad \left\{\begin{aligned} \mathbf{J} &= \sigma\mathbf{E} \\ \mathbf{B} &= \mu\mathbf{H} \\ \mathbf{D} &= \varepsilon\mathbf{E} \end{aligned}\right. \qquad \text{...(2)}$$

In this case of ionised medium $\rho = 0$, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$, so Maxwell's equations take the form

$$\left.\begin{aligned} \operatorname{div}\mathbf{E} &= 0 && \text{...(a)} \\ \operatorname{div}\mathbf{H} &= 0 && \text{...(b)} \\ \operatorname{curl}\mathbf{E} &= -\mu_0\frac{\partial\mathbf{H}}{\partial t} && \text{...(c)} \\ \operatorname{curl}\mathbf{H} &= \sigma\mathbf{E} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} && \text{...(d)} \end{aligned}\right\} \qquad \text{...(3)}$$

Taking the curl of equation (3c), we get

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\mu_0\frac{\partial}{\partial t}(\operatorname{curl}\mathbf{H}).$$

Substituting curl **H** from (3d), we get

$$\text{curl curl } \mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\left(\sigma \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\text{curl curl } \mathbf{E} = -\mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{...(4)}$$

Now using vector identity

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}$$

and keeping in mind that div $\mathbf{E} = 0$, equation (4) yields ...(5)

$$\nabla^2 \mathbf{E} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{...(6)}$$

Similarly taking curl of (3d), substituting curl **E** from (3c), using vector identity (6) keeping in mind div $\mathbf{H} = 0$, we obtain

$$\nabla^2 \mathbf{H} - \mu_0 \sigma \frac{\partial \mathbf{H}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \qquad \text{...(7)}$$

Equations (6) and (7) represent wave equations governing electromagnetic fields **E** and **H** in ionised medium. Eqns. (6) and (7) are vector equations of similar form, therefore each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 \psi - \mu_0 \sigma \frac{\partial \psi}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} = 0, \qquad \text{...(8)}$$

where ψ is a scalar and can stand for any one of six components of **E** and **H**.

The plane wave solutions of equations (6), (7) and (8) may be assumed of the form

$$\mathbf{E} = \mathbf{E}_0 \, e^{i \mathbf{k} \cdot \mathbf{r} - i\omega t} \qquad \text{...(9)}$$

$$\mathbf{H} = \mathbf{H}_0 \, e^{i \mathbf{k} \cdot \mathbf{r} - i\omega t} \qquad \text{...(10)}$$

$$\psi = \psi_0 \, e^{i \mathbf{k} \cdot \mathbf{r} - i\omega t} \qquad \text{...(11)}$$

where $\mathbf{E}_0$, $\mathbf{H}_0$ and ψ_0 are complex amplitudes which are constant is space and time, while **k** is a *wave propagation vector* defined as

$$\mathbf{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\hat{n}$$

where $\hat{n}$ is a unit vector along **k** and v is the *phase velocity of the wave.*

Substituting ψ from (11) in (8) and noting that $\nabla^2 \psi = -k^2 \psi$,

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \text{ and } \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi, \quad \text{we get}$$

$$-k^2 \psi + i\omega\mu_0\sigma\psi + \mu_0\varepsilon_0\omega^2\psi = 0$$

$$(k^2 - i\omega\mu_0\sigma - \mu_0\varepsilon_0\omega^2)\,\psi = 0. \qquad \text{...(12)}$$

As ψ is an arbitrary component of field vector, hence equation (12) holds only if

$$k^2 - i\omega\mu_0\sigma - \mu_0\varepsilon_0\omega^2 = 0$$

i.e.

$$k^2 = \mu_0\varepsilon_0\omega^2\left[1 + \frac{i\sigma}{\omega\varepsilon_0}\right]$$

Substituting the value of σ from equation (1), we get

$$k^2 = \mu_0\varepsilon_0\omega^2\left[1 - \frac{n_0 e^2}{m\varepsilon_0\omega^2}\right] = \frac{\omega^2}{c^2}\left[1 - \frac{n_0 e^2}{m\varepsilon_0\omega^2}\right] \qquad \left(\text{since } c^2 = \frac{1}{\mu_0\varepsilon_0}\right)$$

i.e.

$$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\right] \qquad ...(13)$$

where

$$\omega_p^2 = \frac{n_0 e^2}{m\varepsilon_0}. \qquad ...(14)$$

ω_p being known as the *plasma frequency*.

If n is the refractive index, then

$$n = \frac{c}{v}$$

$$\therefore k = \frac{\omega}{v} = \frac{n\omega}{c} \qquad ...(15)$$

Squaring we get

$$k^2 = \frac{n^2\omega^2}{c^2}.$$

Substituting k^2 from (13), we see that the *refractive index* of a *plasma* is given by

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2}. \qquad ...(16)$$

For *high frequency region* $\omega > \omega_p$ the refractive index. $n = \sqrt{[1-(\omega_p/\omega)^2]}$ is real and therefore waves propagate freely.

For *low frequency region* $\omega > \omega_p$, the refractive index n is purely imaginary. Consequently such electromagnetic waves incident on a plasma will be reflected from the surface. So if we write (in) in place of n in equation (15), then

$$k = \frac{in\omega}{c} \qquad ...(17)$$

Hence equations (9) and (10) for low frequency region may be written as

$$\mathbf{E} = \mathbf{E}_0\, e^{ik(\hat{n}\cdot\mathbf{r}) - i\omega t} = \mathbf{E}_0\, e^{-(\omega n/c)(n\cdot\mathbf{r})}\, e^{-i\omega t} \qquad ...(18)$$

$$\mathbf{H} = \mathbf{H}_0\, e^{-(\omega n/c)(\hat{n}\cdot\mathbf{r})}\, e^{-i\omega t} \qquad ...(19)$$

These equations indicate that within the plasma the electromagnetic field vectors **E** and **H** will fall off exponentially with distance from the surface.

The *skin depth or penetration depth for the plasma* will be

$$\delta_{plasma} = \frac{1}{\beta} = \frac{1}{\dfrac{\omega n}{c}} = \frac{1}{\dfrac{\omega}{c}\left(\dfrac{\omega_p^2}{\omega^2} - 1\right)^{1/2}}$$

(since here attenuation parameter $\beta = \omega n/c$)

$$= \frac{c}{\sqrt{(\omega_p^2 - \omega^2)}} \approx \frac{c}{\omega_p} \quad (\text{since } \omega \ll \omega_p).$$

On the laboratory scale plasma densities are in the range $n_0 \approx 10^{10} - 10^{12}$ electrons/m^3. This means $\omega_p = 6 \times 10^{10} - 6 \times 10^{12}$ sec^{-1}, so that typical penetration depths are of the order of 0·5 cm – 5×10^{-3} cm for low frequency fields. the expulsion of fields from within a plasma is a well known effect in controlled thermonuclear proceccsses.

Ex. 11. (a) *Show that the critical frequency for propagation of electromagnetic waves in plasma is given by*

$$f_C = 9\sqrt{n_0}$$

where n_0 is the number of electrons/m^3.

(b) *Calculate the plasma frequency and maximum penetration depth for a plasma containing 10^{18} electrons/m^3.*

Solution. (a) We know that the transmission is possible in plasma only when the refractive index is real.

The refractive index n is given by

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2} \quad i.e.\ n = \sqrt{\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \qquad \text{where } \omega_p = \sqrt{\frac{n_0 e^2}{m\varepsilon_0}}.$$

If refractive index n is real, then

$$\sqrt{\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \geq 0 \ i.e.\ \omega^2 \geq \omega_p^2.$$

If ω_o is the critical angular frequency for propagation of waves in a plasma, then

$$\omega_o = \omega_p = \sqrt{\frac{n_0 e^2}{m\varepsilon_0}}$$

$$\therefore \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2}\sqrt{\frac{n_0 e^2}{m\varepsilon_0}} = \frac{1}{2 \times 3{\cdot}14}\sqrt{\left(\frac{n_0 \times (1{\cdot}6 \times 10^{-19})^2}{9 \times 10^{-31} \times 8{\cdot}85 \times 10^{-18}}\right)} = 9\sqrt{n_0}$$

(b) $f_C = 9\sqrt{(n_0)} = 9 \times 10^{18} = 9 \times 10^9$ c/s = **9000 Mega Hertz.**

Thus peneration depth, $$\delta = \frac{c}{\sqrt{(\omega_p^2 - \omega^2)}}.$$

$\therefore$ Maximum penetration depth, $$\delta_{\max} = \frac{c}{\omega_p} = \frac{3 \times 10^8}{2\pi \times 9 \times 10^9}\ m$$

$$= 0{\cdot}53 \times 10^{-2}\ \text{m} = \mathbf{0{\cdot}53\ cm.}$$

8.18. Polarisation of Electromagnetic Waves

An electromagnetic wave in which the electric field vector **E** maintains a fixed direction relative to direction of propagation, is said to be *linearly polarised with polarisation vector* **E**. In the forgoing section we have seen that electric and magnetic field vectors are generally perpendicular to each other and also perpendicular to direction of propagation of electromagnetic wave. Hence such electromagnetic waves are plane polarised. The plane containing the direction of propagation and field vector **H** is known as the *plane of polarisation* is optics ; while the plane containing field vector **E** and direction of propagation is called the *plane of vibration.*

Any plane polarised wave can be considered to be the sum of two plane-polarised components in perpendicular directions ; but in phase. For example in fig. 8.4. the field vector **E** can be resolved in two mutually perpendicular components $\mathbf{E}_1$ and $\mathbf{E}_2$. These two independent components which are in phase may be expressed as

$$\mathbf{E}_1 = \mathbf{n}\, E_1^0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

$$\mathbf{E}_2 = \mathbf{n}_2\, E_2^0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \qquad ...(1)$$

∴ The general solution of plane-polarised wave may be expressed as ...(2)

$$E(\mathbf{r}, t) = (\mathbf{n}_1 E_1^0 + n_2 E_2^0)\, e^{i\mathbf{k}\cdot\mathbf{r}-\omega t} \qquad ...(3)$$

Here E_1^0 and E_2^0 must have the same phase so that the wave might be plane polarised. The polarisation vector $\mathbf{E}$ has magnitude $E_0 = \sqrt{[(E_1^0)^2 + (E_2^0)^2]}$ and makes an angle $\theta = \tan^{-1}(E_2^0/E_1^0)$ with $\mathbf{n}_1$.

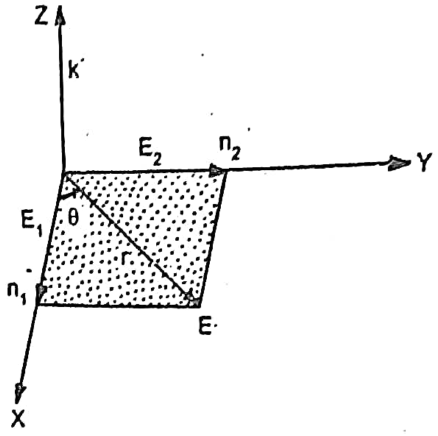


Fig. 8.6 Electric field of a linearly polarised wave.

If E_1^0 and E_2^0 have different phases, the wave (3) is *elliptically polarised.* To understand its meaning let us consider the simplest case of *circular polarisation.* If E_1^0 and E_2^0 have the same real magnitude E_0 say, but differ in phase by 90°, then equation (3) may be expressed as

$$\mathbf{E}(\mathbf{r}, t) = E^0(\mathbf{n}_1 \pm i\,\mathbf{n}_2)\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \qquad ...(4)$$

If we choose the axes so that the wave is propagating in the positive Z-direction, while E_1 and E_2 are along X and Y directions repsectively. Then the components of actual electric field, obtained by taking real part of (4) are

$$\left.\begin{aligned} E_x(\mathbf{r}, t) &= E_0 \cos(kz - \omega t) \\ E_y(\mathbf{r}, t) &= \mp E_0 \sin(kz - \omega t) \end{aligned}\right\} \qquad ...(5)$$

Squaring equations (5) and adding, we obtain

$$E_x^2 + E_y^2 = E_0^2 \quad i.e. \quad \frac{E_x^2}{E_0^2} + \frac{E_y^2}{E_0^2} = 1 \qquad ...(6)$$

which is the equation of a circle. Thus at a fixed point in space, the fields (5) are such that the electric field vector is constant in magnitude, but sweeps around in a circle at a frequency ω. For the upper sign $(\mathbf{n}_1 + i\,\mathbf{n}_2)$, the rotation is anticlockwise when the observer is facing into the oncoming wave. This is the called left handed circularly polarised light and sometimes this wave is said of have positve helicity (since

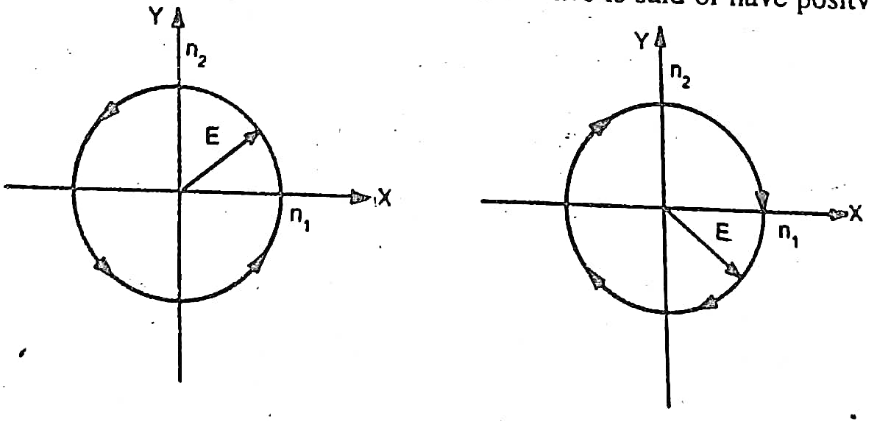


Fig. 8.7 Circularly polarised wave

the wave has a positive projection to angular momentum on Z-axis, the direction of propagation). For the lower sign $(\mathbf{n}_1 + i\,\mathbf{n}_2)$, the rotation of $\mathbf{E}$ is clockwise, then looking into the wave; this wave is called right handed circularly polarised light and is said to have *negative helicity.*

If the amplitudes E_1^0 and E_2^0 are not equal in magnitude then we may write equation (5) as

$$\left.\begin{aligned} E_x(\mathbf{r}, t) &= E_1^0 \cos(kz - \omega t) \\ E_y(\mathbf{r}, t) &= \pm E_2^0 \cos(kz - \omega t) \end{aligned}\right\} \qquad \text{...(7)}$$

Squaring and adding, we obtain

$$\frac{E_x^2}{(E_1^0)^2} + \frac{E_y^2}{(E_2^0)^2} = 1 \qquad \text{...(8)}$$

which is the equation of ellipse. This indicates that the resultant wave is elliptically polarised as shown in fig. 8.8 for a given point in space.

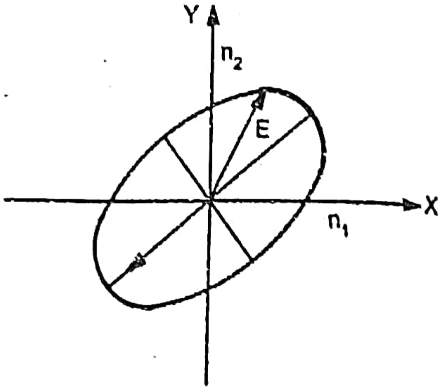


Fig. 8.8 Elliptically polarised wave

QUESTIONS AND PROBLEMS

1. State Ampere's circuital law. Discuss why and how it was modified to include displacement current. **(Meerut 1974)**
2. Establish Maxwell's equations for the electromagnetic fields and obtain an expression for Poynting vector. **(Rohilkhand 1987, 83; Agra 1975)**
3. (a) Write down Maxwell's field equations in differential and integral forms and explain their physical meaning. **(Rohilkhand 1999, 96, 86)**
   (b) State and prove Poynting's theorem relating the flow of energy at a point in space in an electromagnetic field. **(Rohilkhand 1985, 81; Agra 1976)**
4. Establish Maxwell's relations for electromagnetic field and explain their physical meaning. **(Rohilkhand 1986)**
5. Obtain the Poynting theorem for the conservation of energy in an electromagnetic field and discuss the physical meaning of each term in the resulting equation. **(Meerut 1973)**
6. Define Poynting vector. Discuss theoretically Poynting theorem for the flow of energy in an electromagnetic field. **(Meerut 1980)**
7. State and establish Poynting theorem for conservation of energy in an electromagnetic field. What isthe physical significanceof the poynting vector. **(Meerut 1997)**
8. Establish the law of conservation of energy for the electromagnetic field. Explain clearly the meaning of the Poyntings vector and obtain its dimensions. **(Meerut 1971)**
9. Write down the Maxwell's field equations for a material medium at rest and discuss the empirical basis for these equations. comment on the correspondence of the fields B and H with E and D.

   Obtain the general expression for the electromagnetic energy and explain the meaning of various terms. **(Meerut 1976)**
10. (a) Show Maxwell's relations are empirical relations.

    (b) Discuss physical significance of Poynting vector. **(Meerut 1998)**
11. Show that the law of conservation of energy for electromagnetic fields in non-conducting medium is

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

    where u is the energy density and **S** is Poynting vector.
12. State Maxwell's equations for the electromagnetic field and obtain the wave equations for **E** and **B** in homogeneous isotropic non-conducting medium. **(Meerut 1973)**
13. Write down Maxwell's equations in differential form and discuss the physical significance of each. Give the deviation of these equations. **(Rohilkhand 1996, 88)**
14. Write-Maxwell's equations in differential and integral forms and explain their physical meaning.

    Solve Maxwell's equations in free space to show that

    (a) E and B and the direction of propagation form a set of orthogonal vectors.

    (b) E and B are in phase and in constant ratio.

    (c) Energy flows with velocity of light, **(Meerut 1982)**

15. Obtain Maxwell's equations and deduce an expression for the velocity of propagation of a plane electromagnetic wave in a medium of dielectric constant ε and permeability μ. **(Agra 1985; Meerut 1991, 81)**
16. Derive the electromagnetic wave equation from Maxwell's field equations. Consider plane wave solutions of this equation and prove that the energy density associated with such a wave in a stationary homogeneous non-conducting medium propagates with the same speed with which the fields do. **(Meerut 1976)**
17. Explain the propagation of electromagnetic waves in a conducting medium and hence defining skin depth explain why radio communication with submerged submarine becomes increasing difficult at several skin depths. **(Meerut 1997; Rohilkhand 1996, 86, 99)**
18. Explain the propagation of electromagnetic waves in a conducting medium and whence defining skin depth. Show how at microwave frequencies the performance of a pure silver component and a silver plated brass component become indistinguishable. **(Rohilkhand 99)**
19. Discuss the propagation of electromagnetic waves in an isotropic dielectric medium Show that the waves are of transverse nature. Show that the wave energy is equally shared between electric and magnetic fields. **(Rohilkhand 1997)**
20. Show that for an electromagnetic field, the energy density,

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \mu_0 H^2\right)$$

and poynting vector **P** = **E** × **H**. **(Rohilkhand 1995)**

21. Discuss the propagation of plane electromagnetic waves in an isotropic dielectric medium. Show that electric and magnetic field vectors (E and H) are mutually perpendicular. **(Meerut 1992; Rohilkhand 1995, 92)**
22. Explain the theory of propagation of electromagnetic waves in a conducting medium and also explain why in high frequency circuits current flows only on surface of conductors. **(Rohilkhand 1991; Meerut 1987, 82)**
23. Establish Maxwell's equations for the propagation of electromagnetic waves in a conducting medium and show that :
    (i) the wave penetrates the conducting medium to a depth δ,
    (ii) inside the conductor the magnetic field vector H lags behind the electric field vector E.
    (Rohilkhand 1979; Agra 76, 71)
24. (a) Write Maxwell's equations for electromagnetic theory and briefly interpret each of them.
    (b) Discuss the propagation of plane electromagnetic wave in an isotropic dielectric medium. Show that P, E and H are perpendicular to each other. **(Agra 1980; Rohilkhand 1982)**
25. Write down Maxwell's equations for electromagnetic field in a homogeneous isotropic dielectric. Solve these equations to get the velocity of propagation of electromagnetic waves. Why do we regard these waves as transverse ? Show that the wave energy is equally shared between electric and magnetic fields. **(Rohilkhand 1980)**
26. Discuss the propagation of high frequency electromagnetic waves in plasma. Show that the plasma frequency is given by

$$f_c = 9\sqrt{n_0}$$

where n_o is electron density.

27. (a) Give a dynamic model of conductivity at high frequencies.
    (b) Discuss briefly the propagation of electromagnetic waves is ionosphere. Show that
    (i) Ionosphere behave as a medium of refractive index

$$n = \left[1 - \left(\frac{81 n_0}{f^2}\right)\right]^{1/2}$$

where n_0 is electron density and f is the frequency.

(ii) The maximum penetration depth is given by

$$\delta_{\max} \approx \frac{c}{\omega_p}$$

where ω_p is characteristic frequency of medium known as plasma frequency.

28. Coment on the statements

    (a) Speed of electromagnetic waves in isotropic dielectric is more than that in free space. **(Rohilkhand 1998, 86)**

    (b) Electromagnetic wave can easily penetrate in a dielectric medium but can not do so in a conducting medium. **(Kanpur 1986)**

29. Write the electromagnetic equations and find an expression for field energy and momentum densities. Explain fully the reasons which induce you to assign the physical meaning of energy and momentum densities to the expressions obtained. **(Meerut 1983)**

30. Write down the equation for electromagnetic wave propagation in a conducting media for sinusoidal time variation of the field and obtain expression for attenuation and phase constant.

31. In a conducting medium the uniform phase wave is given by

$$E(X) = E_0 e^{-rx}, \quad H(X) = H_0 e^{-rx}$$

Use Maxwell's equations to show that

$$\eta H_0 = \hat{X} \times E_0, \quad E_0 = \eta \hat{X} \times H_0$$

$$\hat{n} . E_0 = 0 \quad \hat{n} . H_0 = 0$$

(Kanpur 1986)

32. (a) Maxwell's electromagnetic theory is backbone of electrodynamics. Justify and discuss this statement.

    (b) Give an account of propagation of electrodymagnetic waves in anisotropic dielectric medium. **(Agra 1981)**

33. (a) Starting from Gauss's law for electric fields, derive Maxwell's equation is differential form based on this law.

34. Write short notes on

    (i) Displacement current. **(Agra 1976)**

    (ii) Maxwell's equations in electromagnetic fields. **(Rohilkhand 1991)**

    (iii) Poynting theorem **(Rohilkhand 1997, 92, 78; Agra 72)**

    (iv) Polarisation of electromagnetic waves. **(Rohilkhand 1990)**

    (v) Skin effect **(Meerut 1998)**

PROBLEMS

1. A plane electromagnetic wave travelling in Z-direction in an unbounded lossless dielectric medium with $K_m = 1$ and $K_c = 3$ has a peak electric field strength E = 6 volt/m. Find (i) the velocity of wave, (ii) intrinsic impedance of medium, (iii) peak value of magnetic field strength, (iv) peak value of Poynting vector.

   [**Ans.** (i) $v = 1{\cdot}70 \times 10^8$ m/s, (ii) 217·65 ohm, (iii) $2{\cdot}76 \times 10^{-2}$ Amp/m, (iv) 0·654 watt/m^2]

2. At frequency 10^6 Hertz, the conductivities of silver and aluminium are respectively given by

$$\sigma\,(\text{Silver}) = 6{\cdot}17 \times 10^7 \text{ mho/m}$$

$$\sigma\,(\text{Aluminium}) = 3{\cdot}72 \times 10^7 \text{ mho/m.}$$

   Assuming that the permeability of these materials to be equal to that of free space, Calculate values of skin depths in silver and aluminium.

3. A good dielectric is roughly defined as one for which $\sigma/\omega\varepsilon$ is equal to or less then 0·1. If conductivity σ and permeability ε of fresh water are respectively 10^{-3} mho/m and $\frac{9 \times 10^{-9}}{4\pi}$ F/m; find the frequency range for which the fresh water is a good dielectric.

4. A good conductor is roughly defined as one for which $\sigma/\omega\varepsilon$ is equal to or greater than 10. If σ and ε for sea water are 4 mho/m and $(9 \times 10^{-9}/4\pi)$ F/m find the frequency range for which sea water is good conductor.

   [**Ans.** (a) $v \geq 2{\cdot}2$ mHz, (b) $v \leq 88$ mHz]

5. A plane 1·59 GHz wave with r.m.s. electric field E = 3 V/m is travelling in a medium with constants $\sigma = 0{\cdot}01$ mho/m, $(\mu_r) = \varepsilon_r = 1$. Find the r.m.s. values of

   (i) conduction current density (ii) displacement current density (iii) total current density.

   [**Ans.** (i) ·03 A/m^2 (ii) – 0·265 j A/m^2 (iii) (·03 – ·026 j) A/m^2]

6. (a) A 10 GHz plane wave travelling in free space has an amplitude $E_x = 1$ V/m.

   (i) Find the phase velocity, the wavelength and propagation constant.

   (ii) Determine the characteristic impedance of the medium.

(iii) Find the amplitude and direction of magnetic field intensity.

(b) Repeat calculations of (i) of (a) if the wave is travelling in a lossless unbounded medium, having a permeability the same as free space but the permitivity four times that of free space.

7. A plane travelling waves has a peak electric field. $E_0 = 6\ V/m$. If the medium is lossless with $\varepsilon_r = 3$, $\mu_r = 1$, find
   (i) velocity of wave,
   (ii) peak Poynting vector,
   (iii) average Poynting vector,
   (iv) characteristic impedance of medium; and
   (v) peak value of magnetic field H.
8. A laser been has a power of 20 GW and a diameter of 2 mm. Find the peak values of E and B.